*“**A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 6/10**.”*

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

*“**A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10**.”*,

which has been published under the **‘VIXRA’ Mathematics subheading:-** *‘**Functions and Analysis**’*.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 6/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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N.B. Page 158 (cont.)

(b) the partial derivative of $f(q)$ with respect to y,

$$\frac{\partial}{\partial y}(f(q)) = \frac{\partial}{\partial y}\begin{cases}[f(q)]_1 \\ \vdots \\ [f(q)]_n\end{cases} = \begin{cases}\frac{\partial}{\partial y}([f(q)]_1) \\ \vdots \\ \frac{\partial}{\partial y}([f(q)]_n)\end{cases},$$

provided that the constituent derivatives, $\frac{\partial}{\partial y}([f(q)]_1), \ldots\ldots\ldots, \frac{\partial}{\partial y}([f(q)]_n)$, are simultaneously defined;

(c) the partial derivative of $f(q)$ with respect to $\hat{x}$,

$$\frac{\partial}{\partial \hat{x}}(f(q)) = \frac{\partial}{\partial \hat{x}}\begin{cases}[f(q)]_1 \\ \vdots \\ [f(q)]_n\end{cases} = \begin{cases}\frac{\partial}{\partial \hat{x}}([f(q)]_1) \\ \vdots \\ \frac{\partial}{\partial \hat{x}}([f(q)]_n)\end{cases},$$

provided that the constituent derivatives, $\frac{\partial}{\partial \hat{x}}([f(q)]_1), \ldots\ldots\ldots, \frac{\partial}{\partial \hat{x}}([f(q)]_n)$, are simultaneously defined;

(d) the partial derivative of $f(q)$ with respect to $\hat{y}$,

$$\frac{\partial}{\partial \hat{y}}(f(q)) = \frac{\partial}{\partial \hat{y}}\begin{cases} [f(q)]_1 \\ \vdots \\ [f(q)]_n \end{cases} = \begin{cases} \frac{\partial}{\partial \hat{y}}([f(q)]_1) \\ \vdots \\ \frac{\partial}{\partial \hat{y}}([f(q)]_n) \end{cases} ,$$

provided that the constituent derivatives, $\frac{\partial}{\partial \hat{y}}([f(q)]_1), \ldots\ldots\ldots, \frac{\partial}{\partial \hat{y}}([f(q)]_n)$, are simultaneously defined.

<u>Theorem TII-3</u>.

Let there exist a <u>single-valued</u> quaternion hypercomplex function,

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i\,v_1(x,y,\hat{x},\hat{y}) + j\,u_2(x,y,\hat{x},\hat{y}) + k\,v_2(x,y,\hat{x},\hat{y}),$$

which is also defined on an <u>arc, C, embedded in q-space</u>, such that we obtain

$$f(q(t)) = u_1(x(t),y(t),\hat{x}(t),\hat{y}(t)) + i\,v_1(x(t),y(t),\hat{x}(t),\hat{y}(t)) + j\,u_2(x(t),y(t),\hat{x}(t),\hat{y}(t)) + k\,v_2(x(t),y(t),\hat{x}(t),\hat{y}(t)) .$$

Henceforth, we may prove that the differential formula,

$$\frac{d}{dt}(f(q(t))) = \frac{\partial}{\partial x}(f(q))\frac{dx}{dt} + \frac{\partial}{\partial y}(f(q))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(f(q))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(f(q))\frac{d\hat{y}}{dt},$$

is always valid, whenever the partial derivatives, $\frac{\partial}{\partial x}(f(q)), \frac{\partial}{\partial y}(f(q)), \frac{\partial}{\partial \hat{x}}(f(q))$ and $\frac{\partial}{\partial \hat{y}}(f(q))$, are simultaneously defined.

* * *

PROOF :-

From the established theorems on differentiating single-valued quaternion hypercomplex functions (viz. the author's first paper [5]), we recall that the parametric first derivative with respect to 't',

$$\frac{d}{dt}(f(q(t))) = \frac{d}{dt}(u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))) + i\frac{d}{dt}(v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))) + j\frac{d}{dt}(u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))) + k\frac{d}{dt}(v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))) \quad .$$

Similarly, from the established theorems on differentiating functions of several real variables, we deduce that

$$\frac{d}{dt}(u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))) = \frac{\partial}{\partial x}(u_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(u_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(u_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(u_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \quad ;$$

$$\frac{d}{dt}(v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))) = \frac{\partial}{\partial x}(v_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(v_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(v_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(v_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \quad ;$$

$$\frac{d}{dt}(u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))) = \frac{\partial}{\partial x}(u_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(u_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(u_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(u_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \quad ;$$

$$\frac{d}{dt}(v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))) = \frac{\partial}{\partial x}(v_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(v_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(v_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(v_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \quad ,$$

and hence, after making the relevant algebraic substitutions, it is evident that the derivative,

