The Strange Attractor Structure of Turbulence and Effective Field Theories

Ervin Goldfain

Abstract

Recent work has conjectured that, under general boundary conditions, non-equilibrium Renormalization Group flows are likely to end up on *strange attractors*. If this conjecture is true, effective field theories must necessarily reflect the properties of these attractors. We start from the observation that, seemingly disparate concepts such as the Berry phase, gauge potentials and the curvature tensor of General Relativity (GR), share a common geometric foundation. Developing further, we posit that the dynamics of gauge and gravitational fields may be derived from the global attributes of strange attractors. The motivation behind this ansatz is that the Navier-Stokes equations bridge the gap between fluid turbulence, on the one hand, and the mathematics of GR and gauge theory, on the other.

Key words: strange attractors, turbulence, Berry phase, gauge theory, General Relativity.

1. Introduction

We have recently found that, under general boundary conditions, non-equilibrium Renormalization Group flows are prone to evolve to *strange attractors* [1, 2]. It is known that these attractors provide realistic models for the onset of chaos in nonlinear dynamics, as well as for the transition to turbulence in fluids described by the Navier-Stokes equations [3, 4]. Here, we start from the idea that, seemingly disparate concepts of quantum physics and classical field theory – namely, the Berry phase, gauge potentials and the connection coefficients of General Relativity (GR) – share common grounds with fluid turbulence and its roots in the geometry of strange attractors. Proceeding along this route, we further find that the non-abelian gauge groups of the Standard Model (SM) spring out from sequential bifurcations of Quantum Electrodynamics. Likewise, fermion families follow a bifurcation pattern in ascending order of mass. As a result, the entire flavor and mass content of the SM replicates the *transition to turbulence* in fluid dynamics via the (nearly) self-similar formation of eddies. Moreover, we find that the onset of turbulence in GR equations is compatible with the Cantor Dust picture of Dark Matter on astronomical scales [ ].

The paper is organized as follows: as preamble to the main text, sections two and three outline the geometric foundation of Berry phase, gauge and gravitational fields. Elaborating from the complex Ginzburg-Landau theory, section four lays out the path from the Navier-Stokes equation to the laminar regime of classical electrodynamics and GR. The transition to the full gauge and flavor spectrum of the SM via progressive turbulence, as well as the unfolding of the Cantor Dust texture of Dark Matter above the SM scale, form the object of section five. Summary and concluding remarks are detailed in the last section.

The reader is cautioned upfront on the provisional nature of this investigation. Given the plethora of unsettled questions related to the topics discussed here, these findings must be taken with a “grain of salt”. Further studies are required to back up, consolidate or rule out our conclusions.

2. Preamble #1: Berry phase in quantum physics

A quantum system adiabatically transported around a closed path C in the space of external parameters acquires a non-vanishing phase (*Berry phase*, BP in short). Since BP is exclusively path-dependent, it provides key insights into the geometric structure of quantum mechanics and quantum field theory (QFT). The BP concept is closely tied to *holonomy*, that is, the extent to which some of variables change as other variables or parameters defining a system return to their initial values [5, 6].

Consider a quantum system described by the time-independent Hamiltonian $H(t)$, whose associated eigenstate is $|\psi(t)\rangle$ and which is embedded in a slowly changing environment. After a periodic evolution of the environmental parameters ($t \to t+T$), the eigenstate returns to itself, apart from a phase angle,

$$|\psi(t)\rangle = e^{i\alpha}|\psi(0)\rangle \tag{1}$$

If ω denotes the eigenvalue of $|\psi(t)\rangle$, a generalization of the phase angle $\alpha = \omega T$ in units of $\hbar = 1$ is given by the "*dynamical phase*"

$$\gamma_d = \int_0^T \omega(t)\,dt = \int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle\,dt \tag{2}$$

Berry has shown that there is a time-independent (but contour dependent) supplemental "*geometric phase*" entering the phase angle, namely,

$$\alpha = \gamma_d + \gamma(C) \tag{3}$$

where

$$\gamma(C) = \int_C \langle\psi\,|\,i\nabla\psi\rangle\,dr \tag{4}$$

The dynamical phase γ_d encodes information about the duration associated with the cyclic evolution of the complex vector $|\psi(t)\rangle$. By contrast, (4) encodes information about the geometry of the environment where the transport takes place.

3. Preamble #2: The geometry of gauge and gravitational fields

The gauge field concept may be built from a straightforward *geometric* interpretation [7, 8]. Consider the parallel transport of a complex vector $|\psi\rangle$ round a closed rectangular loop. The difference between the value of $|\psi\rangle$ at the starting point ($|\psi\rangle_0$) and at the end point $|\psi\rangle_0 \to |\psi\rangle_f$ is given by

$$\Delta\psi = \psi_f - \psi_0 = -ig\,\Delta S^{\mu\nu} F_{\mu\nu} \psi \tag{5}$$

in which $\Delta S^{\mu\nu}$ denotes the area enclosed by the rectangle and the strength of the gauge field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{6}$$

Echoing the formation of the Berry phase, the effect of parallel transport is to induce a non-vanishing rotation of $|\psi\rangle$ in internal space proportional to the strength of the gauge field. Likewise, the curvature tensor of GR may be motivated through similar arguments. Taking a vector V^κ on a round trip by parallel transport, the difference between the initial and final components of the vector amounts to

$$\Delta V^\kappa = \frac{1}{2} R^\kappa_{\lambda\mu\nu} V^\lambda \Delta S^{\mu\nu} \tag{7}$$

This equation faithfully replicates (5) and signals the presence of a gravitational field, via the curvature tensor $R^\kappa_{\lambda\mu\nu}$. The geometric analogy between gauge theory and General Relativity is captured in the table below.

