"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 5/10."

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author's previous submission, namely -

"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.",

which has been published under the **'VIXRA' Mathematics subheading:-** ***'Functions and Analysis'***.

II. COPY OF AUTHOR'S ORIGINAL PAPER – PART 5/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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Theorem TI-7, it likewise follows that the exponential function,

$$\exp(\log[Qq + (1-q^2)^{1/2}] + Q2n\pi) =$$

$$\exp(\log[Qq + (1-q^2)^{1/2}])\exp(Q2n\pi) = Qq + (1-q^2)^{1/2} \Longrightarrow$$

$$\log[Qq + (1-q^2)^{1/2}] + Q2n\pi = \log[Qq + (1-q^2)^{1/2}].$$

Subsequently, we may write

$$w = -Q\log\left[\frac{Q(Qq - (1-q^2)^{1/2})}{(Qq + (1-q^2)^{1/2})(Qq - (1-q^2)^{1/2})}\right]$$

$$= -Q\log\left[\frac{Q(Qq - (1-q^2)^{1/2})}{Q^2q^2 - Qq(1-q^2)^{1/2} + (1-q^2)^{1/2}Qq - (1-q^2)}\right]$$

$$= -Q\log\left[\frac{Q(Qq - (1-q^2)^{1/2})}{-q^2 - Qq(1-q^2)^{1/2} + Qq(1-q^2)^{1/2} - (1-q^2)}\right]$$

$$= -Q\log\left[\frac{Q(Qq - (1-q^2)^{1/2})}{-q^2 - (1-q^2)}\right]$$

$$= -Q\log\left[\frac{Q^2q - Q(1-q^2)^{1/2}}{-q^2 - 1 + q^2}\right]$$

$$= -Q\log\left[\frac{-q - Q(1-q^2)^{1/2}}{-1}\right]$$

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$$= -Q\log\left[\frac{-(q + Q(1-q^2)^{1/2})}{-1}\right]$$

$$= -Q\log[q + Q(1-q^2)^{1/2}].$$

However, since we had originally postulated that the inverse function,

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$$w = \cos^{-1}(q) \Longrightarrow q = \cos(w),$$

we therefore obtain the result,

$\cos^{-1}(q) = -Q\log[q + Q(1-q^2)^{1/2}]$, as required. Q.E.D.

(c) Let the variable,

$$w = \tan^{-1}(q) \Longrightarrow q = \tan(w) = \sin(w)/\cos(w),$$

such that

(i) $w = u_1 + iv_1 + ju_2 + kv_2 = U + \left(\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)V$

$$\Longrightarrow u_1 = U,\ v_1 = \frac{yV}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}},\ u_2 = \frac{\hat{x}V}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}} \text{ and } v_2 = \frac{\hat{y}V}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}};$$

(ii) the variables, $U, V \in \mathbb{R}$.

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Hence, it follows that the auxiliary function,

$$Q^* = \frac{\dfrac{iyV + j\hat{x}V + k\hat{y}V}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{\sqrt{\dfrac{y^2V^2}{y^2+\hat{x}^2+\hat{y}^2} + \dfrac{\hat{x}^2V^2}{y^2+\hat{x}^2+\hat{y}^2} + \dfrac{\hat{y}^2V^2}{y^2+\hat{x}^2+\hat{y}^2}}}$$

$$= \frac{V}{|V|}\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= I^*Q,$$

where $Q = \dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$ and $I^* = V/|V| = \pm 1 \Longrightarrow I^{*2} = 1.$

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Subsequently, we may write

$$q = \tan(w)$$

$$= \sin(w)/\cos(w)$$

$$= \left(\frac{\exp(I^*Qw) - \exp(-I^*Qw)}{2I^*Q}\right) \Big/ \left(\frac{\exp(I^*Qw) + \exp(-I^*Qw)}{2}\right)$$

$$= -I^*Q\left(\exp(I^*Qw) - \exp(-I^*Qw)\right)\left(\exp(I^*Qw) + \exp(-I^*Qw)\right)^{-1}$$

$$\therefore I^*Qq = qI^*Q =$$

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$$-I^{*2}Q^{2}\left(\exp(I^{*}Qw) - \exp(-I^{*}Qw)\right)\left(\exp(I^{*}Qw) + \exp(-I^{*}Qw)\right)^{-1},$$

by virtue of Theorem TI-10, such that

$$I^{*}Qq = \left(\exp(I^{*}Qw) - \exp(-I^{*}Qw)\right)\left(\exp(I^{*}Qw) + \exp(-I^{*}Qw)\right)^{-1}$$

$$\therefore I^{*}Qq\left(\exp(I^{*}Qw) + \exp(-I^{*}Qw)\right) = \exp(I^{*}Qw) - \exp(-I^{*}Qw)$$

$$\therefore I^{*}Qq\exp(I^{*}Qw) + I^{*}Qq\exp(-I^{*}Qw) = \exp(I^{*}Qw) - \exp(-I^{*}Qw)$$

$$\therefore (I^{*}Qq - 1)\exp(I^{*}Qw) + (I^{*}Qq + 1)\exp(-I^{*}Qw) = 0$$

$$\therefore (I^{*}Qq - 1)\left(\exp(I^{*}Qw)\right)^{2} + (I^{*}Qq + 1)\exp(-I^{*}Qw)\exp(I^{*}Qw) = 0$$

$$\therefore (I^{*}Qq - 1)\left(\exp(I^{*}Qw)\right)^{2} + (I^{*}Qq + 1)\exp(-I^{*}Qw + I^{*}Qw) = 0$$

$$\therefore (I^{*}Qq - 1)\left(\exp(I^{*}Qw)\right)^{2} + (I^{*}Qq + 1)\exp(0) = 0$$

$$\therefore (I^{*}Qq - 1)\left(\exp(I^{*}Qw)\right)^{2} + I^{*}Qq + 1 = 0$$

$$\therefore (I^{*}Qq - 1)\left(\exp(I^{*}Qw)\right)^{2} = -(I^{*}Qq + 1)$$

$$\therefore \left(\exp(I^{*}Qw)\right)^{2} = -\frac{(I^{*}Qq + 1)}{I^{*}Qq - 1}$$

$$\therefore \exp(2I^{*}Qw) = -\frac{(I^{*}Qq - I^{*2}Q^{2})}{I^{*}Qq + I^{*2}Q^{2}}$$