$$
\begin{aligned}
\frac{d}{dt}(f(q(t))) = {} & \frac{\partial}{\partial x}(u_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(u_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \\
& \frac{\partial}{\partial \hat{x}}(u_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(u_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \\
& + i\left[\begin{array}{l} \frac{\partial}{\partial x}(v_1(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(v_1(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \\ \frac{\partial}{\partial \hat{x}}(v_1(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(v_1(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \end{array}\right] \\
& + j\left[\begin{array}{l} \frac{\partial}{\partial x}(u_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(u_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \\ \frac{\partial}{\partial \hat{x}}(u_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(u_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \end{array}\right] \\
& + k\left[\begin{array}{l} \frac{\partial}{\partial x}(v_2(x,y,\hat{x},\hat{y}))\frac{dx}{dt} + \frac{\partial}{\partial y}(v_2(x,y,\hat{x},\hat{y}))\frac{dy}{dt} + \\ \frac{\partial}{\partial \hat{x}}(v_2(x,y,\hat{x},\hat{y}))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(v_2(x,y,\hat{x},\hat{y}))\frac{d\hat{y}}{dt} \end{array}\right] \\
= {} & \left[\begin{array}{l} \frac{\partial}{\partial x}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial x}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial x}(u_2(x,y,\hat{x},\hat{y})) + \\ k\frac{\partial}{\partial x}(v_2(x,y,\hat{x},\hat{y})) \end{array}\right]\frac{dx}{dt} + \\
& \left[\begin{array}{l} \frac{\partial}{\partial y}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial y}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial y}(u_2(x,y,\hat{x},\hat{y})) + \\ k\frac{\partial}{\partial y}(v_2(x,y,\hat{x},\hat{y})) \end{array}\right]\frac{dy}{dt} + \\
& \left[\begin{array}{l} \frac{\partial}{\partial \hat{x}}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial \hat{x}}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial \hat{x}}(u_2(x,y,\hat{x},\hat{y})) + \\ k\frac{\partial}{\partial \hat{x}}(v_2(x,y,\hat{x},\hat{y})) \end{array}\right]\frac{d\hat{x}}{dt} + \\
& \left[\begin{array}{l} \frac{\partial}{\partial \hat{y}}(u_1(x,y,\hat{x},\hat{y})) + i\frac{\partial}{\partial \hat{y}}(v_1(x,y,\hat{x},\hat{y})) + j\frac{\partial}{\partial \hat{y}}(u_2(x,y,\hat{x},\hat{y})) + \\ k\frac{\partial}{\partial \hat{y}}(v_2(x,y,\hat{x},\hat{y})) \end{array}\right]\frac{d\hat{y}}{dt} \\
= {} & \frac{\partial}{\partial x}(f(q))\frac{dx}{dt} + \frac{\partial}{\partial y}(f(q))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(f(q))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(f(q))\frac{d\hat{y}}{dt},
\end{aligned}
$$

upon noting the provisions of Theorem TII-2, as required. Q.E.D.

Theorem TII-4.

Let there exist two single-valued quaternion hypercomplex functions, $\phi_1(q)$ and $\phi_2(q)$. In the circumstances, the validity of the following formulae, namely —

(a) $\frac{\partial}{\partial s}(\phi_1(q) + \phi_2(q)) = \frac{\partial}{\partial s}(\phi_1(q)) + \frac{\partial}{\partial s}(\phi_2(q))$;

(b) $\frac{\partial}{\partial s}(\phi_1(q)\phi_2(q)) = \phi_1(q)\frac{\partial}{\partial s}(\phi_2(q)) + \frac{\partial}{\partial s}(\phi_1(q))\phi_2(q)$;

(c) $\frac{\partial}{\partial s}(\phi_2(q)\phi_1(q)) = \phi_2(q)\frac{\partial}{\partial s}(\phi_1(q)) + \frac{\partial}{\partial s}(\phi_2(q))\phi_1(q)$;

(d) $$\frac{\partial}{\partial s}(\phi_1(q)/\phi_2(q)) = \begin{cases} \phi_1(q)\frac{\partial}{\partial s}[\overline{\phi_2(q)}/|\phi_2(q)|^2] + \frac{\partial}{\partial s}(\phi_1(q))[\overline{\phi_2(q)}/|\phi_2(q)|^2] , \\ [\overline{\phi_2(q)}/|\phi_2(q)|^2]\frac{\partial}{\partial s}(\phi_1(q)) + \frac{\partial}{\partial s}[\overline{\phi_2(q)}/|\phi_2(q)|^2]\phi_1(q) \end{cases}$$

where the partial derivative,

$$\frac{\partial}{\partial s}[\overline{\phi_2(q)}/|\phi_2(q)|^2] = \frac{|\phi_2(q)|^2\frac{\partial}{\partial s}(\overline{\phi_2(q)}) - \frac{\partial}{\partial s}(|\phi_2(q)|^2)\overline{\phi_2(q)}}{|\phi_2(q)|^4} \qquad (\phi_2(q) \neq 0) ,$$

may be established, $\forall s \in \{x, y, \hat{x}, \hat{y}\}$, provided that the functions, $\phi_1(q)$ and $\phi_2(q)$, are differentiable in x, y, $\hat{x}$ and $\hat{y}$ respectively.

* * *

PROOF:-

To initiate the proof of this theorem, we accordingly write the functions, $\phi_1(q)$ and $\phi_2(q)$, as

$\phi_1(q) = U_{11} + iV_{11} + jU_{21} + kV_{21}$,

$\phi_2(q) = U_{12} + iV_{12} + jU_{22} + kV_{22}$,

where the real variable functions, $U_{11}, \ldots\ldots\ldots, V_{22}$, are differentiable in $x, y, \hat{x}$ and $\hat{y}$ respectively.

