The Wave Function of the Universe near the Big-Bang Singulatrity and the Generalized Uncertainty Principle

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Abstract

We investigate the Friedmann equations and the Wheeler-DeWitt equations in three dimensional pure gravity under the Generalized Uncertainty Principle (GUP) effects. In addition we study the wave functions near the Big-Bang singularity as the solutions of the deformed Wheeler-DeWitt equation in momentum space. The resulting wave functions are represented as the Mathieu functions. The GUP is considered in the context of the Snyder non-commutative space.

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1 Introduction

Quantum Gravity is the most fascinating and mysterious theory which our human beings have never seen yet. The Big-Bang singularity is expected to be well described by quantum gravity. In our previous report[1] we investigated the wave function of the universe near the Big-Bang singularity in three dimensional pure gravity.

On the other hand, the existence of a minimal observable length has long been suggested in quantum gravity as well as string theory[6][7]. The quantum effects of gravity at the scale of this minimal length become as important and essential as the electroweak and strong interactions. In addition the standard Heisenberg commutation relation in the quantum mechanics is deformed by the existence of a minimal length, since according to the usual uncertainty principle, the length can be measured to an arbitrary precision if momentum is not measured. Several approaches of the GUP(Generalized Uncertainty Principle) have been studied by many researchers[9][11][17]. In one dimensional case, the simplest form of the GUP is represented in the context of the Snyder non-commutative space[2]. Non-commutative geometries are widely considered as plausible candidates for describing physics at the Planck scale [19] and have natural connections with string theory [20] and the three dimensionally extended object.[24]

In this report we consider the deformed Friedmann equations and the deformed Wheeler-DeWitt equations in the context of the Snyder non-commutative space. The Snyder approach is relevant since it can be related to some of DSR (doubly special relativity) models[3][16] and has some motivations from loop quantum gravity [4] as well as two-time physics [5]. If the GUP is considered in FRW universe, the cosmological bounce is reported to be allowed in an isotropic flat model[14] . The GUP is related to quantum gravity fundamentally , since the uncertainty principle is essential in quantum mechanics. The wave function near the Big-Bang singularity is deformed by including the GUP effects , some information of which may be observed by the gravitational wave projects. Recently it is reported that the gravitational wave originated from the Big-bang quantum fluctuation of space-time may be detected experimentally (LiteBIRD)[23]

Our SET-UP as a toy model:

- We deal with three dimensional gravity with no matter
- FRW model
- We consider the Friedmann equation as the Einstein equation in homogeneous and isotropic case.
- The quantized equation of the Friedman equation is looked at as the Wheeler-DeWitt equation.
- We consider the Wheeler-DeWitt equation as the quantum equation of the Einstein equation.

The purpose of this report is that the deformed Friedmann equations and the deformed Wheeler DeWitt equations under the GUP effects are calculated and the wave functions as the solutions of the deformed Wheeler-DeWitt equations near the big-bang singularity are obtained.

The structure of this report is as follows. In the next section we review briefly the GUP in the context of the Snyder non-commutative geometry. In section 3 we recall the derivation of the non-deformed Friedmann equations and the non-deformed Wheeler-DeWitt equations in three dimensional space-time.[1] In section 4 we derive the deformed Friedmann equations and the deformed Wheeler-DeWitt equations and the wave functions as the solutions of the deformed Wheeler DeWitt equations are studied near the Big Bang singularity. In section 5 we end the report with a brief conclusion. In Appendix the Mathieu Functions are explained in brief.

2 The Generalized Uncertainty Principle

We review the GUP in the context of the Snyder non-commutative space briefly[14][15] The existence of a minimal length requires the deformation of the Heisenberg 's uncertainty principle, since if such a length exists, the ordinary Heisenberg's Uncertainty principle no longer valids. Various approaches to the deformation of the Heisenberg 's Uncertainty Principle have been studied[9][11] .We will concentrate on the case of the Snyder non-commutative space.

This uncertainty principle might be generalized in such a way that a fundamental uncertainty on position is increased by new momentum- dependent terms.