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$$\therefore \exp(2I^*Qw) = -\frac{I^*Q(q - I^*Q)}{I^*Q(q + I^*Q)}$$

$$\therefore \exp(2I^*Qw) = -\frac{(q - I^*Q)}{q + I^*Q} \Longrightarrow$$

$$2I^*Qw = \log\left(-\frac{(q - I^*Q)}{q + I^*Q}\right)$$

$$\therefore 2I^*Qw = \log\left(\frac{I^*Q - q}{I^*Q + q}\right)$$

$$\therefore -I^{*2}Q^2w = -\tfrac{1}{2}I^*Q\log\left(\frac{I^*Q - q}{I^*Q + q}\right)$$

$$\therefore w = -\tfrac{1}{2}I^*Q\log\left(\frac{I^*Q - q}{I^*Q + q}\right)$$

$$= \tfrac{1}{2}I^*Q\left[-\log\left(\frac{I^*Q - q}{I^*Q + q}\right)\right]$$

$$= \tfrac{1}{2}I^*Q\log\left(\frac{I^*Q + q}{I^*Q - q}\right),$$

by virtue of Theorem TI-6, since it is evident that the exponential function,

$$\exp\left(-\log\left(\frac{I^*Q - q}{I^*Q + q}\right)\right) = \left[\exp\left(\log\left(\frac{I^*Q - q}{I^*Q + q}\right)\right)\right]^{-1}$$

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$$= \left[\frac{I^*Q - q}{I^*Q + q}\right]^{-1}$$

$$= \frac{I^*Q + q}{I^*Q - q} \implies$$

$$-\log\left(\frac{I^*Q - q}{I^*Q + q}\right) = \log\left(\frac{I^*Q + q}{I^*Q - q}\right).$$

However, since we had originally postulated that the inverse function,

$$w = \tan^{-1}(q) \implies q = \tan(w),$$

we therefore obtain the result,

$$\tan^{-1}(q) = \tfrac{1}{2} I^*Q \log\left(\frac{I^*Q + q}{I^*Q - q}\right).$$

We now wish to prove that

$$\tan^{-1}(q) = \tfrac{1}{2} I^*Q \log\left(\frac{I^*Q + q}{I^*Q - q}\right) = \tfrac{1}{2} Q \log\left(\frac{Q + q}{Q - q}\right),$$

where $I^* = \pm 1$. To do this, let us consider the separate cases where $I^* = 1$ and $I^* = -1$. Firstly, by putting $I^* = 1$, we observe that

$$\tfrac{1}{2} I^*Q \log\left(\frac{I^*Q + q}{I^*Q - q}\right) = \tfrac{1}{2} Q \log\left(\frac{Q + q}{Q - q}\right).$$

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Secondly, by putting $I^* = -1$, we likewise note that

$$\tfrac{1}{2}I^*Q\log\left(\frac{I^*Q+q}{I^*Q-q}\right) = -\tfrac{1}{2}Q\log\left(\frac{-Q+q}{-Q-q}\right)$$

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$$= -\tfrac{1}{2}Q\log\left(\frac{-(Q-q)}{-(Q+q)}\right)$$

$$= -\tfrac{1}{2}Q\log\left(\frac{Q-q}{Q+q}\right)$$

$$= \tfrac{1}{2}Q\left[-\log\left(\frac{Q-q}{Q+q}\right)\right]$$

$$= \tfrac{1}{2}Q\log\left(\frac{Q+q}{Q-q}\right),$$

by virtue of Theorem TI-6, since it is evident that the exponential function,

$$\exp\left(-\log\left(\frac{Q-q}{Q+q}\right)\right) = \left[\exp\left(\log\left(\frac{Q-q}{Q+q}\right)\right)\right]^{-1}$$

$$= \left[\frac{Q-q}{Q+q}\right]^{-1}$$

$$= \frac{Q+q}{Q-q} \Longrightarrow$$

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$$-\log\left(\frac{Q-q}{Q+q}\right) = \log\left(\frac{Q+q}{Q-q}\right).$$

Clearly, we have proven that the function,

$$\tfrac{1}{2}I^*Q\log\left(\frac{I^*Q+q}{I^*Q-q}\right) = \tfrac{1}{2}Q\log\left(\frac{Q+q}{Q-q}\right),$$

whenever $I^* = \pm 1$, and therefore it naturally follows that the <u>inverse tangent</u> function,

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$$\tan^{-1}(q) = \tfrac{1}{2}I^*Q\log\left(\frac{I^*Q+q}{I^*Q-q}\right) = \tfrac{1}{2}Q\log\left(\frac{Q+q}{Q-q}\right),$$

as required. Q.E.D.

For the sake of completeness, we will define the inverse trigonometric functions, $\sec^{-1}(q)$, $\mathrm{cosec}^{-1}(q)$ and $\cot^{-1}(q)$, as follows:–

Definition DI–19.

Let there exist three <u>inverse trigonometric</u> quaternion hypercomplex functions, $\sec^{-1}(q)$, $\mathrm{cosec}^{-1}(q)$ and $\cot^{-1}(q)$. In the circumstances, we postulate that

(a) the <u>inverse secant</u> function,

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$$w = \sec^{-1}(q) \Longrightarrow q = \sec(w) = 1/\cos(w),$$

(b) the <u>inverse cosecant</u> function,

$$w = \operatorname{cosec}^{-1}(q) \Longrightarrow q = \operatorname{cosec}(w) = 1/\sin(w),$$

(c) the <u>inverse cotangent</u> function,

$$w = \cot^{-1}(q) \Longrightarrow q = \cot(w) = 1/\tan(w).$$

From this definition, the reader will note that we have neither specified the respective domains of these particular functions nor have we defined them in terms of functions with which we are already familiar. Our next theorem, however, will address both of these issues simultaneously.

<u>Theorem TI-26.</u>

Let there exist the inverse trigonometric functions, $\sec^{-1}(q)$, $\operatorname{cosec}^{-1}(q)$ and $\cot^{-1}(q)$, as previously defined. Subsequently, we may prove that the following formulae, namely -

(a) $\sec^{-1}(q) = \cos^{-1}(q^{-1})$,

(b) $\operatorname{cosec}^{-1}(q) = \sin^{-1}(q^{-1})$,

(c) $\cot^{-1}(q) = \tan^{-1}(q^{-1})$,

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are valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}$.

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PROOF:-

(a) From Definitions DI-11 and DI-19, we recall that the inverse secant function,

$w = \sec^{-1}(q) \Longrightarrow q = \sec(w)$

$\therefore q^{-1} = (\sec(w))^{-1}$

$\therefore q^{-1} = (1/\cos(w))^{-1} = \cos(w)$, $\forall q \in \mathbb{H} - \{0\}$.

However, in accordance with Definition DI-18, we likewise deduce that

$w = \cos^{-1}(q^{-1}) \Longrightarrow q^{-1} = \cos(w) \Longrightarrow q = \sec(w)$,

whence it is evident that the inverse function,

$w = \sec^{-1}(q) = \cos^{-1}(q^{-1})$, $\forall q \in \mathbb{H} - \{0\}$, as required. Q.E.D.