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(a) In view of the above stated requirements, it therefore follows that the function sum,

$$\phi_1(q) + \phi_2(q) = U_{11} + iV_{11} + jU_{21} + kV_{21} + U_{12} + iV_{12} + jU_{22} + kV_{22}$$

$$= U_{11} + U_{12} + i(V_{11} + V_{12}) + j(U_{21} + U_{22}) + k(V_{21} + V_{22}),$$

and hence the partial derivative,

$$\frac{\partial}{\partial x}(\phi_1(q) + \phi_2(q)) = \frac{\partial}{\partial x}(U_{11} + U_{12}) + i\frac{\partial}{\partial x}(V_{11} + V_{12}) + j\frac{\partial}{\partial x}(U_{21} + U_{22}) + k\frac{\partial}{\partial x}(V_{21} + V_{22})$$

$$= \frac{\partial U_{11}}{\partial x} + \frac{\partial U_{12}}{\partial x} + i\frac{\partial V_{11}}{\partial x} + i\frac{\partial V_{12}}{\partial x} + j\frac{\partial U_{21}}{\partial x} + j\frac{\partial U_{22}}{\partial x} + k\frac{\partial V_{21}}{\partial x} + k\frac{\partial V_{22}}{\partial x}$$

$$= \frac{\partial U_{11}}{\partial x} + i\frac{\partial V_{11}}{\partial x} + j\frac{\partial U_{21}}{\partial x} + k\frac{\partial V_{21}}{\partial x} + \frac{\partial U_{12}}{\partial x} + i\frac{\partial V_{12}}{\partial x} + j\frac{\partial U_{22}}{\partial x} + k\frac{\partial V_{22}}{\partial x}$$

$$= \frac{\partial}{\partial x}(\phi_1(q)) + \frac{\partial}{\partial x}(\phi_2(q)) \qquad \underline{(1)},$$

by virtue of Theorem TII-2. Similarly, we perceive that the derivations of the differential formulae,

$$\frac{\partial}{\partial y}(\phi_1(q) + \phi_2(q)) = \frac{\partial}{\partial y}(\phi_1(q)) + \frac{\partial}{\partial y}(\phi_2(q)) \qquad \underline{(2)},$$

$$\frac{\partial}{\partial \hat{x}}(\phi_1(q) + \phi_2(q)) = \frac{\partial}{\partial \hat{x}}(\phi_1(q)) + \frac{\partial}{\partial \hat{x}}(\phi_2(q)) \qquad \underline{(3)},$$

$$\frac{\partial}{\partial \hat{y}}(\phi_1(q) + \phi_2(q)) = \frac{\partial}{\partial \hat{y}}(\phi_1(q)) + \frac{\partial}{\partial \hat{y}}(\phi_2(q)) \qquad \underline{(4)},$$

are completely analogous to that of Eq. (1), insofar as we replace the partial differential operator, $\frac{\partial}{\partial x}$, by the operators, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \hat{x}}$ and $\frac{\partial}{\partial \hat{y}}$, respectively. The amalgamation of Eqs. (1) – (4) thus yields the formula,

$$\frac{\partial}{\partial s}(\phi_1(q) + \phi_2(q)) = \frac{\partial}{\partial s}(\phi_1(q)) + \frac{\partial}{\partial s}(\phi_2(q)), \ \forall s \in \{x, y, \hat{x}, \hat{y}\},$$

as required. <u>Q.E.D.</u>

(b) From the established properties of quaternion products, we perceive that the product function,

$$\phi_1(q)\phi_2(q) = (U_{11} + iV_{11} + jU_{21} + kV_{21})(U_{12} + iV_{12} + jU_{22} + kV_{22})$$

$$= U_{11}U_{12} - V_{11}V_{12} - U_{21}U_{22} - V_{21}V_{22} + i(U_{11}V_{12} + U_{12}V_{11} + U_{21}V_{22} - U_{22}V_{21}) + j(U_{11}U_{22} - V_{11}V_{22} + U_{21}U_{12} + V_{21}V_{12}) + k(U_{11}V_{22} + U_{22}V_{11} - U_{21}V_{12} + U_{12}V_{21}).$$

Hence, the partial derivative,

$$\begin{aligned}\frac{\partial}{\partial x}(\phi_1(q)\phi_2(q)) = {} & \frac{\partial}{\partial x}(U_{11}U_{12} - V_{11}V_{12} - U_{21}U_{22} - V_{21}V_{22}) + \\ & i\frac{\partial}{\partial x}(U_{11}V_{12} + U_{12}V_{11} + U_{21}V_{22} - U_{22}V_{21}) + \\ & j\frac{\partial}{\partial x}(U_{11}U_{22} - V_{11}V_{22} + U_{21}U_{12} + V_{21}V_{12}) + \\ & k\frac{\partial}{\partial x}(U_{11}V_{22} + U_{22}V_{11} - U_{21}V_{12} + U_{12}V_{21})\end{aligned}$$

$$= \frac{\partial}{\partial x}(U_{11}U_{12}) - \frac{\partial}{\partial x}(V_{11}V_{12}) - \frac{\partial}{\partial x}(U_{21}U_{22}) - \frac{\partial}{\partial x}(V_{21}V_{22}) +$$

$$i\left[\frac{\partial}{\partial x}(U_{11}V_{12}) + \frac{\partial}{\partial x}(U_{12}V_{11}) + \frac{\partial}{\partial x}(U_{21}V_{22}) - \frac{\partial}{\partial x}(U_{22}V_{21})\right] +$$

$$j\left[\frac{\partial}{\partial x}(U_{11}U_{22}) - \frac{\partial}{\partial x}(V_{11}V_{22}) + \frac{\partial}{\partial x}(U_{21}U_{12}) + \frac{\partial}{\partial x}(V_{21}V_{12})\right] +$$

$$k\left[\frac{\partial}{\partial x}(U_{11}V_{22}) + \frac{\partial}{\partial x}(U_{22}V_{11}) - \frac{\partial}{\partial x}(U_{21}V_{12}) + \frac{\partial}{\partial x}(U_{12}V_{21})\right]$$

$$= U_{11}\frac{\partial}{\partial x}(U_{12}) + \frac{\partial}{\partial x}(U_{11})U_{12} - \left(V_{11}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(V_{11})V_{12}\right) -$$

$$\left(U_{21}\frac{\partial}{\partial x}(U_{22}) + \frac{\partial}{\partial x}(U_{21})U_{22}\right) - \left(V_{21}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(V_{21})V_{22}\right) +$$

$$i\left[\begin{aligned} & U_{11}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{11})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{11}) + \frac{\partial}{\partial x}(U_{12})V_{11} + \\ & U_{21}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(U_{21})V_{22} - \left(U_{22}\frac{\partial}{\partial x}(V_{21}) + \frac{\partial}{\partial x}(U_{22})V_{21}\right) \end{aligned}\right] +$$

