Riemann Zeta/Dirichlet Eta Zeros Characteristics Using Value of Re(eta) when Re(eta)=Im(eta).

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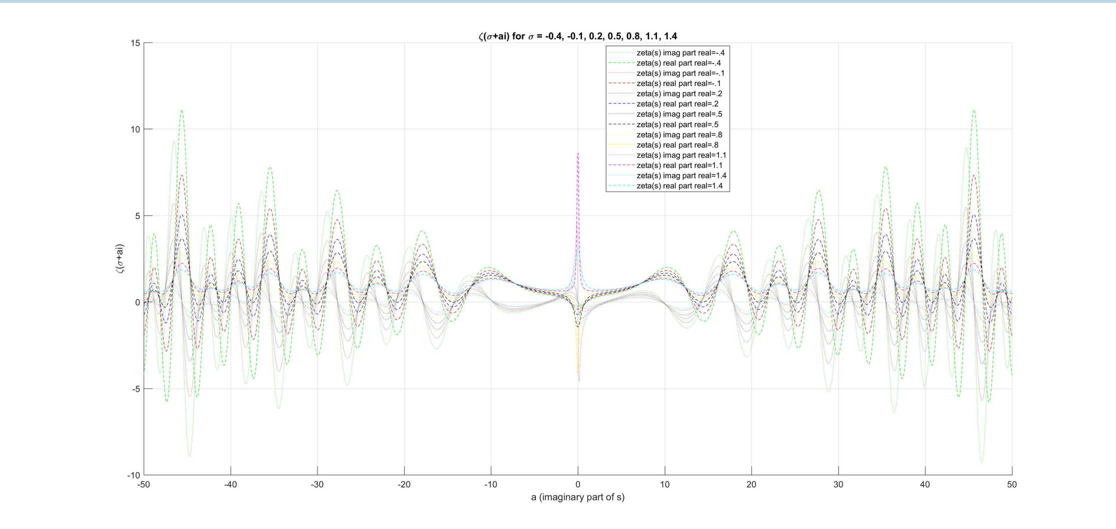
Problem: Zeta Function Zeros

A key question unresolved in Riemann's 1859 paper [1] (translation in [2]) is the nature of the roots of the Riemann xi function - are they all real? - which is equivalent to asking if the Riemann Zeta function (see below) roots all have real parts equal to 1/2.



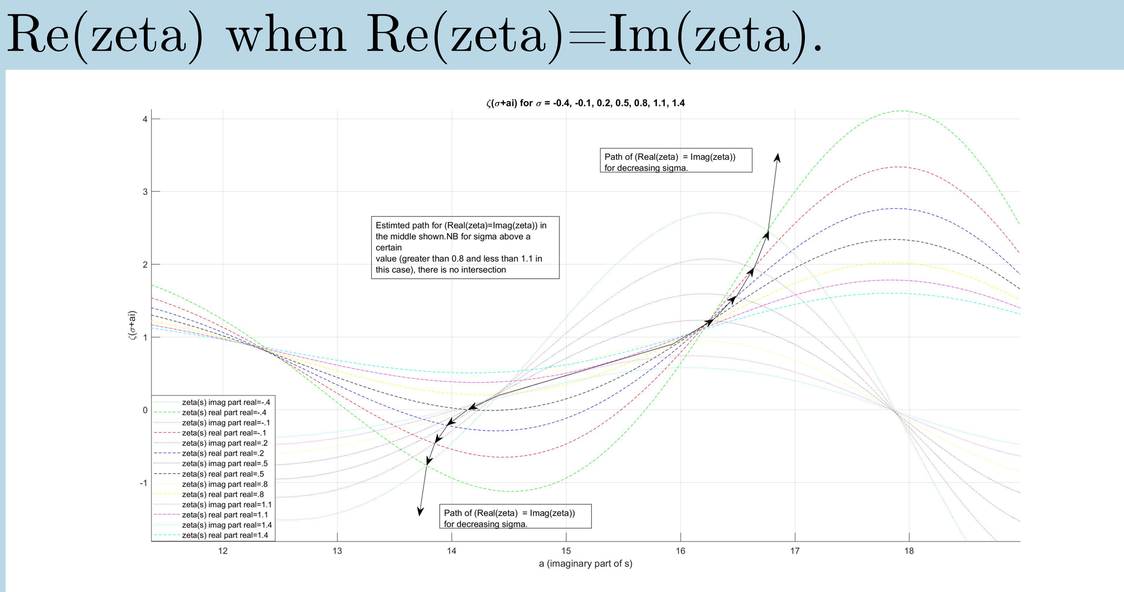
Overview

Triggered by an observation of the characteristics of the Zeta function, we will look at the Dirichlet Eta function (same zeros as the Zeta function but converges in the critical strip), investigate the implicit functions of Re(eta)=Im(eta) (noting the regions where σ and a are functions of each other and show that since the derivative of Re(eta) does not change sign when Re(eta)=Im(eta), then the eta function only has a single zero on each separate curve. Combined with the property of the Riemann xi function (xi(s) = xi(1-s)), this means that all the zeros have real part 1/2.