(b) From Definitions DI-11 and DI-19, we similarly perceive that the inverse cosecant function,

$w = \operatorname{cosec}^{-1}(q) \Longrightarrow q = \operatorname{cosec}(w)$

$\therefore q^{-1} = (\mathrm{cosec}(w))^{-1}$

$\therefore q^{-1} = (1/\sin(w))^{-1} = \sin(w)\ ,\ \forall q \in \mathbb{H} - \{0\}.$

However, in accordance with Definition DI-18, we likewise deduce that

$$w = \sin^{-1}(q^{-1}) \Longrightarrow q^{-1} = \sin(w) \Longrightarrow q = \mathrm{cosec}(w),$$

whence it is evident that the inverse function,

$w = \mathrm{cosec}^{-1}(q) = \sin^{-1}(q^{-1})$, $\forall q \in \mathbb{H} - \{0\}$, as required. Q.E.D.

(c) Finally, from Definitions DI-11 and DI-19, we recall that the inverse cotangent function,

$$w = \cot^{-1}(q) \Longrightarrow q = \cot(w)$$

$\therefore q^{-1} = (\cot(w))^{-1}$

$\therefore q^{-1} = (1/\tan(w))^{-1} = \tan(w)$, $\forall q \in \mathbb{H} - \{0\}.$

However, in accordance with Definition DI-18, we likewise deduce that

$$w = \tan^{-1}(q^{-1}) \Longrightarrow q^{-1} = \tan(w) \Longrightarrow q = \cot(w),$$

whence it is evident that the inverse function,

$w = \cot^{-1}(q) = \tan^{-1}(q^{-1})$, $\forall q \in \mathbb{H} - \{0\}$, as required. Q.E.D.

We conclude our discussion of the inverse trigonometric quaternion hypercomplex functions with the following remarks:-

(a) The results of the preceding Theorem TI-25 are analogous with Eqs. (1-104), (1-105) & (1-106) insofar as the complex number, i, is replaced by the quaternion variable,

$$Q = \frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}} .$$

(b) By direct contrast, however, neither Sales & Einar Hille [2] nor Churchill et al. [1] have provided real and complex variable analogues of the results enunciated in Theorem TI-26. Admittedly, the former authors did describe the real variable functions, $\sec^{-1}(x)$, $\operatorname{cosec}^{-1}(x)$ and $\cot^{-1}(x)$, as being the inverses of $\sec(x)$, $\operatorname{cosec}(x)$ and $\cot(x)$ respectively.

9. The Inverse Hyperbolic Functions.

From complex variable analysis, we recall that the inverse hyperbolic functions, $\sinh^{-1}(z)$, $\cosh^{-1}(z)$ and $\tanh^{-1}(z)$, are defined in the following manner:-

(a) the inverse hyperbolic sine function,

$$w = \sinh^{-1}(z) \Longrightarrow z = \sinh(w) \qquad (1-107);$$

(b) the inverse hyperbolic cosine function,

$$w = \cosh^{-1}(z) \Longrightarrow z = \cosh(w) \qquad (1-108);$$

(c) the inverse hyperbolic tangent function,

$$w = \tanh^{-1}(z) \Longrightarrow z = \tanh(w) \qquad (1-109).$$

Moreover, Churchill et al. [1] provide us with three supplementary formulae for $\sinh^{-1}(z)$, $\cosh^{-1}(z)$ and $\tanh^{-1}(z)$, namely –

$$\sinh^{-1}(z) = \log[z + (z^2+1)^{1/2}] \qquad (1-110);$$

$$\cosh^{-1}(z) = \log[z + (z^2-1)^{1/2}] \qquad (1-111);$$

$$\tanh^{-1}(z) = \tfrac{1}{2}\log\left(\frac{1+z}{1-z}\right) \qquad (1-112).$$

Hence, the purpose of our next definition and theorem is to derive the quaternion analogues of Eqs. $(1-107) \longrightarrow (1-112)$.

Definition DI-20.

Let there exist three inverse hyperbolic quaternion hypercomplex functions, $\sinh^{-1}(q)$, $\cosh^{-1}(q)$ and $\tanh^{-1}(q)$, whereupon $q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H}$. In the circumstances, we postulate that

(a) the inverse hyperbolic sine function,

$$w = \sinh^{-1}(q) \Longrightarrow q = \sinh(w),$$

(b) the <u>inverse hyperbolic cosine</u> function,

$$w = \cosh^{-1}(q) \Longrightarrow q = \cosh(w),$$

(c) the <u>inverse hyperbolic tangent</u> function,

$$w = \tanh^{-1}(q) \Longrightarrow q = \tanh(w).$$

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<u>Theorem TI-27.</u>

Let there exist the <u>inverse hyperbolic</u> functions, $\sinh^{-1}(q)$, $\cosh^{-1}(q)$ and $\tanh^{-1}(q)$, as previously defined. Subsequently, we may prove that the following formulae, namely -

(a) $\sinh^{-1}(q) = \log(q + (q^2+1)^{1/2})$,

(b) $\cosh^{-1}(q) = \log(q + (q^2-1)^{1/2})$,

(c) $\tanh^{-1}(q) = \frac{1}{2}\log\left(\dfrac{1+q}{1-q}\right)$,

are likewise valid, whenever $q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H}$.

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<u>PROOF</u>:-

From Definitions DI-12, DI-13 and DI-20, we initially recall that

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(i) the <u>inverse hyperbolic sine</u> function,

$$w = \sinh^{-1}(q) \Longrightarrow q = \sinh(w)$$

$$= \frac{\exp(w) - \exp(-w)}{2},$$

(ii) the <u>inverse hyperbolic cosine</u> function,

$$w = \cosh^{-1}(q) \Longrightarrow q = \cosh(w)$$

$$= \frac{\exp(w) + \exp(-w)}{2},$$

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(iii) the <u>inverse hyperbolic tangent</u> function,

$$w = \tanh^{-1}(q) \Longrightarrow q = \tanh(w)$$

$$= \frac{\sinh(w)}{\cosh(w)} \qquad (\cosh(w) \neq 0)$$

$$= \frac{\exp(w) - \exp(-w)}{\exp(w) + \exp(-w)},$$

such that

$$w = u_1 + iv_1 + ju_2 + kv_2$$

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$$= u_1 + QV,$$

whenever $v_1 = \dfrac{yV}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$, $u_2 = \dfrac{\hat{z}V}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$ and

$$v_2 = \frac{\hat{y}V}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} \Longrightarrow Q = \frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}.$$

We will now examine each of these inverse hyperbolic functions separately.