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$$j\left[\begin{aligned} & U_{11}\frac{\partial}{\partial x}(U_{22}) + \frac{\partial}{\partial x}(U_{11})U_{22} - \left(V_{11}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(V_{11})V_{22}\right) + \\ & U_{21}\frac{\partial}{\partial x}(U_{12}) + \frac{\partial}{\partial x}(U_{21})U_{12} + V_{21}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(V_{21})V_{12} \end{aligned}\right] +$$

$$k\left[\begin{aligned} & U_{11}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(U_{11})V_{22} + U_{22}\frac{\partial}{\partial x}(V_{11}) + \frac{\partial}{\partial x}(U_{22})V_{11} - \\ & \left(U_{21}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{21})V_{12}\right) + U_{12}\frac{\partial}{\partial x}(V_{21}) + \frac{\partial}{\partial x}(U_{12})V_{21} \end{aligned}\right]$$

$$= U_{11}\frac{\partial}{\partial x}(U_{12}) + \frac{\partial}{\partial x}(U_{11})U_{12} - V_{11}\frac{\partial}{\partial x}(V_{12}) - \frac{\partial}{\partial x}(V_{11})V_{12} -$$

$$U_{21}\frac{\partial}{\partial x}(U_{22}) - \frac{\partial}{\partial x}(U_{21})U_{22} - V_{21}\frac{\partial}{\partial x}(V_{22}) - \frac{\partial}{\partial x}(V_{21})V_{22} +$$

$$i\left[\begin{array}{l} U_{11}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{11})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{11}) + \frac{\partial}{\partial x}(U_{12})V_{11} + \\ U_{21}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(U_{21})V_{22} - U_{22}\frac{\partial}{\partial x}(V_{21}) - \frac{\partial}{\partial x}(U_{22})V_{21} \end{array}\right] +$$

$$j\left[\begin{array}{l} U_{11}\frac{\partial}{\partial x}(U_{22}) + \frac{\partial}{\partial x}(U_{11})U_{22} - V_{11}\frac{\partial}{\partial x}(V_{22}) - \frac{\partial}{\partial x}(V_{11})V_{22} + \\ U_{21}\frac{\partial}{\partial x}(U_{12}) + \frac{\partial}{\partial x}(U_{21})U_{12} + V_{21}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(V_{21})V_{12} \end{array}\right] +$$

$$k\left[\begin{array}{l} U_{11}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(U_{11})V_{22} + U_{22}\frac{\partial}{\partial x}(V_{11}) + \frac{\partial}{\partial x}(U_{22})V_{11} - \\ U_{21}\frac{\partial}{\partial x}(V_{12}) - \frac{\partial}{\partial x}(U_{21})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{21}) + \frac{\partial}{\partial x}(U_{12})V_{21} \end{array}\right]$$

$$= U_{11}\frac{\partial}{\partial x}(U_{12}) - V_{11}\frac{\partial}{\partial x}(V_{12}) - U_{21}\frac{\partial}{\partial x}(U_{22}) - V_{21}\frac{\partial}{\partial x}(V_{22}) +$$

$$\frac{\partial}{\partial x}(U_{11})U_{12} - \frac{\partial}{\partial x}(V_{11})V_{12} - \frac{\partial}{\partial x}(U_{21})U_{22} - \frac{\partial}{\partial x}(V_{21})V_{22} +$$

$$i(U_{11}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{12})V_{11} + U_{21}\frac{\partial}{\partial x}(V_{22}) - \frac{\partial}{\partial x}(U_{22})V_{21}) +$$

$$i(\frac{\partial}{\partial x}(U_{11})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{11}) + \frac{\partial}{\partial x}(U_{21})V_{22} - U_{22}\frac{\partial}{\partial x}(V_{21})) +$$

$$j(U_{11}\frac{\partial}{\partial x}(U_{22}) - V_{11}\frac{\partial}{\partial x}(V_{22}) + U_{21}\frac{\partial}{\partial x}(U_{12}) + V_{21}\frac{\partial}{\partial x}(V_{12})) +$$

$$j(\frac{\partial}{\partial x}(U_{11})U_{22} - \frac{\partial}{\partial x}(V_{11})V_{22} + \frac{\partial}{\partial x}(U_{21})U_{12} + \frac{\partial}{\partial x}(V_{21})V_{12}) +$$

$$k(U_{11}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(U_{22})V_{11} - U_{21}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{12})V_{21}) +$$

$$k(\frac{\partial}{\partial x}(U_{11})V_{22} + U_{22}\frac{\partial}{\partial x}(V_{11}) - \frac{\partial}{\partial x}(U_{21})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{21}))$$