2.1 Snyder deformed Heisenberg algebras

We review following[14].
We consider a n-dimensional non-commutative (deformed)Euclidean space such that the commutator between the coordinates has the non-trivial structure

$$[\tilde{q}_i, \tilde{q}_j] = \alpha M_{ij} \tag{1}$$

$$\{i, j\} \in \{1..n\} \tag{2}$$

where $\tilde{q}_i$ are the non-commutative coordinates and α is the deformation parameter with dimension of a squared length, and $[A, B] \equiv AB - BA$. We demand that the rotation generators M_{ij} satisfy the ordinary $SO(n)$ algebra and that the translation group is not deformed.

$$\begin{aligned} M_{ij} &= -M_{ji} = i(q_i p_j - q_j p_i) & (3) \\ [M_{ij}, M_{kl}] &= \delta_{jk} M_{il} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik} + \delta_{il M_{jk}} & (4) \\ [p_i, p_j] &= 0 & (5) \end{aligned}$$

In order to preserve the rotational symmetry the commutators between the rotation generators M_{ij} and the coordinates $\tilde{q}_k$, as well as between M_{ij} and the momentum p_k, have to be undeformed. We assume the relations

$$[M_{ij}, \tilde{q}_k] = \delta_{jk} \tilde{q}_i - \delta_{ik} \tilde{q}_j \tag{6}$$

$$[M_{ij}, p_k] = \delta_{jk} p_i - \delta_{ik} p_j \tag{7}$$

This way the (Euclidean) Snyder space [8]is dealt with . The above relations do not uniquely fix the commutators between $\tilde{q}_i$ and p_j. The most general $SO(n)$ covariant realization for $\tilde{q}_i$ is given by

$$\tilde{q}_i = q_i \varphi_1(\alpha p^2) + \alpha (q_j p_j) p_i \varphi_2(\alpha p^2) \tag{8}$$

where the convention $a_i b_i = \sum_i a_i b_i$ is adopted and φ_1 and φ_2 are finite functions. In order to recover the ordinary Heisenberg algebra as $\alpha = 0$, the boundary condition reads $\varphi_1(0) = 1$ The commutator between $\tilde{q}_i$ and p_j arises from the realizaion (8) and reads

$$[\tilde{q}_i, p_j] = i(\delta_{ij} \varphi_1 + \alpha p_i p_j \varphi_2) \tag{9}$$

From the above relation we obtain the generalized uncertainty principle (GUP) underlying the Snyder non-commutative space as

$$\Delta\tilde{q}_i\Delta p_j \geqslant \frac{1}{2}|\delta_{ij}\langle\varphi_1\rangle + \alpha\langle p_i p_j \varphi_2\rangle| \tag{10}$$

In the above equation the ordinary Heisenberg framework is recovered in the $\alpha \to 0$ limit. The deformation of the only commutator between the spatial coordinates leads to infinitely many realizations of the algebra. We should note that , if $\varphi \neq 0$, compatible observables no longer exist.

Interestingly , for one-dimensional systems, this picture is almost uniquely fixed. The most general realization is given by

$$\tilde{q} = q\varphi(\alpha p^2) = q\sqrt{1-\alpha p^2} \tag{11}$$

The commutation relation is given by inserting (11) into (9) :

$$[\tilde{q}, p] = [q\sqrt{1-\alpha p^2}, p] = (qp - pq)\sqrt{1-\alpha p^2} = [q,p]\sqrt{1-\alpha p^2} = i\sqrt{1-\alpha p^2} \tag{12}$$

where we used the ordinary quantum mechanical commutation relation:

$$[q,p] = i, \;\; [q,q] = [p,p] = 0 \tag{13}$$

In one dimension, the simplest form of GUP in the context of the Snyder non-commutative space can be written as

$$\Delta q\Delta p \geq \frac{1}{2}| < \sqrt{1-\alpha p^2} > | \tag{14}$$

The above relation has three cases:

$$\alpha < 0, \;\; \alpha = 0, \;\; \alpha > 0 \tag{15}$$

1)$\alpha < 0$, minimal uncertainty relation : At the first order in α the string theory result is recovered

$$\Delta q \gtrsim (\frac{1}{\Delta p} + l_s^2\Delta p) \tag{16}$$

where l_s is the string length.

$$l_s = (-\alpha/2)^{1/2} \tag{17}$$

2)$\alpha = 0$, Standard Heisenberg unceraínty principle
3)$\alpha > 0$, A vanishing uncertainty principle in the non-commutative coordinate is allowed and appears as soon as Δp reaches the critical value :

$$(\Delta p)^* = \sqrt{(1-\alpha) < p >)/\alpha} \tag{18}$$

Thus the deformed quantum mechanical commutation relation can be written as

$$[q, p] = i\sqrt{1 - \alpha p^2} \tag{19}$$

where we wrote $\tilde{q}$ as q. The above deformation parameter α has the freedom of the signature.
We can conclude that a minimum length($\alpha < 0$) or a maximal momentum($\alpha > 0$) is predicted by the Snyder-deformed relation. The effects of the modified Heisenberg uncertainty relations including the Snyder-deformed relation has been studied . Among them the implications of a deformed Heisenberg algebra on the Friedmann cosmological model and the Wheeler-DeWitt equations after the investigation of non-deformed cases in the next section.