Gauge Theory	General Relativity
Gauge transformation	Coordinate transformation
Gauge group	Group of coordinate transformations
Gauge potential A_μ	Connection coefficient $\Gamma_{\mu\nu}^{\kappa}$
Field strength $F_{\mu\nu}$	Curvature tensor $R_{\lambda\mu\nu}^{\kappa}$

Comparison between the geometry of gauge and gravitational fields

Considerations developed in sections two and three show that, seemingly disparate concepts of quantum physics and classical field theory – namely, the Berry phase, gauge potentials and the connection coefficients of GR emerge from a common geometric foundation. Drawing on the findings of [ ] and working outside traditional approaches to field unification, we further explore how QFT and GR may be linked to the universal geometry of strange attractors and fluid turbulence.

4. Fluid flows and effective field theories

The object of this section is to show that the roots of classical electrodynamics and GR may be traced back to the *complex Ginzburg-Landau equation (CGLE).* To gradually introduce the main ideas and in the interest of clarity, we partition the section in three paragraphs. First paragraph outlines the derivation of the Navier-Stokes equation from CGLE and the role of kinematic viscosity in the transition from laminar flows to turbulence. The second and third paragraphs detail the Navier-Stokes formulation of classical electrodynamics and GR, respectively, with emphasis on the steady approach to turbulence through progressive changes of the energy scale.

4.1 CGLE and the Navier-Stokes equation

We start by recalling that the CGLE highlights many key properties of out-of-equilibrium nonlinear systems with space-time dependence. Specifically, as universal paradigm for the emergence of complex behavior, CGLE describes the generic onset of chaos, turbulence and spatio-temporal patterns in extended dynamical systems [ ]. It assumes the standard form

$$\partial_t z = \varepsilon z + (1 + i c_1)\Delta z - (1 - i c_3) z |z|^{2\sigma} \tag{8}$$

in which z is a complex-valued field, the parameters ε and σ are positive and the coefficients c_1 and c_3 are real [ ]. The *nonlinear Schrödinger equation* (NSE) is a particular embodiment of the CGLE in the limit $\varepsilon \to 0$, namely [ ],

$$i \partial_t z = -c_1 \Delta z + c_3 |z|^{2\sigma} z \tag{9}$$

In what follows, we work in 1+1 space-time and assume $\sigma = 1$. In its original formulation and natural units ($\hbar = 1$), the quantum-mechanical version of (9) reads

$$i \frac{\partial}{\partial t} z(x,t) = [-\frac{1}{2m}\nabla^2 + V(x,t)]\, z(x,t) \tag{10}$$

where $V(x,t)$ is the potential function. The so-called *Madelung transformation* enables one to turn (10) into the quantum Euler equation for compressible potential flows [ ]. For the sake of completeness, we repeat here the formal derivation of this transformation. Taking the complex-valued field in the canonical form,

$$z(x,t)=\sqrt{\frac{\rho(x,t)}{m}}\exp[iS(x,t)] \tag{11}$$

and substituting it into (10) leads to

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0 \tag{12}$$

$$\frac{d\vec{u}}{dt}=\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\frac{1}{m}\nabla(Q+V) \tag{13}$$

Here, $\vec{u}(x,t)$ is the flow velocity, $\rho=m|z|^2$ stands for the mass density and

$$Q=-\frac{1}{2m}\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} \tag{14}$$

is the Bohm potential [ ]. The flow velocity and its associated probability current are given by

$$\vec{u}(x,t)=\frac{1}{m}\nabla S=-\frac{i}{m}\frac{\nabla z}{z} \tag{15}$$

$$\vec{j}=\rho\vec{u}=\frac{1}{2mi}[z^*(\nabla z)-z(\nabla z^*)] \tag{16}$$

Since the Schrödinger equation is conservative, the Madelung transformation naturally leads to the Euler equation, which is exclusively valid for inviscid flows. To account for fluid viscosity and arrive at the Navier-Stokes equation, one needs to either appeal to an extended version of the NSE containing non-conservative terms *or* bring up the *kinematic viscosity* – a concept linked to the mass of quantum particles as in [ ]

$$\nu = \frac{1}{2m} \tag{17}$$

On account of (17) and for incompressible flows, the Navier-Stokes equation that mirrors (13) is given by [ ]

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} \tag{18}$$

where p is the pressure. An alternative expression for the (18) may be obtained using the identity

$$\vec{u}\cdot\nabla\vec{u} = \frac{1}{2}\nabla u^2 - \vec{u}\times\vec{\omega} \tag{19}$$

where $\vec{\omega} = \nabla\times\vec{u}$ represents the vorticity vector [ ]. The corresponding Navier-Stokes equation reads

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\omega}\times\vec{u} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu\nabla^2\vec{u} \tag{20}$$

Using (15), the phase of the field amounts to

$$S = m\int u\,dx \tag{21}$$

such that

$$z(x,t) = \sqrt{\frac{\rho(x,t)}{m}}\exp(-im\int u\,dx) \tag{22}$$

... (*text to follow*) ...