**A new solution of a singular quadratic Liénard equation with Sundman and Lie Symmetry analysis**

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**Abstract**

A fairly and simple exact solution of a well-known singular quadratic Liénard type equation is developed in this paper. The solution is compared with those obtained by Sundman and Lie Symmetry analysis. It is found that the result are in nice agreement.

**Introduction**

Consider the equation

$$\ddot{x} - \frac{2}{3x}\dot{x}^2 + \frac{\omega^2}{3}x = 0 \tag{1}$$

which is a Liénard II type harmonic nonlinear oscillator equation, where $x$ is the function of time $t$ and $\omega$ is a free parameter.

This equation is investigated by Orhan and Özer in **[1]** wherein, his first integral of the form $A(t,x)\dot{x} + B(t,x)$ is found and the invariant solution is obtained. The aim of this paper is first to show that the equation (1) is a particular case of a more general one that belongs to the general class of quadratic Liénard type equations introduced by Akande et al. **[2, 3, 4]** and then determine in a straightforward fashion an exact solution. Second to analyze the equation (1) in one part from Lie Symmetry point of view and in second part by Sundman Symmetry to show that the obtained solutions are the same one computed by application of the nonlinear differential theory introduced by Akande et al. **[2, 3, 4]**

**2- Solution using the generalized Sundman transformation developed by Akande et al. [2, 3, 4]**

Let us consider the theory of nonlinear differential equations recently introduced by Akande et al. **[2, 3, 4].** In this way, let's take into account the general class of quadratic Liénard type equations **[2, 3, 4].**

$$\ddot{x} + \left( l\frac{g'(x)}{g(x)} - \gamma\frac{f'(x)}{f(x)} \right)\dot{x}^2 + b\dot{x}f^{\gamma}(x) + a^2\frac{f^{2\gamma}(x)\int g^{l}(x)dx}{g^{l}(x)} = 0 \tag{2}$$

which is found by applying the transformation

$$y(\tau) = \int g^{l}(x)dx\,; \qquad d\tau = f^{\gamma}(x)dt \tag{3}$$

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to the equation of harmonic oscillator

$$y''(\tau) + a^2 y(\tau) = 0 \tag{4}$$

For b=0, $f(x) = g(x) = x^2$, the equation (2) is rewriting in this form

$$\ddot{x} + 2(l-\gamma)\frac{\dot{x}^2}{x} + \frac{a^2}{2l+1}x^{4\gamma+1} = 0 \tag{5}$$

The choice $\gamma = 0$, $l = \frac{-1}{3}$ and $a^2 = \frac{\omega^2}{9}$, leads to the equation (1) studied in **[1]**

This equation is then a particular case of the generalized equation (5).To find the solution, let us consider (3) which after substitution of $f(x)$, $g(x)$ and $\gamma$, becomes

$$y(\tau) = 3x^{\frac{1}{3}},$$

and

$d\tau = dt$ which reduces to

$\tau = t + k$ , with $k$ a constant

Since $y(\tau)$ is solution of (4), then

$$y(\tau) = A_1 \sin[a\tau + \alpha] \tag{6}$$

so that one may obtain the solution to (1) in the form

$$x(t) = \frac{A_1^3}{27}\sin^3\left(\frac{\omega}{3}t + A_2\right) \tag{7}$$

where $A_2 = \alpha + \frac{\omega}{3}$, and $k = 1$

## 3-Lie point symmetry analysis

### 3.1-Eight symmetries of (1)

Consider the one parameter Lie group of infinitesimal transformation in the variables

$$\begin{cases} T = t + \varepsilon\xi(t,x) + 0(\varepsilon^2) \\ X = x + \varepsilon\eta(t,x) + 0(\varepsilon^2) \end{cases}$$

where $\xi(t,x)$ and $\eta(t,x)$ are the infinitesimal symmetries defined by the determining equations **[5],**