3 The non-deformed FRW equation and the non-deformed Wheeler-DeWitt equation

First we recall the ordinary non-deformed FRW cosmological model in three dimensional space-time following the previous report[1] The action of pure gravity with no matter in three dimensions

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R - 2\Lambda) \tag{20}$$

where R is the scalar carvature and Λ is the cosmological constant. We first consider the equations governing homogeneous and isotropic universe in minisuperspace model. In terms of the (2+1) dimensional Friedmann-Robertson-Walker (FRW) metric:

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right) \\ &= -dt^2 + a^2(t)d\sigma^2 \\ &= g_{\mu\nu}dx^\mu dx^\nu \end{aligned} \tag{21}$$

where k is a constant , the value of which equals $+1, 0$ or -1 with appropriate choice of units for r. For $k = -1, 0$ the space is infinite (open) , and for $k = +1$ the space is finite (closed). The above FRW metric has to be satisfied with the Einstein equations (pure Gravity case)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 0 \tag{22}$$

We calculate the scalar carvature R and substitute it into the action:

$$\begin{aligned} S &= \frac{1}{16\pi G} \int d^3x \sqrt{-g}(R - 2\Lambda) \\ &= \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[2 \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} + 2\frac{\ddot{a}}{a} \right) - 2\Lambda \right] \end{aligned} \tag{23}$$

Integrating the $\ddot{a}$ term in the resulting expression by parts with respect to t [3], we obtain the Lagrangian:

$$L(a, \dot{a}) = \frac{1}{2} a(\dot{a}^2 - k + \Lambda a^2) \tag{24}$$

We find the conjugated momentum

$$p = \frac{\partial L}{\partial \dot{a}} = a\dot{a} \tag{25}$$

Then we obtain the Hamiltonian as follows:

$$H = p\dot{a} - L = \frac{1}{2a}(p^2 + ka^2 - \Lambda a^4) \tag{26}$$

Here we can check the value of Hamiltonian is equal to zero.

$$\begin{aligned} H &= \frac{1}{2a}(p^2 + ka^2 - \Lambda a^4) \\ &= \frac{1}{2a}(a^2\dot{a}^2 + ka^2 - \Lambda a^4) \\ &= \frac{1}{2a^5} \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \Lambda \right) \\ &= 0 \end{aligned} \tag{27}$$

Now we define the Poisson Bracket:

$$\{A, B\}_p \equiv \frac{\partial A}{\partial a}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial a} \tag{28}$$

Of course

$$\{a,p\}_p = \frac{\partial a}{\partial a}\frac{\partial p}{\partial p} - \frac{\partial a}{\partial p}\frac{\partial p}{\partial a} = 1 \tag{29}$$

The time evolution of a and p is known as the canonical equations of Hamilton

$$\dot{a} = \{a,H\} = \frac{\partial H}{\partial p} = \frac{p}{a} \tag{30}$$

$$\dot{p} = \{p,H\} = -\frac{\partial H}{\partial a} = \frac{p^2}{2a^2} - \frac{k}{2} + \frac{3}{2}\Lambda a^2 \tag{31}$$

where $\dot{a} = \frac{da}{dt}$ and t is the proper time. The initial condition is set as follows

$$\lim_{t\to 0} a(t) = 0 \tag{32}$$

We can obtain the Friedmann equation from the equations ((12)(16)(17)):

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{k}{a^2} \tag{33}$$

Next we derive the Wheeler-DeWitt equation by the conventional canonical quantization procedure as follows :
In the Hamiltonian, p is replaced by $\frac{\hbar}{i}\frac{\partial}{\partial a}$ and the wave function $\psi(a)$ is operated to H.

$$\left(-\hbar^2\frac{d^2}{da^2} + U(a)\right)\psi(a) = 0 \tag{34}$$

$$U(a) = -\Lambda a^4 + ka^2 \tag{35}$$

Next we obtain the Wheeler-DeWitt equation in the momentum representation. Quantum mechanics in momentum space is treated[12] In the Hamiltonian , a is replaced by $-\frac{\hbar}{i}\frac{\partial}{\partial p}$(minus signature is derived from $[a,p] = i\hbar$)and it is obtained

$$H\psi(p) = 0 \tag{36}$$

namely

$$(p^2 + ka^2 - \Lambda a^4)\psi(p) = 0 \tag{37}$$

After calculation by $a \to -\frac{\hbar}{i}\frac{\partial}{\partial p}$ we obtain

$$\hbar^4\Lambda\frac{\partial^4\psi(p)}{\partial p^4} + \hbar^2 k\frac{\partial^2\psi(p)}{\partial p^2} - p^2\psi(p) = 0 \tag{38}$$