$$= \begin{bmatrix} U_{11}\frac{\partial}{\partial x}(U_{12}) - V_{11}\frac{\partial}{\partial x}(V_{12}) - U_{21}\frac{\partial}{\partial x}(U_{22}) - V_{21}\frac{\partial}{\partial x}(V_{22}) + \\ i(U_{11}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{12})V_{11} + U_{21}\frac{\partial}{\partial x}(V_{22}) - \frac{\partial}{\partial x}(U_{22})V_{21}) + \\ j(U_{11}\frac{\partial}{\partial x}(U_{22}) - V_{11}\frac{\partial}{\partial x}(V_{22}) + U_{21}\frac{\partial}{\partial x}(U_{12}) + V_{21}\frac{\partial}{\partial x}(V_{12})) + \\ k(U_{11}\frac{\partial}{\partial x}(V_{22}) + \frac{\partial}{\partial x}(U_{22})V_{11} - U_{21}\frac{\partial}{\partial x}(V_{12}) + \frac{\partial}{\partial x}(U_{12})V_{21}) \end{bmatrix} +$$

$$\begin{bmatrix} \frac{\partial}{\partial x}(U_{11})U_{12} - \frac{\partial}{\partial x}(V_{11})V_{12} - \frac{\partial}{\partial x}(U_{21})U_{22} - \frac{\partial}{\partial x}(V_{21})V_{22} + \\ i(\frac{\partial}{\partial x}(U_{11})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{11}) + \frac{\partial}{\partial x}(U_{21})V_{22} - U_{22}\frac{\partial}{\partial x}(V_{21})) + \\ j(\frac{\partial}{\partial x}(U_{11})U_{22} - \frac{\partial}{\partial x}(V_{11})V_{22} + \frac{\partial}{\partial x}(U_{21})U_{12} + \frac{\partial}{\partial x}(V_{21})V_{12}) + \\ k(\frac{\partial}{\partial x}(U_{11})V_{22} + U_{22}\frac{\partial}{\partial x}(V_{11}) - \frac{\partial}{\partial x}(U_{21})V_{12} + U_{12}\frac{\partial}{\partial x}(V_{21})) \end{bmatrix}$$

$$= (U_{11} + iV_{11} + jU_{21} + kV_{21})(\frac{\partial}{\partial x}(U_{12}) + i\frac{\partial}{\partial x}(V_{12}) + j\frac{\partial}{\partial x}(U_{22}) + k\frac{\partial}{\partial x}(V_{22})) +$$

$$(\frac{\partial}{\partial x}(U_{11}) + i\frac{\partial}{\partial x}(V_{11}) + j\frac{\partial}{\partial x}(U_{21}) + k\frac{\partial}{\partial x}(V_{21}))(U_{12} + iV_{12} + jU_{22} + kV_{22})$$

$$= \phi_1(q)\frac{\partial}{\partial x}(\phi_2(q)) + \frac{\partial}{\partial x}(\phi_1(q))\phi_2(q) \qquad \underline{(1)},$$

by virtue of Theorem TII−2 and the established properties of quaternion products. Similarly, we perceive that the derivations of the differential formulae,

$$\frac{\partial}{\partial y}(\phi_1(q)\phi_2(q)) = \phi_1(q)\frac{\partial}{\partial y}(\phi_2(q)) + \frac{\partial}{\partial y}(\phi_1(q))\phi_2(q) \qquad \underline{(2)},$$

$$\frac{\partial}{\partial \bar{x}}(\phi_1(q)\phi_2(q)) = \phi_1(q)\frac{\partial}{\partial \bar{x}}(\phi_2(q)) + \frac{\partial}{\partial \bar{x}}(\phi_1(q))\phi_2(q) \qquad \underline{(3)},$$

$$\frac{\partial}{\partial \bar{y}}(\phi_1(q)\phi_2(q)) = \phi_1(q)\frac{\partial}{\partial \bar{y}}(\phi_2(q)) + \frac{\partial}{\partial \bar{y}}(\phi_1(q))\phi_2(q) \qquad \underline{(4)},$$

are completely analogous to that of Eq.(1), insofar as we replace the partial differential operator, $\frac{\partial}{\partial x}$, by the operators, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \hat{x}}$ and $\frac{\partial}{\partial \hat{y}}$, respectively. The amalgamation of Eqs.(1)–(4) thus yields the formula,

$$\frac{\partial}{\partial s}(\phi_1(q)\phi_2(q)) = \phi_1(q)\frac{\partial}{\partial s}(\phi_2(q)) + \frac{\partial}{\partial s}(\phi_1(q))\phi_2(q), \ \forall s \in \{x, y, \hat{x}, \hat{y}\},$$

as required. Q.E.D.

(c) The proof of this part of the theorem is completely analogous with the preceding part (b), insofar as the positions of the component functions, $\phi_1(q)$ and $\phi_2(q)$, have merely been interchanged with respect to the product functions, $\phi_1(q)\phi_2(q)$ and $\phi_2(q)\phi_1(q)$. Q.E.D.