$$\xi_{xx} + \frac{2}{3x}\xi_x = 0 \tag{8}$$

$$\eta_{xx} - \frac{2}{3x}\eta_x + \frac{2}{3x^2}\eta - 2\xi_{tx} = 0 \tag{9}$$

$$2\eta_{tx} - \frac{4}{3x}\eta_t - \xi_{tt} - \omega^2 x\xi_x = 0 \tag{10}$$

$$\eta_{tt} + \eta g_x - g\eta_x + 2g\xi_t = 0 \tag{11}$$

From (9),

$$\xi(t,x) = 3b(t)x^{\frac{1}{3}} + a(t) \tag{12}$$

Replacing (12) in (10) and doing some algebraic manipulations, one can find

$$\eta(t,x) = 9\dot{b}(t)x^{4/3} + xc(t) + x^{2/3}d(t) \tag{13}$$

The next step is to calculate $a(t)$, $b(t)$, $c(t)$ and $d(t)$. To achieve this, let us replace $\xi(t,x)$ and $\eta(t,x)$ in (10) and (11) and setting the coefficient of $x^m$ equal to zero, one may have a following system of differential equations

$$\begin{cases} \ddot{b} + \dfrac{\omega^2}{9}b = 0 \\ \ddot{a} = 2\dot{c} \\ \ddot{d} + \dfrac{\omega^2}{9}d = 0 \\ \ddot{c} + \dfrac{2\omega^2}{9}\dot{a} = 0 \end{cases} \tag{14}$$

The integration of the equations of the previous system leads to the expression of $a(t)$, $b(t)$, $c(t)$ and $d(t)$, viz

$$a(t) = \left(\frac{3}{2\omega}\right)^2 a_1 + a_2 \sin\left(\frac{2\omega t}{3}\right) + a_3 \cos\left(\frac{2\omega t}{3}\right) \tag{15}$$

$$b(t) = b_1 \sin\left(\frac{\omega}{3}t\right) + b_2 \cos\left(\frac{\omega}{3}t\right) \tag{16}$$

$$c(t) = c_1 + \frac{\omega}{3}a_2 \cos\left(\frac{2\omega t}{3}\right) - \frac{\omega}{3}a_3 \sin\left(\frac{2\omega t}{3}\right) \tag{17}$$

and

$$d(t) = d_1 \sin\left(\frac{\omega}{3}t\right) + d_2 \cos\left(\frac{\omega}{3}t\right) \tag{18}$$

Thus the infinitesimal symmetries are

$$\xi(t,x) = 3x^{\frac{1}{3}} b_1 \sin\left(\frac{\omega}{3}t\right) + 3x^{\frac{1}{3}} b_2 \cos\left(\frac{\omega}{3}t\right) + \left(\frac{3}{2\omega}\right)^2 a_1 + a_2 \sin\left(\frac{2\omega t}{3}\right) + a_3 \cos\left(\frac{2\omega t}{3}\right) \quad (19)$$

and

$$\eta(t,x) = 3\omega x^{4/3} b_1 \cos\left(\frac{\omega t}{3}\right) - 3\omega x^{4/3} b_2 \sin\left(\frac{\omega t}{3}\right) + 3xc_1 + x\omega a_2 \cos\left(\frac{2\omega t}{3}\right) - x\omega a_3 \sin\left(\frac{2\omega t}{3}\right) +$$

$$+ x^{2/3} d_1 \sin\left(\frac{\omega}{3}t\right) + x^{2/3} d_2 \cos\left(\frac{\omega}{3}t\right) \quad (20)$$

The eight symmetries are then given by

$$X_1 = \left(\frac{3}{2\omega}\right)^2 \partial_t$$

$$X_2 = \sin\left(\frac{2\omega}{3}t\right)\partial_t + x\omega \cos\left(\frac{2\omega}{3}t\right)\partial_x$$

$$X_3 = \cos\left(\frac{2\omega}{3}t\right)\partial_t - x\omega \sin\left(\frac{2\omega}{3}t\right)\partial_x$$

$$X_4 = 3x^{1/3} \sin\left(\frac{\omega}{3}t\right)\partial_t + 3\omega x^{4/3} \cos\left(\frac{\omega}{3}t\right)\partial_x$$

$$X_5 = 3x^{1/3} \cos\left(\frac{\omega}{3}t\right)\partial_t - 3\omega x^{4/3} \sin\left(\frac{\omega}{3}t\right)\partial_x$$