NB s=$\sigma + ai$ for eta(s) and zeta(s). The critical strip is defined as $0 < \sigma < 1$. Riemann proved that there are no zeros outside the critical strip, so we focus only on the critical strip.

Riemann Zeta Observation

The figure below shows in detail the real and imaginary components of the Zeta function for varying σ and a around a known zero. Note the visible zero at $\sigma = 1/2$ and the way the function changes for different values of σ - especially the always positive derivative for the value of Re(zeta) when Re(zeta)=Im(zeta).



References

[1] Riemann, B., 'Gesammelte Werke.'(Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953)

[2] Edwards, H.M., 'Riemann's Zeta Function'(Dover Publications, 2001)

[3] DLMF, https://dlmf.nist.gov/25.2.E3

[4] DLMF, https://dlmf.nist.gov/1.5

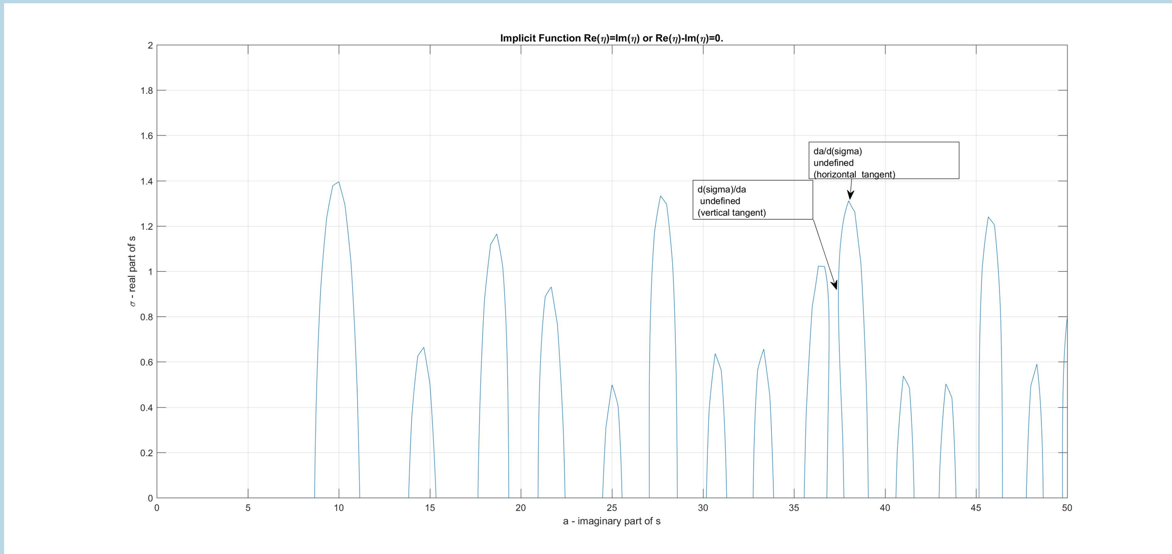
[5] N. Oo, W.-S. Gan, 'On harmonic addition theorem', International Journal of Computer and Communication Engineering, vol. 1, no. 3, 2012.

Eta Components, Implicit Functions and Derivatives

The Dirichlet η function [3] is related to the ζ function by $\eta(s)$=(1-2$^{(1-s)}$)$\zeta(s)$ and is convergent (uniformly not absolutely) for $\sigma > 0$, so it can be used to explore the zeros of $\zeta(s)$ in the critical strip.

η(s) = $\Sigma_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n^s}$ = (1-$\frac{1}{2^s}$+$\frac{1}{3^s}$-...) = 1-$\frac{cos(alog(2))}{2^\sigma}$+$\frac{cos(alog(3))}{3^\sigma}$-...+i($\frac{sin(alog(2))}{2^\sigma}$-$\frac{sin(alog(3))}{3^\sigma}$-...)

When Re($\eta(s)$) = Im($\eta(s)$) then Re($\eta(s)$)-Im($\eta(s)$)=0, describing an implicit function that relates σ and a when Re($\eta(s)$) = Im($\eta(s)$). This function is illustrated below.



The implicit function theorem [4] tells us that the above expression (being continuously differentiable) describes a curve with neighbourhoods where σ is a function of a, except where $\frac{d\sigma}{da}$ is undefined as the denominator is zero, alternatively where a is a function of σ, except where $\frac{da}{d\sigma}$ is undefined where the denominator is zero.

Differentiating and rearranging: $\frac{d\sigma}{da}$ = (-$\frac{log(2)sin(alog(2))}{2^\sigma}$+$\frac{log(3)sin(alog(3))}{3^\sigma}$+$\frac{log(2)cos(alog(2))}{2^\sigma}$-$\frac{log(3)cos(alog(3))}{3^\sigma}$+...) /(+$\frac{log(2)sin(alog(2))}{2^\sigma}$-$\frac{log(3)sin(alog(3))}{3^\sigma}$+$\frac{log(2)cos(alog(2))}{2^\sigma}$-$\frac{log(3)cos(alog(3))}{3^\sigma}$+...)