(a) In view of the criteria specified by Theorem TI-10, we perceive that

$$q = \sinh(w)$$

$$= \frac{\exp(w) - \exp(-w)}{2}$$

$$\therefore \exp(w) - \exp(-w) = 2q$$

$$\therefore (\exp(w))^2 - \exp(-w)\exp(w) = 2q\exp(w)$$

$$\therefore (\exp(w))^2 - \exp(-w+w) - 2q\exp(w) = 0$$

$$\therefore (\exp(w))^2 - 2q\exp(w) - \exp(0) = 0$$

$$\therefore (\exp(w))^2 - 2q\exp(w) - 1 = 0$$

$$\therefore (\exp(w))^2 - 2q\exp(w) = 1$$

$$\therefore (\exp(w))^2 - 2q\exp(w) + q^2 = q^2 + 1$$

$$\therefore (\exp(w) - q)^2 = q^2 + 1$$

$$\therefore \exp(w) - q = (q^2+1)^{1/2}$$

$$\therefore \exp(w) = q + (q^2+1)^{1/2} \Longrightarrow$$

$$w = \log(q + (q^2+1)^{1/2}).$$

However, since we had originally stated that the inverse function,

$$w = \sinh^{-1}(q) \Longrightarrow q = \sinh(w),$$

we accordingly obtain the result,

$\sinh^{-1}(q) = \log(q + (q^2+1)^{1/2})$, as required. Q.E.D.

(b) Similarly, in view of the previously established criteria from part (a) of this theorem, we likewise perceive that

$$q = \cosh(w)$$

$$= \frac{\exp(w) + \exp(-w)}{2}$$

$$\therefore \exp(w) + \exp(-w) = 2q$$

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$$\therefore (\exp(w))^2 + \exp(-w)\exp(w) = 2q\exp(w)$$

$$\therefore (\exp(w))^2 - 2q\exp(w) + \exp(-w+w) = 0$$

$$\therefore (\exp(w))^2 - 2q\exp(w) + \exp(0) = 0$$

$$\therefore (\exp(w))^2 - 2q\exp(w) + 1 = 0$$

$$\therefore (\exp(w))^2 - 2q\exp(w) = -1$$

$$\therefore (\exp(w))^2 - 2q\exp(w) + q^2 = q^2 - 1$$

$$\therefore (\exp(w) - q)^2 = q^2 - 1$$

$$\therefore \exp(w) - q = (q^2-1)^{1/2}$$

$$\therefore \exp(w) = q + (q^2-1)^{1/2} \Longrightarrow$$

$$w = \log(q + (q^2-1)^{1/2}).$$

However, since we had originally stated that the inverse function,

$$w = \cosh^{-1}(q) \Longrightarrow q = \cosh(w),$$

we accordingly obtain the result,

$\cosh^{-1}(q) = \log(q + (q^2-1)^{1/2})$, as required. Q.E.D.

(c) Finally, in view of the previous considerations arising from parts (a) and (b) of this theorem, we deduce that

$$q = \tanh(w)$$

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$$= \frac{\exp(w) - \exp(-w)}{\exp(w) + \exp(-w)}$$

$$\therefore q(\exp(w) + \exp(-w)) = \exp(w) - \exp(-w)$$

$$\therefore q\exp(w) + q\exp(-w) = \exp(w) - \exp(-w)$$

$$\therefore (q-1)\exp(w) + (q+1)\exp(-w) = 0$$

$$\therefore (q-1)(\exp(w))^2 + (q+1)\exp(-w)\exp(w) = 0$$

$$\therefore (q-1)(\exp(w))^2 + (q+1)\exp(-w+w) = 0$$

$$\therefore (q-1)(\exp(w))^2 + (q+1)\exp(0) = 0$$

$$\therefore (q-1)\exp(2w) + q+1 = 0$$

$$\therefore \exp(2w) = -\frac{(q+1)}{q-1}$$

$$\therefore \exp(2w) = -\frac{(1+q)}{-(1-q)}$$

$$\therefore \exp(2w) = \frac{1+q}{1-q} \Longrightarrow$$

$$2w = \log\left(\frac{1+q}{1-q}\right)$$

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$$\therefore\ w = \tfrac{1}{2}\log\left(\frac{1+q}{1-q}\right).$$

However, since we had originally stated that the inverse function,

$$w = \tanh^{-1}(q) \Longrightarrow q = \tanh(w),$$

we accordingly obtain the result,

$$\tanh^{-1}(q) = \tfrac{1}{2}\log\left(\frac{1+q}{1-q}\right),$$ as required. <u>Q.E.D.</u>

For the sake of completeness, we will define the inverse hyperbolic functions, $\operatorname{sech}^{-1}(q)$, $\operatorname{cosech}^{-1}(q)$ and $\coth^{-1}(q)$, as follows:-

<u>Definition DI-21.</u>

Let there exist three <u>inverse hyperbolic</u> quaternion hypercomplex functions, $\operatorname{sech}^{-1}(q)$, $\operatorname{cosech}^{-1}(q)$ and $\coth^{-1}(q)$. In the circumstances, we postulate that

(a) the <u>inverse hyperbolic secant</u> function,

$$w = \operatorname{sech}^{-1}(q) \Longrightarrow q = \operatorname{sech}(w),$$

(b) the <u>inverse hyperbolic cosecant</u> function,

$w = \operatorname{cosech}^{-1}(q) \Longrightarrow q = \operatorname{cosech}(w)$,

(c) the <u>inverse hyperbolic cotangent</u> function,

$w = \coth^{-1}(q) \Longrightarrow q = \coth(w)$.

From this definition, the reader will note that we have neither specified the respective domains of these particular functions nor have we defined them in terms of functions with which we are already familiar. Our next theorem, however, will address both of these issues simultaneously.

<u>Theorem TI-28</u>.

Let there exist the inverse hyperbolic functions, $\operatorname{sech}^{-1}(q)$, $\operatorname{cosech}^{-1}(q)$ and $\coth^{-1}(q)$, as previously defined. Subsequently, we may prove that the following formulae, namely –

(a) $\operatorname{sech}^{-1}(q) = \cosh^{-1}(q^{-1})$,

(b) $\operatorname{cosech}^{-1}(q) = \sinh^{-1}(q^{-1})$,

(c) $\coth^{-1}(q) = \tanh^{-1}(q^{-1})$,

are valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}$.