Here we treat the expansion of the potential near $a = 0$.

$$\begin{aligned} U(a) &= -|\Lambda|a^4 + ka^2 \\ &= U(0) + \acute{U}(0)a + \frac{1}{2!}U^{(2)}(0)a^2 + \frac{1}{3!}U^{(3)}(0)a^3 + \frac{1}{4!}U^{(4)}(0)a^4 + \\ &= ka^2 - |\Lambda|a^4 + \\ &\approx ka^2 \end{aligned} \tag{39}$$

In this case the Wheeler-DeWitt equations are in position space

$$\left(-\hbar^2 \frac{d^2}{da^2} + ka^2\right)\psi(a) = 0 \tag{40}$$

This equation means the wave function does not depend on Λ near $a = 0$, though it is natural. Namely around the big bang singularity the wave function of the universe does not depend on the space time geometry (AdS, dS or Minkowski). More speculated, near the big-bang singularity the wave function may be represented as

$$\psi \sim A\psi_{AdS} + B\psi_{dS} + C\psi_{flat} \tag{41}$$

If we set

$$a \equiv \sqrt{\frac{\hbar}{2}}x$$

we obtain

$$\frac{d^2\psi(x)}{dx^2} - \frac{k}{4}x^2\psi(x) = 0 \tag{42}$$

This is the Weber equation and the solutions are represented by the Weber functions.

Similarly in momentum space

$$(p^2 + ka^2)\psi(p) = 0 \tag{43}$$

After calculation by $a \to -\frac{\hbar}{i}\frac{\partial}{\partial p}$ we obtain

$$\hbar^2 k \frac{\partial^2\psi(p)}{\partial p^2} - p^2\psi(p) = 0 \tag{44}$$

If we set

$$p \equiv \sqrt{\frac{\hbar}{2}}y$$

We obtain

$$\frac{d^2\psi(y)}{dx^2} - \frac{1}{4k}y^2\psi(y) = 0 \tag{45}$$

This is the Weber equation too, and the solutions are given by the Weber functions. For details see [1]

4 The deformed FRW equation and the deformed Wheeler-DeWitt equation

In this section we derive the deformed FRW equation and the deformed Wheeler-deWitt equation under the GUP.

4.1 The deformed FRW equation

At first we have to check the deformed Poisson Bracket resulted from the deformed quantum mechanical commutation relation.

$$\begin{array}{ccc} \frac{1}{i\hbar}[a,p]=1 & \Longleftrightarrow & \frac{1}{i\hbar}[a,p]_D=\sqrt{1-\alpha p^2} \\ \Updownarrow & & \Updownarrow \\ \{A,B\}_p=\frac{\partial A}{\partial a}\frac{\partial B}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial B}{\partial a} & \Longleftrightarrow & \{A,B\}_{Dp}=\left(\frac{\partial A}{\partial a}\frac{\partial B}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial B}{\partial a}\right)\sqrt{1-\alpha p^2} \end{array} \tag{46}$$

At first we derive the deformed FRW equation. After that we find the deformed Wheeler-deWitt equation. The deformed Poisson Bracket has to be antisymmmetric, bilinear, and satisfy the Leibniz rules as well as the Jacobi identity.
The deformed Poisson bracket in the two-dimensional phase space is

$$\{A,B\}_{Dp}=\left(\frac{\partial A}{\partial a}\frac{\partial B}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial B}{\partial a}\right)\sqrt{1-\alpha p^2} \tag{47}$$

The time evolution of scale factor a and its conjugate momentum p_a , namely the Heisenberg's equation of motion, is modified via the relation (21).

$$\dot{a} = \{a,H\}_{Dp}=\frac{\partial H}{\partial p_a}\sqrt{1-\alpha p_a^2}=\frac{p_a}{a}\sqrt{1-\alpha p_a^2} \tag{48}$$

$$\dot{p} = \{p_a,H\}_{Dp}=-\frac{\partial H}{\partial a}\sqrt{1-\alpha p_{a^2}}=\left(\frac{p_a^2}{2a^2}-\frac{k}{2}-\frac{3}{2}\Lambda a^2\right)\sqrt{1-\alpha p_a^2} \tag{49}$$

Thus we can obtain the deformed FRW equation from the above equations of motion.

$$\left(\frac{\dot{a}}{a}\right)^2=\left(\Lambda-\frac{k}{a^2}\right)\left(1-\alpha(\Lambda a^4-ka^2)\right) \tag{50}$$

The above deformed FRW equation leads to the standard FRW equation (32) as $\alpha\to 0$.