Totally differentiating Re($\eta(s)$): D(Re(η)) =dσ $\frac{}{da}$ ($\frac{log(2)cos(alog(2))}{2^\sigma}$-$\frac{log(3)cos(alog(3))}{3^\sigma}$ +...)+ $\frac{log(2)sin(alog(2))}{2^\sigma}$ -$\frac{log(3)sin(alog(3))}{3^\sigma}$ +...

A similar process can be completed for $\frac{da}{d\sigma}$.

In addition, a similar process is valid for Im($\eta(s)$).

Harmonic Addition Theorem, Riemann Xi and Conclusion

Noting the Harmonic Addition Theorem [5]:

Given $x_s(t)$=$\Sigma_{i=1}^{L}\alpha_i sin(\omega_0 t + \phi_i)$ or $x_c(t)$=$\Sigma_{i=1}^{L}\alpha_i cos(\omega_0 t + \phi_i)$, it is possible to find β and Ψ so that $x_s(t) = \beta sin(\omega_0 t + \Psi)$ or $x_c(t) = \beta cos(\omega_0 t + \Psi)$, where:

$\beta = (\Sigma_{i=1}^{L}\alpha_i^2 + 2\Sigma_{i=1}^{L-1}\Sigma_{j=i+1}^{L}\alpha_i\alpha_j cos(\phi_i - \phi_j))^{\frac{1}{2}}$ and: Ψ = arg($\frac{\Sigma_{i=1}^{L}\alpha_i sin\phi_i}{\Sigma_{i=1}^{L}\alpha_i cos\phi_i}$), $-\pi < \Psi \leq \pi$

NB β as an amplitude does not change sign as σ and a are varied (without rearranging series), but can have a minimum of zero.

Applying this theorem to the derivative expressions above by substituting log(2) for ω_0 and a for t and noticing that the α_i and ϕ_i terms are identical for both the sin and cos series (and noting that both the series in the expression for Ψ converge as they are phase shifted versions of the sin and cos series and that we can determine β from the value of Ψ and the convergent series of the real or imaginary component of eta - even though it may be difficult to show the convergence of the series for β directly):

$\frac{d\sigma}{da}$ = ((-$\beta sin(log(2)a + \Psi)$+$\beta cos(log(2)a + \Psi)$)/($\beta cos(log(2)a + \Psi)$+$\beta sin(log(2)a + \Psi)$)) Exp 1

D(Re($\eta(s)$)) =$\beta csc(log(2)a + \Psi + \frac{\pi}{4})/2^{\frac{1}{2}}$ Exp 2

And: $\frac{da}{d\sigma}$ = ((-$\beta sin(log(2)a+\Psi)$-$\beta cos(log(2)a+\Psi)$)/(-$\beta cos(log(2)a+\Psi)$+$\beta sin(log(2)a+\Psi)$)) Exp 3

D(Re($\eta(s)$)) = $-\beta csc(log(2)a + \Psi - \frac{\pi}{4})/2^{\frac{1}{2}}$ Exp 4

The csc function has no zeros (and is undefined in between sections of alternating all positive values and all negative values) . All expressions are valid for all σ and a values for $\eta(s)$ (and describe a single valued function for each σ,a input) - except those points where $\frac{d\sigma}{da}$ and $\frac{da}{d\sigma}$ are undefined.

Exp 1 describes a number of curves with neighbourhoods where σ is a function of a, except where Exp 1 is undefined when the denominator is zero. Exp 2 gives the derivative of the function which describes the value of the real part of η(s) in those neighbourhoods, which is pos(neg) in one neighbourhood where σ is a function of a (ie the value of the real part of η(s) incs(decs) for inc a), is undefined at the same points where Exp 1 is undefined and is neg(pos) in the adjacent neighbourhood (ie the value of the real part of η(s) incs(decs) for dec a). This means that each separate curve segment describing the value of the real part of η(s) when Re($\eta(s)$) = Im($\eta(s)$) always has a pos(neg) derivative. The same argument holds for Exps 3 and 4 (except that a is now a function of σ)

This means that the separate curve segments described by Exp2 and Exp 4 either have all pos or all neg derivatives (not changing sign, although individual segments might have pos or neg derivatives) - which means that they can only have a single zero per curve. This, in turn, means that there can be only one zero in the local region of any particular value of a.

To conclude, noting that one of the key properties of the Riemann $\xi(s)$ function (which has the same zeros as $\zeta(s)$ and $\eta(s)$) is that $\xi(s)$=$\xi(1-s)$, this means that for each value of a there is at most 1 zero, at $\sigma = 1/2$ for $\zeta(s)$ and $\eta(s)$ (ie a real zero in the case of $\xi(s)$). Riemann Hypothesis Proved.