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<u>PROOF</u>:–

(a) From Definitions DI-13 and DI-21, we recall that the <u>inverse hyperbolic secant</u> function,

$$w = \operatorname{sech}^{-1}(q) \Longrightarrow q = \operatorname{sech}(w)$$

$$\therefore q^{-1} = (\operatorname{sech}(w))^{-1}$$

$$\therefore q^{-1} = (1/\cosh(w))^{-1} = \cosh(w), \ \forall q \in \mathbb{H} - \{0\}.$$

However, in accordance with Definition DI-20, we likewise deduce that

$$w = \cosh^{-1}(q^{-1}) \Longrightarrow q^{-1} = \cosh(w) \Longrightarrow q = \operatorname{sech}(w),$$

whence it is evident that the inverse function,

$$w = \operatorname{sech}^{-1}(q) = \cosh^{-1}(q^{-1}), \ \forall q \in \mathbb{H} - \{0\},$$ as required. <u>Q.E.D.</u>

(b) From Definitions DI-13 and DI-21, we similarly perceive that the <u>inverse hyperbolic cosecant</u> function,

$$w = \operatorname{cosech}^{-1}(q) \Longrightarrow q = \operatorname{cosech}(w)$$

$$\therefore q^{-1} = (\operatorname{cosech}(w))^{-1}$$

$$\therefore q^{-1} = (1/\sinh(w))^{-1} = \sinh(w), \ \forall q \in \mathbb{H} - \{0\}.$$

However, in accordance with Definition DI-20, we likewise deduce that

$w = \sinh^{-1}(q^{-1}) \Longrightarrow q^{-1} = \sinh(w) \Longrightarrow q = \operatorname{cosech}(w),$

whence it is evident that the inverse function,

$w = \operatorname{cosech}^{-1}(q) = \sinh^{-1}(q^{-1}),\ \forall q \in \mathbb{H} - \{0\}$, as required. Q.E.D.

(c) Finally, from Definitions DI-13 and DI-21, we recall that the inverse hyperbolic cotangent function,

$w = \coth^{-1}(q) \Longrightarrow q = \coth(w)$

$\therefore q^{-1} = (\coth(w))^{-1}$

$\therefore q^{-1} = (\cosh(w)/\sinh(w))^{-1} = \sinh(w)/\cosh(w) = \tanh\ (w),\ \forall q \in \mathbb{H} - \{0\}.$

However, in accordance with Definition DI-20, we likewise deduce that

$w = \tanh^{-1}(q^{-1}) \Longrightarrow q^{-1} = \tanh(w) \Longrightarrow q = \coth(w),$

whence it is evident that the inverse function,

$w = \coth^{-1}(q) = \tanh^{-1}(q^{-1}),\ \forall q \in \mathbb{H} - \{0\}$, as required. Q.E.D.

We conclude our discussion of the inverse hyperbolic quaternion hypercomplex functions with the following remarks:-

(a) The results of the preceding Theorem TI-27 are completely analogous with Eqs. (1-110), (1-111) & (1-112).

(b) By direct contrast, however, neither Salas & Einar Hille [2] nor Churchill et al. [1] have provided real and complex variable analogues of the results enunciated in Theorem TI-28. The former authors had justified such an omission on the grounds that $\sinh^{-1}(x)$, $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$ are the only real variable inverse hyperbolic functions which play a significant rôle in modern analysis.

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II. Further Calculus of Quaternion Hypercomplex Functions

The author in his first paper [5] initially developed the calculus of quaternion (hypercomplex) functions by deriving suitable analogues of several definitions and theorems arising from real and complex variable analysis (viz. Salas and Einar Hille [2] and Churchill et al. [1]). Hence, the purpose of this section is to provide the reader with a deeper understanding of this particular theory after we have expounded the following topics, namely –

(a) the notion of limits and continuity applied to multi-valued quaternion hypercomplex functions;
(b) further properties of the first order derivatives of quaternion hypercomplex functions;
(c) further properties of the indefinite and definite integrals of quaternion hypercomplex functions.

1. The Notion of Limits and Continuity applied to Multi-valued Quaternion Hypercomplex Functions.

From Part 1 of Section I, the reader will recall that we had both invoked the notion of a functional array and then defined some of its properties as a means of representing the constituent values of a multi-valued quaternion (hypercomplex) function. We can now take this concept a stage further by including the notions of limits and continuity as follows:-

Definition DII-1.

Let there exist a multi-valued quaternion-hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 & \Longrightarrow \# f(q) = n. \\ \vdots \\ \vdots \\ [f(q)]_n \end{cases}$$

In the circumstances, we postulate the existence of the limit of 'f', as $q \to q_0$, namely -

$$\lim_{q \to q_0}(f(q)) = \lim_{q \to q_0}\left(\begin{cases} [f(q)]_1 \\ \vdots \\ \vdots \\ [f(q)]_n \end{cases}\right) = \begin{cases} \lim_{q \to q_0}([f(q)]_1) \\ \vdots \\ \vdots \\ \lim_{q \to q_0}([f(q)]_n) \end{cases},$$

provided that the constituent limits, $\lim_{q \to q_0}([f(q)]_1), \ldots\ldots, \lim_{q \to q_0}([f(q)]_n)$, are simultaneously defined.

Theorem TII–1.

Let there exist a multi-valued quaternion hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 \Longrightarrow \# f(q) = n. \\ \vdots \\ [f(q)]_n \end{cases}$$

Subsequently, it may be proven that this function is always continuous at q_0, that is to say –

$$\lim_{q \to q_0}(f(q)) = f(q_0),$$

if and only if the constituent single-valued functions, $[f(q)]_1, \ldots\ldots, [f(q)]_n$, are also continuous at q_0.

* * *

PROOF:–

We firstly observe that the multi-valued function,

$$f(q) = \begin{cases} [f(q)]_1 \\ \vdots \\ [f(q)]_n \end{cases} \Longrightarrow f(q_0) = \begin{cases} [f(q_0)]_1 \\ \vdots \\ [f(q_0)]_n \end{cases} ,$$

and similarly, from Definition DII-1, we recall that

$$\lim_{q \to q_0}(f(q)) = \begin{cases} \lim_{q \to q_0}([f(q)]_1) \\ \vdots \\ \lim_{q \to q_0}([f(q)]_n) \end{cases} .$$

However, since the <u>continuity</u> at q_0 of the <u>single-valued</u> functions, $[f(q)]_1, \ldots\ldots, [f(q)]_n$, also implies the existence of the limits,

$$\lim_{q \to q_0}([f(q)]_1) = [f(q_0)]_1 , \ldots\ldots\ldots , \lim_{q \to q_0}([f(q)]_n) = [f(q_0)]_n ,$$

it therefore follows that the limit,

$$\lim_{q \to q_0}(f(q)) = \begin{cases} \lim_{q \to q_0}([f(q)]_1) \\ \vdots \\ \lim_{q \to q_0}([f(q)]_n) \end{cases} = \begin{cases} [f(q_0)]_1 \\ \vdots \\ [f(q_0)]_n \end{cases} = f(q_0),$$

and hence we have shown that the function, $f(q)$, is likewise continuous at q_0. <u>Q.E.D.</u>

If we consider the particular case of a <u>multi-valued</u> quaternion function, $f(q)$, which is also defined on an <u>arc, C,</u> embedded in <u>q-space</u>, then we can reiterate the outcome of Theorem TII-1 in terms of the next definition:-