4.2 The deformed Wheeler-DeWitt equation

Next we derive the deformed Wheeler-DeWitt equation.
The Hilbert space representation of the deformed Heisenberg algebra (12) is constructed.

Such a relation is represented in the momentum space , where the a (a corresponds to $\tilde{q}$) and p operators act on the wave function in momentum representation ;

$$\psi(p) =< p|\psi > \tag{51}$$

as

$$p\psi(p) = p\psi(p), \;\; a\psi(p) = i\hbar\sqrt{1-\alpha p^2}\partial_p\psi(p) \tag{52}$$

on a dense domain D of smooth functions. The Wheeler-DeWitt equation is defined as

$$H\psi(p) = 0 \tag{53}$$

where the Hamiltonian H is:

$$H = \frac{1}{2a}(p^2 + ka^2 - \Lambda a^4) \tag{54}$$

Thus the Wheeler-DeWitt equation is

$$(p^2 + ka^2 - \Lambda a^4)\psi(p) = 0 \tag{55}$$

where we set

$$U(a) = ka^2 - \Lambda a^4 \tag{56}$$

Then the Wheeler-DeWitt equation is

$$(p^2 + U(a))\psi(p) = 0 \tag{57}$$

The above equation can be calculated by use of (44), for example,

$$\begin{aligned} a^2\psi(p) &= aa\psi(p) = ai\hbar\sqrt{1-\alpha p^2}\partial_p\psi(p) \\ &= i\hbar\sqrt{1-\alpha p^2}\partial_p(i\hbar\sqrt{1-\alpha p^2}\partial_p\psi(p)) \\ &= -\hbar^2\sqrt{1-\alpha p^2}\{\partial_p(\sqrt{1-\alpha p^2})\partial_p\psi(p) + \sqrt{1-\alpha p^2}\partial_p^2\psi(p)\} \\ &= \hbar^2\alpha p\partial_p\psi(p) - \hbar^2(1-\alpha p^2)\partial_p^2\psi(p) \end{aligned} \tag{58}$$

Similarly,

$$\begin{aligned} a^4\psi(p) = \hbar^4\alpha^2 p\partial_p\psi(p) &+ \hbar^4(7\alpha^2p^2 - 4\alpha)\partial_p^2\psi(p) - 6\hbar^4\alpha p(1-\alpha p^2)\partial_p^3\psi(p) \\ &+ \hbar^4(1-\alpha p^2)^2\partial_p^4\psi(p) \end{aligned} \tag{59}$$

Then we can obtain

$$\hbar^4\Lambda\frac{\partial^4\psi(p)}{\partial^4 p}-\frac{6\hbar^4\Lambda\alpha p}{(1-\alpha p^2)}\frac{\partial^3\psi(p)}{\partial^3 p}+\frac{\hbar^4 k(1-\alpha p^2)\Lambda(7\alpha^2p^2-4\alpha)}{(1-\alpha p^2)^2}\frac{\partial^2\psi(p)}{\partial^2 p}$$
$$+\frac{\hbar^4(\Lambda\alpha^2 p-k\alpha p)}{(1-\alpha p^2)^2}\frac{\partial\psi}{\partial p}+\frac{p^2}{(1-\alpha p^2)^2}\psi(p)=0 \tag{60}$$

The above deformed Wheeler-DeWitt equation leads to the standard Wheeler-DeWitt equation as $\alpha\to 0$.
By the way we are interested in the wave function near the Big-Bang singularity, so we expand $U(a)$ around $a=0$ as follows

$$\begin{aligned}U(a) &= -|\Lambda|a^4+ka^2 \\ &= U(0)+\acute{U}(0)a+\frac{1}{2!}U^{(2)}(0)a^2+\frac{1}{3!}U^{(3)}(0)a^3+\frac{1}{4!}U^{(4)}(0)a^4+ \\ &= ka^2-|\Lambda|a^4+ \\ &\approx ka^2\end{aligned} \tag{61}$$

where we assume $|\Lambda|$ is very smal, $|\Lambda|\ll 1$.
Near the singularity the Wheeler-DeWitt equation does not depend on Λ, that is not surprising , since around the birth of space-time the topology of space-time may not be shaped geometrically. Then the Wheeler-DeWitt equation is