(d) Since the quotient function,

$$\phi_1(q)/\phi_2(q) = \begin{cases} \phi_1(q)\overline{\phi_2(q)}/|\phi_2(q)|^2 & (\phi_2(q) \neq 0), \\ \overline{\phi_2(q)}\phi_1(q)/|\phi_2(q)|^2 & \end{cases}$$

we immediately recall from Definition DII-6 that the partial derivative of $\phi_1(q)/\phi_2(q)$ with respect to $s \in \{x, y, \hat{x}, \hat{y}\}$,

$$\frac{\partial}{\partial s}(\phi_1(q)/\phi_2(q)) = \frac{\partial}{\partial s}\begin{cases} \phi_1(q)\overline{\phi_2(q)}/|\phi_2(q)|^2 \\ \overline{\phi_2(q)}\phi_1(q)/|\phi_2(q)|^2 \end{cases}$$

$$= \begin{cases} \frac{\partial}{\partial s}(\phi_1(q)\overline{\phi_2(q)}/|\phi_2(q)|^2) \\ \frac{\partial}{\partial s}(\overline{\phi_2(q)}\phi_1(q)/|\phi_2(q)|^2) \end{cases}$$

$$= \begin{cases} \frac{\partial}{\partial s}(\phi_1(q)(\overline{\phi_2(q)}/|\phi_2(q)|^2)) \\ \frac{\partial}{\partial s}((\overline{\phi_2(q)}/|\phi_2(q)|^2)\phi_1(q)) \end{cases}$$

$$= \begin{cases} \phi_1(q)\frac{\partial}{\partial s}[\overline{\phi_2(q)}/|\phi_2(q)|^2] + \frac{\partial}{\partial s}(\phi_1(q))[\overline{\phi_2(q)}/|\phi_2(q)|^2] \ , \\ [\overline{\phi_2(q)}/|\phi_2(q)|^2]\frac{\partial}{\partial s}(\phi_1(q)) + \frac{\partial}{\partial s}[\overline{\phi_2(q)}/|\phi_2(q)|^2]\phi_1(q) \end{cases}$$

by virtue of the preceding part (b) of this theorem.

Finally, in view of these same considerations, we likewise deduce that the partial derivative,

$$\frac{\partial}{\partial s}[\overline{\phi_2(q)}/|\phi_2(q)|^2] = \overline{\phi_2(q)}\frac{\partial}{\partial s}[1/|\phi_2(q)|^2] + \frac{\partial}{\partial s}(\overline{\phi_2(q)})[1/|\phi_2(q)|^2]$$

$$= \overline{\phi_2(q)}\left(\frac{-\frac{\partial}{\partial s}(|\phi_2(q)|^2)}{|\phi_2(q)|^4}\right) + \frac{\frac{\partial}{\partial s}(\overline{\phi_2(q)})}{|\phi_2(q)|^2}$$

$$= \frac{|\phi_2(q)|^2\frac{\partial}{\partial s}(\overline{\phi_2(q)}) - \frac{\partial}{\partial s}(|\phi_2(q)|^2)\overline{\phi_2(q)}}{|\phi_2(q)|^4} \qquad (\phi_2(q) \neq 0),$$

$\forall s \in \{x, y, \hat{x}, \hat{y}\}$, as required. <u>Q.E.D.</u>

<u>Theorem TII-5.</u>

Let there exist a <u>monomial</u> quaternion hypercomplex function,

$f(q) = q^n$, $\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots, \pm\infty\} = \mathbb{Z}$,

such that its domain,

$\mathrm{dom}(f) \subset \mathbb{H}$.

In the circumstances, it may be proven that the <u>partial derivative of this function with respect to x</u>,

$\frac{\partial}{\partial x}(f(q)) = \frac{\partial}{\partial x}(q^n) = nq^{n-1}$,

likewise exists, $\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm\infty\} = \mathbb{Z}$.

* * *

<u>PROOF</u>:-

For the purposes of proving this theorem, we will invoke the principle of mathematical induction and thus demonstrate that the partial differential formula,

$\frac{\partial}{\partial x}(q^n) = nq^{n-1}$,

is valid for all integer values of n in general.

<u>(a)</u> Consider the case, $n = -1$. Hence, from Theorem TII-4, we deduce that

$\frac{\partial}{\partial x}(q^{-1}) = \frac{\partial}{\partial x}(\bar{q}/|q|^2)$

$$= \frac{|q|^2 \frac{\partial}{\partial x}(\bar{q}) - \frac{\partial}{\partial x}(|q|^2)\bar{q}}{|q|^4}$$

$$= \frac{|q|^2 \frac{\partial}{\partial x}(x - iy - j\hat{x} - k\hat{y}) - \frac{\partial}{\partial x}(x^2 + y^2 + \hat{x}^2 + \hat{y}^2)\bar{q}}{|q|^4}$$

$$= \frac{|q|^2 - 2x \cdot \bar{q}}{|q|^4}$$

$$= \frac{|q|^2 - (\bar{q} + q)\bar{q}}{|q|^4}$$

$$= \frac{|q|^2 - \bar{q}^2 - q\bar{q}}{|q|^4}$$

$$= \frac{|q|^2 - \bar{q}^2 - |q|^2}{|q|^4}$$

$$= -\bar{q}^2/|q|^4 = -(\bar{q}/|q|^2)^2 = -(q^{-1})^2 = -q^{-2}.$$

(b) Consider the case, $n = 0$. Clearly, it is evident that

$$\frac{\partial}{\partial x}(q^0) = \frac{\partial}{\partial x}(1) = 0.$$

(c) Consider the case, $n = 1$. Clearly, it is evident that

$$\frac{\partial}{\partial x}(q) = \frac{\partial}{\partial x}(x + iy + j\hat{x} + k\hat{y})$$

$$= \frac{\partial}{\partial x}(x) + i\frac{\partial}{\partial x}(y) + j\frac{\partial}{\partial x}(z) + k\frac{\partial}{\partial x}(\hat{y})$$

$$= 1.$$

(d) Consider the case, $n = 2$. Hence, from Theorem TII–4, we deduce that

$$\frac{\partial}{\partial x}(q^2) = q\frac{\partial}{\partial x}(q) + \frac{\partial}{\partial x}(q)\,q$$

$$= q \cdot 1 + 1 \cdot q$$

$$= q + q = 2q.$$

(e) Consider the case, $n = 3$. Hence, from Theorem TII–4, we deduce that

$$\frac{\partial}{\partial x}(q^3) = \frac{\partial}{\partial x}(q \cdot q^2)$$

$$= q\frac{\partial}{\partial x}(q^2) + \frac{\partial}{\partial x}(q)\,q^2$$

$$= q(2q) + 1 \cdot q^2$$

$$= 2q^2 + q^2 = 3q^2.$$

From the specific cases (a) – (e) outlined above, it would appear that the partial derivative with respect to x of the monomial function,

$$f(q) = q^n,$$

is given by the general formula,

$$\frac{\partial}{\partial x}(f(q)) = \frac{\partial}{\partial x}(q^n) = nq^{n-1}, \ \forall n \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots, \pm \infty\} = \mathbb{Z}.$$