<u>Definition DII-2.</u>

Let there exist a <u>multi-valued</u> quaternion (hypercomplex) function, $f(q)$, which is also defined on an <u>arc, C,</u> embedded in <u>q-space</u>, such that we obtain

$$f(q(t)) = \left\{ \begin{array}{l} [f(q(t))]_1 \\ \vdots \\ [f(q(t))]_n \end{array} \right. \Longrightarrow \#f(q(t)) = n.$$

Subsequently, we postulate the existence of the <u>parametric limit</u> of 'f', as $t \to t_0$, denoted by

$$\lim_{t\to t_0}\left(f(q(t))\right) = \lim_{t\to t_0}\left(\left\{ \begin{array}{l} [f(q(t))]_1 \\ \vdots \\ [f(q(t))]_n \end{array} \right.\right) = \left\{ \begin{array}{l} \lim_{t\to t_0}\left([f(q(t))]_1\right) \\ \vdots \\ \lim_{t\to t_0}\left([f(q(t))]_n\right) \end{array} \right.$$

$$= \left\{ \begin{array}{l} [f(q(t_0))]_1 \\ \vdots \\ [f(q(t_0))]_n \end{array} \right. = f(q(t_0)),$$

insofar as we let the single-valued functions, $[f(q(t))]_1, \ldots\ldots\ldots, [f(q(t))]_n$, be continuous at t_0, thereby inferring that the function, $f(q(t))$, is also continuous at t_0.

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We conclude our discussion of this topic by remarking that, in view of the functional array notation employed in Definitions DI-1 and DI-3, the multi-valued functions, $f(q)$ and $f(q(t))$, may be respectively written as —

$$f(q) = \{[f(q)]_{j^*};\ \forall j^* = 1, \ldots\ldots\ldots, n]$$

$$\Longrightarrow \#f(q) = \#\{[f(q)]_{j^*};\ \forall j^* = 1, \ldots\ldots\ldots, n] = n \qquad (2\text{-}1)$$

AND

$$f(q(t)) = \{[f(q(t))]_{j^*};\ \forall j^* = 1, \ldots\ldots\ldots, n]$$

$$\Longrightarrow \#f(q(t)) = \#\{[f(q(t))]_{j^*};\ \forall j^* = 1, \ldots\ldots\ldots, n] = n \qquad (2\text{-}2).$$

2. Further Properties of the First Order Derivatives of Quaternion Hypercomplex Functions.

In this part of Section II, we will

(a) enunciate various definitions and theorems whose purpose is to expand upon those techniques of differentiation initially developed by the author in his first paper [5]

AND

(b) compare these definitions and theorems with their more familiar analogues from real and complex variable analysis.

<u>Definition DII – 3</u>.

Let there exist a <u>multi-valued</u> quaternion hypercomplex function, $f(q)$, which is also defined on an <u>arc, C, embedded in q-space</u>, such that we obtain

$$f(q(t)) = \begin{cases}[f(q(t))]_1 \\ \vdots \\ [f(q(t))]_n\end{cases} \Longrightarrow \#f(q(t)) = n.$$

Subsequently, we postulate the existence of the <u>parametric first derivative of 'f', with respect to 't'</u>, in other words –

$$\frac{d}{dt}\left(f(q(t))\right) = \frac{d}{dt}\left(\begin{cases}[f(q(t))]_1 \\ \vdots \\ [f(q(t))]_n\end{cases}\right) = \begin{cases}\frac{d}{dt}\left([f(q(t))]_1\right) \\ \vdots \\ \frac{d}{dt}\left([f(q(t))]_n\right)\end{cases},$$

provided that the constituent derivatives, $\frac{d}{dt}\left([f(q(t))]_1\right), \ldots\ldots, \frac{d}{dt}\left([f(q(t))]_n\right)$, are simultaneously defined.

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Definition DII-4.

Let there exist a multi-valued quaternion hypercomplex function, $f(q)$, which is also defined on an arc, C, embedded in q-space, such that we obtain

$$f(q(t)) = \begin{cases} [f(q(t))]_1 \\ \vdots \\ [f(q(t))]_n \end{cases} \Longrightarrow \#f(q(t)) = n.$$

Subsequently, we postulate the existence of the first derivative of 'f', with respect to 'q', thus restricted to an arc, C, which is accordingly denoted by

$$\left[\frac{d}{dq}\right]_C(f(q)) = \left[\frac{d}{dq}\right]_C\left(\begin{cases} [f(q)]_1 \\ \vdots \\ [f(q)]_n \end{cases}\right) = \begin{cases} \left[\frac{d}{dq}\right]_C([f(q)]_1) \\ \vdots \\ \left[\frac{d}{dq}\right]_C([f(q)]_n) \end{cases},$$

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provided that the constituent derivatives, $\left[\frac{d}{dq}\right]_C([f(q)]_1), \ldots\ldots, \left[\frac{d}{dq}\right]_C([f(q)]_n)$, are simultaneously defined.

Definition DII-5.

Let there exist a single-valued quaternion hypercomplex function,

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$$f(q) = f(x+iy+j\hat{x}+k\hat{y}) = u_1(x,y,\hat{x},\hat{y}) + iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y}) .$$

Subsequently, we postulate the existence of

(a) the <u>partial derivative of $f(q)$ with respect to x,</u>

$$\frac{\partial}{\partial x}(f(q)) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x+iy+j\hat{x}+k\hat{y}) - f(x+iy+j\hat{x}+k\hat{y})}{\delta x}\right],$$

such that

$$\left|\frac{f(x+\delta x+iy+j\hat{x}+k\hat{y}) - f(x+iy+j\hat{x}+k\hat{y})}{\delta x} - \frac{\partial}{\partial x}(f(q))\right| < \epsilon ,$$

whenever $0 < |\delta x| < \delta$, $\forall \delta, \epsilon > 0$;

(b) the <u>partial derivative of $f(q)$ with respect to y,</u>

$$\frac{\partial}{\partial y}(f(q)) = \lim_{\delta y \to 0}\left[\frac{f(x+i(y+\delta y)+j\hat{x}+k\hat{y}) - f(x+iy+j\hat{x}+k\hat{y})}{\delta y}\right],$$

such that

$$\left|\frac{f(x+i(y+\delta y)+j\hat{x}+k\hat{y}) - f(x+iy+j\hat{x}+k\hat{y})}{\delta y} - \frac{\partial}{\partial y}(f(q))\right| < \epsilon ,$$

whenever $0 < |\delta y| < \delta$, $\forall \delta, \epsilon > 0$;

(c) the <u>partial derivative</u> of $f(q)$ with respect to $\hat{x}$,

$$\frac{\partial}{\partial \hat{x}}(f(q)) = \lim_{\delta \hat{x} \to 0} \left[\frac{f(x + iy + j(\hat{x} + \delta\hat{x}) + k\hat{y}) - f(x + iy + j\hat{x} + k\hat{y})}{\delta \hat{x}} \right] ,$$

such that

$$\left| \frac{f(x + iy + j(\hat{x} + \delta\hat{x}) + k\hat{y}) - f(x + iy + j\hat{x} + k\hat{y})}{\delta\hat{x}} - \frac{\partial}{\partial \hat{x}}(f(q)) \right| < \epsilon ,$$

whenever $0 < |\delta\hat{x}| < \delta$, $\forall \delta, \epsilon > 0$;