$$(p^2+ka^2)\psi(p)=0 \tag{62}$$

So the equation is made very simple like a harmonic oscillator. It should be noticed that around $a=0$ the Wheeler-DeWitt equation does not depend on Λ, namely it dose not depend on the shape of the space-time (AdS, dS, flat).
By use of (50), namely the canonical quantization procedure ($a\to i\hbar\sqrt{1-\alpha p^2}\frac{\partial}{\partial p}$), we obtain the final deformed Wheeler-DeWitt equation near the Big-Bang singularity

$$\hbar^2 k\frac{\partial^2\psi(p)}{\partial p^2}-\hbar^2 k\frac{\alpha p}{1-\alpha p^2}\frac{\partial\psi}{\partial p}-\frac{p^2}{1-\alpha p^2}\psi(p)=0 \tag{63}$$

The above equation is the well known differential equation called the Mathieu equation or the modified Mathieu equation in the algebraic form[13][21].
(1) : When $k=1$, the equation is

$$\hbar^2\frac{\partial^2\psi(p)}{\partial p^2}-\hbar^2\frac{\alpha p}{1-\alpha p^2}\frac{\partial\psi}{\partial p}-\frac{p^2}{1-\alpha p^2}\psi(p)=0 \tag{64}$$

Here we set

$$\sqrt{\alpha}p=t \tag{65}$$

We obtain

$$\frac{d^2\psi}{dt^2} - \frac{t}{1-t^2}\frac{d\psi}{dt} - \frac{\frac{1}{(\hbar\alpha)^2}t^2}{1-t^2}\psi = 0 \tag{66}$$

The original Mathieu equation is procedured by

$$t = \cos z \tag{67}$$

Then we obtain

$$\frac{d^2\psi}{dz^2} - \frac{1}{2(\hbar\alpha)^2}(1+\cos 2z)\psi = 0 \tag{68}$$

Its solutions can be explicitly written in terms of the Mathieu Function: even periodic and odd periodic respectively $ce_n(z,q), se_n(z,q)$[18] as

$$\psi(z) = Ace_n(z, \frac{1}{2}\frac{1}{2(\hbar\alpha)^2}) + Bse_n(z, \frac{1}{2}\frac{1}{2(\hbar\alpha)^2}) \tag{69}$$

where A and B are integration constants .
(2) : When $k = -1$, the equation is obtained similarly

$$\frac{d^2\psi}{dt^2} - \frac{t}{1-t^2}\frac{d\psi}{dt} + \frac{\frac{1}{(\hbar\alpha)^2}t^2}{1-t^2}\psi = 0 \tag{70}$$

The original Mathieu equation is

$$\frac{d^2\psi}{dz^2} + \frac{1}{2(\hbar\alpha)^2}(1+\cos 2z)\psi = 0 \tag{71}$$

Its solutions are similarly written as

$$\psi(z) = Ace_n(z, -\frac{1}{2}\frac{1}{2(\hbar\alpha)^2}) + Bse_n(z, -\frac{1}{2}\frac{1}{2(\hbar\alpha)^2}) \tag{72}$$

5 Conclusion

We studied the application of the GUP to the FRW cosmology in the context of the Snyder non-commutative space. We calculated the deformed FRW equations and deformed Wheeler-DeWitt equations in three dimensional pure gravity. The resulting equations include the effects of the GUP. The wave functions of the universe near the Big-Bang singularity were also obtained in momentum space. The uncertainty principle is the fundamental property of quantum theory even in the quantum gravity. The extension of the Heisenberg's Uncertainty Principle is essential to quantum gravity from the standpoint of the existence of the minimal length. On the other hand the existence of the Big-Bang singularity means that the emergence of the space-time is very strange and mysterious. We obtained the wave functions satisfying the deformed Wheeler-DeWitt equations near the singularity which were reduced to the Mathieu equations.

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Appendix

We summarize the Mathieu functions and the modified Mathieu functions briefly.[13][22]

Mathieu Function

Canonical form of the Mathieu Equation

$$\frac{d^2y}{dz^2} + (a - 2q\cos 2z)y = 0 \tag{73}$$

If we set

$$t = \cos z \tag{74}$$

We obtain an Algebraic Form of the Mathieu equation

$$\frac{d^2\psi}{dt^2} - \frac{t}{1-t^2}\frac{d\psi}{dt} - \frac{a + 2q - 4qt^2}{1-t^2}\psi = 0 \tag{75}$$