Subsequently, in order to prove that this assertion is valid, we likewise deduce that the partial derivative,

$$\frac{\partial}{\partial x}(q^{n+1}) = \frac{\partial}{\partial x}(q \cdot q^n)$$

$$= q\frac{\partial}{\partial x}(q^n) + \frac{\partial}{\partial x}(q)\, q^n$$

$$= q \cdot nq^{n-1} + 1 \cdot q^n$$

$$= nq^n + q^n$$

$$= (n+1)\, q^n, \ \forall n \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots, \pm \infty\} = \mathbb{Z},$$

as anticipated, and hence the validity of our general formula for such derivatives has now been established. Q.E.D.

Theorem TII-6.

Let the exponential quaternion hypercomplex function, $\exp(q)$, be defined on a domain,

$$\mathrm{dom}(\exp) \subset \mathbb{H}.$$

Subsequently, it may be shown that the partial derivative of $\exp(q)$ with respect to x is given by the formula,

$$\frac{\partial}{\partial x}(\exp(q)) = \exp(q), \quad \forall q \in dom(\exp) \subset \mathbb{H}.$$

* * *

PROOF:-

From Theorem TI-4, we recall that the exponential function, $\exp(q)$, is algebraically expressed as

$$\exp(q) = e^x \cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] e^x \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})$$

and hence, from the previously established theorems on partial differentiation, we perceive that the partial derivative,

$$\frac{\partial}{\partial x}(\exp(q)) = \frac{\partial}{\partial x}(e^x \cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2})) + i\frac{\partial}{\partial x}\left(\frac{y e^x \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) +$$

$$j\frac{\partial}{\partial x}\left(\frac{\hat{x} e^x \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) + k\frac{\partial}{\partial x}\left(\frac{\hat{y} e^x \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= \frac{\partial}{\partial x}(e^x)\cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) +$$

$$\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{\partial}{\partial x}(e^x)\sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})$$

$$= e^x \cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] e^x \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})$$

$= \exp(q)$, $\forall q \in dom(\exp) \subset \mathbb{H}$, as required. Q.E.D.

Theorem TII−7.

Let the trigonometric quaternion hypercomplex functions, $\sin(q)$ and $\cos(q)$, be defined on the domains,

$dom(\sin), dom(\cos) \subset \mathbb{H}$.

Subsequently, it may be shown that the partial derivatives of $\sin(q)$ and $\cos(q)$ with respect to x are respectively given by the formulae,

(a) $\frac{\partial}{\partial x}(\sin(q)) = \cos(q)$, $\forall q \in dom(\sin) \subset \mathbb{H}$,

(b) $\frac{\partial}{\partial x}(\cos(q)) = -\sin(q)$, $\forall q \in dom(\cos) \subset \mathbb{H}$.

*　　　*　　　*

PROOF:-

From Theorem TI−11, we recall that the trigonometric functions, $\sin(q)$ and $\cos(q)$, are algebraically expressed as

$$\sin(q) = \sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right),$$

$$\cos(q) = \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sin(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right).$$

(a) From the previously established theorems on partial differentiation, we perceive that the partial derivative,

$$\frac{\partial}{\partial x}(\sin(q)) = \frac{\partial}{\partial x}\left(\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right) + i\frac{\partial}{\partial x}\left(\frac{y\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) +$$

$$j\frac{\partial}{\partial x}\left(\frac{\hat{x}\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) + k\frac{\partial}{\partial x}\left(\frac{\hat{y}\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= \frac{\partial}{\partial x}(\sin(x))\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{\partial}{\partial x}(\cos(x))\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right](-\sin(x))\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) -$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sin(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$= \cos(q)$, $\forall q \in dom(\sin) \subset \mathbb{H}$, as required. Q.E.D.

(b) In a completely analogous manner to part (a) of this theorem, we likewise deduce that the partial derivative,

$$\frac{\partial}{\partial x}(\cos(q)) = \frac{\partial}{\partial x}(\cos(x)\cosh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})) - i\frac{\partial}{\partial x}\left(\frac{y\sin(x)\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) -$$

$$j\frac{\partial}{\partial x}\left(\frac{\hat{x}\sin(x)\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) - k\frac{\partial}{\partial x}\left(\frac{\hat{y}\sin(x)\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= \frac{\partial}{\partial x}(\cos(x))\cosh(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) -$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{\partial}{\partial x}(\sin(x))\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})$$

$$= -\sin(x)\cosh(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) -$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\cos(x)\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})$$

$$= -\left[\sin(x)\cosh(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\cos(x)\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})\right]$$

$= -\sin(q)$, $\forall q \in dom(\cos) \subset \mathbb{H}$, as required. <u>Q.E.D.</u>

<u>Theorem TII-8</u>.