$$X_6 = 3x\partial_x$$

$$X_7 = x^{2/3} \sin\left(\frac{\omega}{3}t\right)\partial_x$$

and

$$X_8 = x^{2/3} \cos\left(\frac{\omega}{3}t\right)\partial x$$

**3.2- Solution of the equation using Lie Symmetry**

The equation admitting eight parameters is isochronous according to **[5].** Hence there exists an invertible transformation $X = h(x)$, which maps this equation into

$$\ddot{X} + \omega_0{}^2 X = 0$$

Let us find $h(x)$, according to the equation **[5]**

$$\ddot{x} + \frac{h''(x)}{h'(x)}\dot{x}^2 + \omega_0{}^2 \frac{h(x)}{h'(x)} = 0 \qquad (21)$$

Comparing (21) and (1), it leads

$$\frac{h''(x)}{h'(x)} = \frac{-2}{3x}$$

Then $h(x) = 3x^{1/3}$. Let us determine $\omega_0$

Comparing also (21) and (1) gives

$$\omega_0{}^2 \frac{h(x)}{h'(x)} = \frac{\omega^2}{3} x$$

Then $\omega_0 = \frac{\omega}{3}$. Since the solution $X(t)$ is given by

$$X(t) = A_1 \sin(\omega_0 t + A_2)$$

the solution $x(t)$ is then

$$x(t) = \frac{A_1{}^3}{27} \sin^3(\frac{\omega}{3} t + A_2)$$

where $A_1$ and $A_2$ are constants . Such a solution is the same as that obtained previously by generalized Sundman transformation.

**4- Sundman symmetry analysis**

**4.1-Elements of Sundman Symmetry analysis**

The comparison of (1) with **[6]**

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + B(x) = 0$$

leads

$$\frac{1}{2}\phi_x = -\frac{2}{3x}$$

from which

$$\phi(x)=\ln\left(\frac{1}{x}\right)^{4/3}$$

and $B(x)=\dfrac{\omega^2}{3}x$

Consider the general nonlocal transformation **[6]**

$$X=F(x)\ ;\quad dT=G(x)dt$$

$$\text{with}\ \begin{cases} F(x)=\left(\pm\dfrac{2}{\beta^2}\displaystyle\int B(x)e^{\phi(x)}dx\right)^{1/2} \\ G(x)=\dfrac{B(x)e^{\frac{\phi(x)}{2}}}{\left(\pm 2\int B(x)e^{\phi(x)}dx\right)^{1/2}} \end{cases} \tag{22}$$

Replacing $\phi(x)$ and $B(x)$ yields

$$F(x)=\frac{\omega}{\beta}x^{1/3},\ \text{and}\ \ g(x)=\frac{\omega}{3}$$

Thus, the nonlocal transformation of (1) is

$$X=\frac{\omega}{\beta}x^{1/3},\quad \text{and}\ \ dT=\frac{\omega}{3}dt$$

**4.2-Sundman Symmetry of equation (1)**

The Sundman symmetries $\tilde{x}$ and $\tilde{t}$ are determined by the following equations

$$F(\tilde{x})=\pm\sqrt{F^2(x)+C} \tag{23}$$

and

$$G(\tilde{x})d\tilde{t}=\pm G(x)\frac{F(x)}{\sqrt{F^2(x)+C}} \tag{24}$$

where $c$ is a constant, $F$ and $G$ are given respectively by ( 22).

from (23) one may have

$$\frac{\omega}{\beta}\tilde{x}^{1/3}=\pm\sqrt{\left(\frac{\omega^2}{\beta^2}x^{2/3}+c\right)}$$

which may give

$$\tilde{x}=\pm\left(x^{2/3}+\frac{c\beta^2}{\omega^2}\right)^{3/2}$$

On the other hand, substituting $F$ and $G$ in (24) leads to

$$d\widetilde{t} = \pm \frac{x^{1/3}}{\sqrt{x^{2/3} + \frac{c\beta^2}{\omega^2}}} dt$$

By integration one may obtain

$\widetilde{t} = A + \int \frac{x^{1/3}}{\sqrt{x^{2/3} + \frac{c\beta^2}{\omega^2}}} dt$ , where $A$ is a constant.