Requiring the solutions of the original Mathieu equation to be periodic leads to characteristic values for a. Periodic solutions that are even with respect to z have characteristic values a_n, whereas those that are odd have characteristic values b_n. If $q = 0$, then $a_n = b_n = n^2$ and the even and odd solutions are the cosine and sine functions respectively. The even and odd periodic Mathieu functions are respectively

$$ce_n(z,q) = \sum_{k=-\infty}^{+\infty} c_{n,k}(q)\cos[(n+2k)z],\ n = 0,1,2,,, \tag{76}$$

$$se_n(z,q) = \sum_{k=-\infty}^{+\infty} \tilde{c}_{n,k}(q)\sin[(n+2k)z],\ n = 1,2,3,, \tag{77}$$

They are normalized such that

$$\int_0^{2\pi} dz|ce_n(z,q)|^2 = \int_0^{2\pi} dz|se_n(z,q)|^2 = \pi \tag{78}$$

The normalized coefficients can be expanded in formal power series.
For $n \geqq 2$, the even coefficients are

$$c_{n,0}(q) = 1 + \mathcal{O}(q^2) \tag{79}$$

$$c_{n,\pm 1}(q) = \mp \frac{1}{4(n \pm 1)} q + \mathcal{O}(q^2) \tag{80}$$

For $n = 1$, we obtain

$$c_{1,0}(q) = 1 + \mathcal{O}(q^2) \tag{81}$$

$$c_{1,+1} = -\frac{1}{8} q + \mathcal{O}(q^2) \tag{82}$$

For $n = 0$,

$$c_{0,0}(q) = \frac{1}{\sqrt{2}} + \mathcal{O}(q^2) \tag{83}$$

$$c_{0,+1}(q) = -\frac{1}{2\sqrt{2}} q + \mathcal{O}(q^2) \tag{84}$$

All other coefficients are $\mathcal{O}(q^2)$ or higher. The odd coefficients are not the same as the even but they agree up to $\mathcal{O}(q^2)$ with one exception:

$$\tilde{c}_{2,-1}(q) = \mathcal{O}(q^2) \tag{85}$$

Modified Mathieu Equation

In Mathieu equation, if we set

$$z \to iz \tag{86}$$

we will obtain the Modified Mathieu Equation

$$\frac{d^2 y}{dz^2} - (a - 2q \cosh 2z) y = 0 \tag{87}$$

If we set

$$t = \cosh z \tag{88}$$

We obtain an Algbraic form of the Modified Mathieu equation

$$\frac{d^2 \psi}{dt^2} - \frac{t}{1 - t^2} \frac{d\psi}{dt} - \frac{a + 2q - 4qt^2}{1 - t^2} \psi = 0 \tag{89}$$

Namely the Algebraic form of the Modified Mathieu equation is the same as that of the Mathieu equation.
If we set $\zeta = 2\sqrt{q} \cosh z$, then the modified Mathieu equation is changed as

$$(\zeta^2 - 4q) \frac{d^2 y}{d\zeta^2} + \zeta \frac{dy}{d\zeta} - (a + 2q - \zeta^2) y = 0 \tag{90}$$

When $q=0$, then $a=a_n=n^2(\text{even case}), b=b_n=n^2(\text{odd case})$ and the above equation (83) reduces the Bessel differential equation. The even and odd modified Mathieu functions of the first kind are, respectively

$$\begin{aligned} Mc_n^{(1)}(z,q) &= \sum_{-\infty}^{\infty} d_{n,k}(q) J_{n+2k}(2\sqrt{q}\cosh z), n=0,1,2,.., && (91)\\ Ms_n^{(1)}(z,q) &= \tanh z \sum_{-\infty}^{\infty} \tilde{d}_{n,k}(q) J_{n+2k}(2\sqrt{q}\cosh z), n=1,2,3,.., && (92)\end{aligned}$$

The coefficients are related to those in Mathieu functions:

$$d_{n,k}(q) = \rho(q)(-1)^k c_{n,k}(q) \quad (93)$$
$$\tilde{d}_{n,k}(q) = \tilde{\rho}(q)(-1)^k (n+2k)\tilde{c}_{n,k}(q) \quad (94)$$

where $\rho(q)$ and $\tilde{\rho}(q)$ are normalization factors. The modified Mathieu functions of the first kind are normalized to have the same asymptotic form as the Bessel functions, namely

$$Mc_n^{(1)}(z,q) \sim J_n(z), \text{ as } z \to \infty \quad (95)$$
$$Ms_n^{(1)}(z,q) \sim J_n(z), \text{ as } z \to \infty \quad (96)$$

the normalization factors are

$$\rho(q) = \frac{1}{\sum_{-\infty}^{\infty} c_{n,k}(q)} \quad (97)$$
$$\tilde{\rho} = \frac{1}{\sum_{-\infty}^{\infty} (n+2k)\tilde{c}_{n,k}(q)} \quad (98)$$