(d) the <u>partial derivative</u> of $f(q)$ with respect to $\hat{y}$,

$$\frac{\partial}{\partial \hat{y}}(f(q)) = \lim_{\delta \hat{y} \to 0} \left[\frac{f(x + iy + j\hat{x} + k(y + \delta\hat{y})) - f(x + iy + j\hat{x} + k\hat{y})}{\delta \hat{y}} \right] ,$$

such that

$$\left| \frac{f(x + iy + j\hat{x} + k(\hat{y} + \delta\hat{y})) - f(x + iy + j\hat{x} + k\hat{y})}{\delta\hat{y}} - \frac{\partial}{\partial \hat{y}}(f(q)) \right| < \epsilon ,$$

whenever $0 < |\delta\hat{y}| < \delta$, $\forall \delta, \epsilon > 0$.

<u>Theorem TII-2</u>.

Let there exist a <u>single-valued</u> quaternion hypercomplex function, $f(q)$, as previously defined in Definition DII-5. In the circumstances, it may be proven that

(a) the <u>partial derivative</u> of $f(q)$ with respect to x,

$$\frac{\partial}{\partial x}(f(q)) = \frac{\partial}{\partial x}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial x}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial x}(u_2(x,y,\hat{x},\hat{y})) + k\frac{\partial}{\partial x}(v_2(x,y,\hat{x},\hat{y})),$$

provided that the real variable partial derivatives, $\frac{\partial}{\partial x}(u_1(x,y,\hat{x},\hat{y}))$, ..., $\frac{\partial}{\partial x}(v_2(x,y,\hat{x},\hat{y}))$, are simultaneously defined ;

(b) the <u>partial derivative of $f(q)$ with respect to y</u>,

$$\frac{\partial}{\partial y}(f(q)) = \frac{\partial}{\partial y}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial y}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial y}(u_2(x,y,\hat{x},\hat{y})) + k\frac{\partial}{\partial y}(v_2(x,y,\hat{x},\hat{y})),$$

provided that the real variable partial derivatives, $\frac{\partial}{\partial y}(u_1(x,y,\hat{x},\hat{y}))$, ..., $\frac{\partial}{\partial y}(v_2(x,y,\hat{x},\hat{y}))$, are simultaneously defined ;

(c) the <u>partial derivative of $f(q)$ with respect to $\hat{x}$</u>,

$$\frac{\partial}{\partial \hat{x}}(f(q)) = \frac{\partial}{\partial \hat{x}}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial \hat{x}}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial \hat{x}}(u_2(x,y,\hat{x},\hat{y})) + k\frac{\partial}{\partial \hat{x}}(v_2(x,y,\hat{x},\hat{y})),$$

provided that the real variable partial derivatives, $\frac{\partial}{\partial \hat{x}}(u_1(x,y,\hat{x},\hat{y}))$, ..., $\frac{\partial}{\partial \hat{x}}(v_2(x,y,\hat{x},\hat{y}))$, are simultaneously defined ;

(d) the <u>partial derivative of $f(q)$ with respect to $\hat{y}$</u>,

$$\frac{\partial}{\partial \hat{y}}(f(q)) = \frac{\partial}{\partial \hat{y}}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial \hat{y}}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial \hat{y}}(u_2(x,y,\hat{x},\hat{y})) + k\frac{\partial}{\partial \hat{y}}(v_2(x,y,\hat{x},\hat{y})),$$

provided that the real variable partial derivatives, $\frac{\partial}{\partial \hat{y}}(u_1(x,y,\hat{x},\hat{y}))$, ..., $\frac{\partial}{\partial \hat{y}}(v_2(x,y,\hat{x},\hat{y}))$, are simultaneously defined.

* * *

PROOF:-

From Definition DII−5, we deduce that the function,

$$f(q) = f(x+iy+j\hat{x}+k\hat{y}) = u_1(x,y,\hat{x},\hat{y}) + iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y}) \Longrightarrow$$

$$f(x+\delta x+iy+j\hat{x}+k\hat{y}) = u_1(x+\delta x,y,\hat{x},\hat{y}) + iv_1(x+\delta x,y,\hat{x},\hat{y}) + ju_2(x+\delta x,y,\hat{x},\hat{y}) + kv_2(x+\delta x,y,\hat{x},\hat{y}) \ ,$$

$$f(x+i(y+\delta y)+j\hat{x}+k\hat{y}) = u_1(x,y+\delta y,\hat{x},\hat{y}) + iv_1(x,y+\delta y,\hat{x},\hat{y}) + ju_2(x,y+\delta y,\hat{x},\hat{y}) + kv_2(x,y+\delta y,\hat{x},\hat{y}) \ ,$$

$$f(x+iy+j(\hat{x}+\delta\hat{x})+k\hat{y}) = u_1(x,y,\hat{x}+\delta\hat{x},\hat{y}) + iv_1(x,y,\hat{x}+\delta\hat{x},\hat{y}) + ju_2(x,y,\hat{x}+\delta\hat{x},\hat{y}) + kv_2(x,y,\hat{x}+\delta\hat{x},\hat{y}) \ ,$$

$$f(x+iy+j\hat{x}+k(\hat{y}+\delta\hat{y})) = u_1(x,y,\hat{x},\hat{y}+\delta\hat{y}) + iv_1(x,y,\hat{x},\hat{y}+\delta\hat{y}) + ju_2(x,y,\hat{x},\hat{y}+\delta\hat{y}) + kv_2(x,y,\hat{x},\hat{y}+\delta\hat{y}) \ .$$