The Sundman Symmetry of equation (1) is then given by

$$\begin{cases} \widetilde{x} = \pm\left( x^{2/3} + \frac{c\beta^2}{\omega^2} \right)^{3/2} \\ \widetilde{t} = A + \int \frac{x^{1/3}}{\sqrt{x^{2/3} + \frac{c\beta^2}{\omega^2}}} dt \end{cases}$$

### 4.3-Parametric solution of equation (1)

Consider $X = \frac{\omega}{\beta} x^{1/3}$ such that $x = \left(\frac{\beta}{\omega}\right)^3 X^3$ and $dT = \frac{\omega}{3} dt$

The previous transformation maps equation (1) into the equation

$$\frac{d^2 X}{dT^2} + X = 0$$

and its first integral is

$\left(\frac{dX}{dT}\right)^2 + X^2 = I_1$, with $I_1$ a constant. Setting $\frac{dX}{dT} = \tau$ , one has

$\tau^2 + X^2 = I_1$, so that $X(\tau) = \left(I_1 - \tau^2\right)^{3/2}$

Hence $x(\tau) = \left(\frac{\beta}{\omega}\right)^3 \left(I_1 - \tau^2\right)^{3/2}$. Knowing that

$$\frac{dX}{dT} = \frac{dX}{d\tau} \cdot \frac{d\tau}{dT}$$

one may have

$$dT = \frac{-1}{\left(I_1 - \tau^2\right)^{1/2}} d\tau$$

which may yield

$$t(\tau) = -\frac{3}{\omega}\int \left(I_1 - \tau^2\right)^{-1/2} d\tau + c_2$$

Therefore the parametric solution is

$$\begin{cases} x(\tau) = \left(\dfrac{\beta}{\omega}\right)^3 \left(I_1 - \tau^2\right)^{3/2} \\ t(\tau) = -\dfrac{3}{\omega}\displaystyle\int \left(I_1 - \tau^2\right)^{-1/2} d\tau + c_2 \end{cases} \tag{25}$$

$I_1$, $\beta$ and $c_2$ are arbitrary constants and $\tau$ is the parameter.

**4.4- Explicit solution of (1) using Sundman symmetry**

The expression of $t(\tau)$ may be arranged in the form

$$t(\tau) = \frac{-3}{\omega}\int \frac{1}{\sqrt{1-\left(\frac{\tau}{\sqrt{I_1}}\right)^2}} d\tau + c_2$$

which leads to

$$-\frac{\omega}{3}\left(t - c_2\right) = \arcsin\left(\frac{\tau}{\sqrt{I_1}}\right)$$

Then

$$\tau(t) = -\sqrt{I_1}\,\sin\left(\frac{\omega}{3}t + B_1\right) \tag{26}$$

with $B_1 = -\frac{\omega}{3}c_2$

Replacing (26) in (25) yields

$$x(t) = \left(\frac{\beta\sqrt{I_1}}{\omega}\right)^3 \left(1 - \sin^2\left(\frac{\omega}{3}t + B_1\right)\right)$$

Since

$$1 - \sin^2\left(\frac{\omega}{3}t + B_1\right) = \cos^2\left(\frac{\omega}{3}t + B_1\right)$$

and $\cos\left(\frac{\omega}{3}t + B_1\right) = \sin\left(\frac{\omega}{3}t + B_1 + \frac{\pi}{2}\right)$

then

$$x(t) = \left(\frac{\beta\sqrt{I_1}}{\omega}\right)^3 \sin^3\left(\frac{\omega}{3}t + A_2\right)$$

where $A_2 = B_1 + \frac{\pi}{2}$

Now, choosing $\frac{A_1}{3} = \frac{\beta\sqrt{I_1}}{\omega}$, one may finally write $x(t)$ in the form

$$x(t) = \frac{A_1^3}{27}\sin^3\left(\frac{\omega}{3}t + A_2\right)$$

which is nothing but the solution obtained previously by application of the generalized Sundman transformation introduced by Akande et al. **[2, 3, 4]** and Lie symmetry group analysis.

**Conclusion**

A singular quadratic Liénard type equation with well-known solutions has been considered in this contribution. A new solution which appear fairly and simple has been developed within the framework of the generalized sundman transformation. The obtained solution has been found to be identical to those computed from Lie point symmetry group analysis as well as from Sundman symmetry method.

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