The modified Mathieu functions in (84) and (85) are absolutely convergent for $|\cosh z| > 1$, [22] Using the results from the Mathieu functions the coefficients can be expanded in formal power series. The even coefficients for $n \geq 2$ are

$$\begin{aligned} d_{n,0}(q) &= 1 - \frac{1}{2(n^2-1)}q + \mathcal{O}(q^2) && (99)\\ d_{n,\pm1}(q) &= \pm\frac{1}{4(n\pm1)}q + \mathcal{O}(q^2) && (100)\end{aligned}$$

For $n=1$

$$d_{1,0}(q) = 1 + \frac{1}{8}q + \mathcal{O}(q^2) \quad (101)$$
$$d_{1,+1}(q) = \frac{1}{8}q + \mathcal{O}(q^2) \quad (102)$$

for $n=0$

$$d_{0,0}(q) =! + \frac{1}{2}q + \mathcal{O}(q^2) \tag{103}$$

$$d_{0,+1}(q) = \frac{1}{2}q + \mathcal{O}(q^2) \tag{104}$$

All the other even coefficients are $\mathcal{O}(q^2)$ or higher.
The odd coefficients :
For $n \geq 2$

$$\tilde{d}_{n,0}(q) = 1 + \frac{1}{2(n^2-1)}q + \mathcal{O}(q^2) \tag{105}$$

$$\tilde{d}_{n,\pm 1}(q) = \pm\frac{n \pm 2}{4n(n \pm 1)}q + \mathcal{O}(q^2) \tag{106}$$

For $n = 1$

$$\tilde{d}_{1,0}(q) = 1 + \frac{3}{8}q + \mathcal{O}(q^2) \tag{107}$$

$$\tilde{d}_{1,+1}(q) = \frac{3}{8}q + \mathcal{O}(q^2) \tag{108}$$

All the other odd coefficients are $\mathcal{O}(q^2)$ or higher.

Here we check the asymptotic form of the modified Mathieu functions when $z \to \infty$, since the limitation means $p \to$very large and $a \to$ very small. Namely, near the Big Bang singularity we could deal with the wave function.
Now we define $Me^{(1)}_{2m}(z)$, $Me^{(2)}_{2m}(z)$ such that[25]

$$Me^{(1)}_{2m}(z) = \frac{2(-1)^m}{\pi ce_{2m}(0)} \int_0^{\frac{\pi}{2}-i\infty} e^{2i\hbar \cosh z \cos \acute{z}} ce_{2m}(\acute{z}) d\acute{z} \tag{109}$$

$$Me^{(2)}_{2m}(z) = \frac{2(-1)^m}{\pi ce_{2m}(0)} \int_{\frac{\pi}{2}-i\infty}^{\pi} e^{2i\hbar \cosh z \cos \acute{z}} ce_{2m}(\acute{z}) d\acute{z} \tag{110}$$

The above both satisfy the modified Mathieu equation in some eigen values and relate with $Ce_{2m}(z)$

$$Ce_{2m} = (-1)^m \frac{ce_{2m}(0) ce_{2m}(\frac{\pi}{2})}{2A_0^{2m}} [Me^{(1)}_{2m}(z) + Me^{(2)}_{2m}(z)] \tag{111}$$

We can do some procedure of the saddle point method in order to calculate the asymptotic form of $z \to \infty$

The asymptotic form:

$$Me^{(1)}_{2m}(z) \sim \frac{2}{\pi} e^{2i\hbar \cosh z - \frac{\pi}{4} i} \int_0^\infty e^{-\hbar s^2 \cosh z} ds \tag{112}$$

$$= \sqrt{\frac{1}{\pi h \cosh z}} e^{2ih \cosh z - m\pi i - \frac{\pi}{4} i} \tag{113}$$

Similarly

$$Me^{(2)}_{2m}(z) \sim \sqrt{\frac{1}{\pi h \cosh z}} e^{-2ih \cosh z + m\pi i + \frac{\pi}{4} i} \tag{114}$$

We obtain the asymptotic form of $Ce_{2m}(z)$

$$Ce_{2m} \sim \frac{ce_{2m}(0) ce_{2m}(\frac{\pi}{2})}{2A_0^{2m}} \sqrt{\frac{1}{\pi h \cosh z}} \cos\left(2h \cosh z - \frac{\pi}{4}\right) \tag{115}$$

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