(a) Now let the partial derivative of $f(q)$ with respect to x,

$$\frac{\partial}{\partial x}(f(q)) = U_1 + iV_1 + jU_2 + kV_2 \ ,$$

where the variables, $U_1, V_1, U_2, V_2 \in \mathbb{R}$. Since, by virtue of the above definition, the partial derivative,

$$\frac{\partial}{\partial x}(f(q)) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x+iy+j\hat{x}+k\hat{y}) - f(x+iy+j\hat{x}+k\hat{y})}{\delta x}\right] \ ,$$

also implies the existence of the inequality,

$$\left| \frac{f(x+\delta x+iy+j\hat{x}+k\hat{y})-f(x+iy+j\hat{x}+k\hat{y})}{\delta x} - \frac{\partial}{\partial x}(f(q)) \right| < \epsilon,$$

wherever $0<|\delta x|<\delta$, $\forall \delta,\epsilon>0$,

it therefore follows, after making the appropriate algebraic substitutions, that

$$\left| \begin{array}{l} \dfrac{u_1(x+\delta x,y,\hat{x},\hat{y})-u_1(x,y,\hat{x},\hat{y})}{\delta x} + i\left(\dfrac{v_1(x+\delta x,y,\hat{x},\hat{y})-v_1(x,y,\hat{x},\hat{y})}{\delta x}\right) + \\ j\left(\dfrac{u_2(x+\delta x,y,\hat{x},\hat{y})-u_2(x,y,\hat{x},\hat{y})}{\delta x}\right) + k\left(\dfrac{v_2(x+\delta x,y,\hat{x},\hat{y})-v_2(x,y,\hat{x},\hat{y})}{\delta x}\right) \\ -(U_1+iV_1+jU_2+kV_2) \end{array} \right| < \epsilon,$$

wherever $0<|\delta x|<\delta$, $\forall \delta,\epsilon>0$,

$$\therefore \left| \begin{array}{l} \dfrac{u_1(x+\delta x, y, \hat{x}, \hat{y}) - u_1(x, y, \hat{x}, \hat{y})}{\delta x} - U_1 + \\ i\left(\dfrac{v_1(x+\delta x, y, \hat{x}, \hat{y}) - v_1(x, y, \hat{x}, \hat{y})}{\delta x} - V_1\right) + \\ j\left(\dfrac{u_2(x+\delta x, y, \hat{x}, \hat{y}) - u_2(x, y, \hat{x}, \hat{y})}{\delta x} - U_2\right) + \\ k\left(\dfrac{v_2(x+\delta x, y, \hat{x}, \hat{y}) - v_2(x, y, \hat{x}, \hat{y})}{\delta x} - V_2\right) \end{array} \right| < \epsilon,$$

whenever $0 < |\delta x| < \delta$, $\forall \delta, \epsilon > 0$.

Now, from the established properties of quaternion modular inequalities (viz. the author's first paper [5]), we further obtain

$$\left| \frac{u_1(x+\delta x, y, \hat{x}, \hat{y}) - u_1(x, y, \hat{x}, \hat{y})}{\delta x} - U_1 \right| < \epsilon;$$

$$\left| \frac{v_1(x+\delta x, y, \hat{x}, \hat{y}) - v_1(x, y, \hat{x}, \hat{y})}{\delta x} - V_1 \right| < \epsilon;$$

$$\left| \frac{u_2(x+\delta x, y, \hat{x}, \hat{y}) - u_2(x, y, \hat{x}, \hat{y})}{\delta x} - U_2 \right| < \epsilon;$$

$$\left| \frac{v_2(x+\delta x, y, \hat{x}, \hat{y}) - v_2(x, y, \hat{x}, \hat{y})}{\delta x} - V_2 \right| < \epsilon,$$

whenever $0 < |\delta x| < \delta$, $\forall \delta, \epsilon > 0$,

which are precisely the conditions required for the existence of the <u>real variable partial derivatives</u>,

$$U_1 = \lim_{\delta x \to 0}\left[\frac{u_1(x+\delta x, y, \hat{x}, \hat{y}) - u_1(x, y, \hat{x}, \hat{y})}{\delta x}\right] = \frac{\partial}{\partial x}(u_1(x, y, \hat{x}, \hat{y})),$$

$$V_1 = \lim_{\delta x \to 0}\left[\frac{v_1(x+\delta x, y, \hat{x}, \hat{y}) - v_1(x, y, \hat{x}, \hat{y})}{\delta x}\right] = \frac{\partial}{\partial x}(v_1(x, y, \hat{x}, \hat{y})),$$

$$U_2 = \lim_{\delta x \to 0}\left[\frac{u_2(x+\delta x, y, \hat{x}, \hat{y}) - u_2(x, y, \hat{x}, \hat{y})}{\delta x}\right] = \frac{\partial}{\partial x}(u_2(x, y, \hat{x}, \hat{y})),$$

$$V_2 = \lim_{\delta x \to 0}\left[\frac{v_2(x+\delta x, y, \hat{x}, \hat{y}) - v_2(x, y, \hat{x}, \hat{y})}{\delta x}\right] = \frac{\partial}{\partial x}(v_2(x, y, \hat{x}, \hat{y})).$$

Subsequently, we conclude that

$$\begin{aligned}\frac{\partial}{\partial x}(f(q)) &= U_1 + iV_1 + jU_2 + kV_2 \\ &= \frac{\partial}{\partial x}(u_1(x, y, \hat{x}, \hat{y})) + i\frac{\partial}{\partial x}(v_1(x, y, \hat{x}, \hat{y})) + j\frac{\partial}{\partial x}(u_2(x, y, \hat{x}, \hat{y})) + \\ &\quad k\frac{\partial}{\partial x}(v_2(x, y, \hat{x}, \hat{y})),\end{aligned}$$

as required. <u>Q.E.D.</u>

<u>(b)</u> the proof of this part of the theorem is completely analogous with part <u>(a)</u>, insofar as we substitute the increment, δy, for δx and the function, $f(x + i(y+\delta y) + j\hat{x} + k\hat{y})$, for $f(x + \delta x + iy + j\hat{x} + k\hat{y})$. <u>Q.E.D.</u>

<u>(c)</u> the proof of this part of the theorem is completely analogous with part <u>(a)</u>,

insofar as we substitute the increment, $\delta\hat{x}$, for δx and the function, $f(x+iy+j(\hat{x}+\delta\hat{x})+k\hat{y})$, for $f(x+\delta x+iy+j\hat{x}+k\hat{y})$. Q.E.D.

(d) the proof of this part of the theorem is completely analogous with part (a), insofar as we substitute the increment, $\delta\hat{y}$, for δx and the function, $f(x+iy+j\hat{x}+k(\hat{y}+\delta\hat{y}))$, for $f(x+\delta x+iy+j\hat{x}+k\hat{y})$. Q.E.D.

Definition DII−6.

Let there exist a multi-valued quaternion hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 \\ \vdots \\ [f(q)]_n \end{cases} \Longrightarrow \#f(q) = n.$$

In the circumstances, we postulate the existence of

(a) the partial derivative of $f(q)$ with respect to x,

$$\frac{\partial}{\partial x}(f(q)) = \frac{\partial}{\partial x}\begin{cases} [f(q)]_1 \\ \vdots \\ [f(q)]_n \end{cases} = \begin{cases} \frac{\partial}{\partial x}([f(q)]_1) \\ \vdots \\ \frac{\partial}{\partial x}([f(q)]_n) \end{cases},$$

provided that the constituent derivatives, $\frac{\partial}{\partial x}([f(q)]_1), \ldots\ldots\ldots, \frac{\partial}{\partial x}([f(q)]_n)$, are simultaneously defined;

To be continued via the author's next submission, namely -

"__A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 6/10__."

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