Let the <u>hyperbolic</u> quaternion hypercomplex functions, $\sinh(q)$ and $\cosh(q)$, be defined on the domains, dom(sinh), dom(cosh) $\subset \mathbb{H}$.

Subsequently, it may be shown that the partial derivatives of sinh(q) and cosh(q) with respect to x are respectively given by the formulae,

(a) $\frac{\partial}{\partial x}(\sinh(q)) = \cosh(q), \ \forall q \in dom(\sinh) \subset \mathbb{H}$,

(b) $\frac{\partial}{\partial x}(\cosh(q)) = \sinh(q), \ \forall q \in dom(\cosh) \subset \mathbb{H}$.

* * *

PROOF:-

From Theorem TI-17, we recall that the hyperbolic functions, sinh(q) and cosh(q), are algebraically expressed as

$$\sinh(q) = \sinh(x)\cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\cosh(x)\sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2}),$$

$$\cosh(q) = \cosh(x)\cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sinh(x)\sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2}).$$

(a) From the previously established theorems on partial differentiation, we perceive that the partial derivative,

$$\frac{\partial}{\partial x}(\sinh(q)) = \frac{\partial}{\partial x}(\sinh(x)\cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2})) + i\frac{\partial}{\partial x}\left(\frac{y\cosh(x)\sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) +$$

$$j\frac{\partial}{\partial x}\left(\frac{\hat{x}\cosh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) + k\frac{\partial}{\partial x}\left(\frac{\hat{y}\cosh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= \frac{\partial}{\partial x}(\sinh(x))\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{\partial}{\partial x}(\cosh(x))\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \cosh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sinh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$= \cosh(q)$, $\forall q \in \mathrm{dom}(\sinh) \subset \mathbb{H}$, as required. <u>Q.E.D.</u>

<u>(b)</u> In a completely analogous manner to part <u>(a)</u> of this theorem, we likewise deduce that the partial derivative,

$$\frac{\partial}{\partial x}(\cosh(q)) = \frac{\partial}{\partial x}\left(\cosh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right) + i\frac{\partial}{\partial x}\left(\frac{y\sinh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) +$$

$$j\frac{\partial}{\partial x}\left(\frac{\hat{x}\sinh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) + k\frac{\partial}{\partial x}\left(\frac{\hat{y}\sinh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)$$

$$= \frac{\partial}{\partial x}(\cosh(x))\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{\partial}{\partial x}(\sinh(x))\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \sinh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\cosh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$= \sinh(q)$, $\forall q \in dom(\cosh) \subset \mathbb{H}$, as required. $\underline{\underline{Q.E.D.}}$

Theorem TII-9.

Let the logarithmic quaternion hyperbolic function, $\log(q)$, be defined on the domain,

$dom(\log) \subset \mathbb{H} - \{0\}$.

Subsequently, it may be shown that the partial derivative of $\log(q)$, with respect to x, is given by the formula,

$\frac{\partial}{\partial x}(\log(q)) = q^{-1}$, $\forall q \in dom(\log) \subset \mathbb{H} - \{0\}$.

*   *   *

PROOF:-

From Theorem TI-20 and Definition DI-15, we recall that the logarithmic function, $\log(q)$, is algebraically expressed as

$$\log(q) = \log\left(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(2n\pi + \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right]\right),$$

$\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots, \pm\infty\}$, the set of integers, such that the real variable function,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] \in [0, \pi].$$

Hence, from the previously established theorems on partial differentiation, it follows that the partial derivative,

$$\frac{\partial}{\partial x}(\log(q)) = \frac{\partial}{\partial x}\left(\log\left(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\right)\right) +$$

$$i\frac{\partial}{\partial x}\left[\frac{y}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\left(2n\pi + \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right]\right)\right] +$$

$$j\frac{\partial}{\partial x}\left[\frac{\hat{x}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\left(2n\pi + \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right]\right)\right] +$$

$$k\frac{\partial}{\partial x}\left[\frac{\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\left(2n\pi + \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right]\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\log\left(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\right)\right) +$$

$$\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{\partial}{\partial x}\left(2n\pi + \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right]\right)$$

$$= \frac{\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}} \quad +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{-1}{\sqrt{1-(x^2/(x^2+y^2+\hat{x}^2+\hat{y}^2))}}\right)\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= \frac{\frac{1}{2}(x^2+y^2+\hat{x}^2+\hat{y}^2)^{-\frac{1}{2}}\frac{\partial}{\partial x}(x^2+y^2+\hat{x}^2+\hat{y}^2)}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}} \quad +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{-\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\left(\frac{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2})}{x^2+y^2+\hat{x}^2+\hat{y}^2}\right)$$

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$$= \frac{\frac{1}{2}(2x)}{x^2+y^2+\hat{x}^2+\hat{y}^2} \quad -$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\left(\frac{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}-x(\frac{1}{2}(x^2+y^2+\hat{x}^2+\hat{y}^2)^{-1/2}\cdot 2x)}{x^2+y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \frac{x}{x^2+y^2+\hat{x}^2+\hat{y}^2} - \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{x^2+y^2+\hat{x}^2+\hat{y}^2-x^2}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}(x^2+y^2+\hat{x}^2+\hat{y}^2)}\right)$$

$$= \frac{x}{x^2+y^2+\hat{x}^2+\hat{y}^2} - \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{y^2+\hat{x}^2+\hat{y}^2}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}(x^2+y^2+\hat{x}^2+\hat{y}^2)}\right)$$

$$= \frac{x}{x^2+y^2+\hat{x}^2+\hat{y}^2} - \frac{(iy+j\hat{x}+k\hat{y})(y^2+\hat{x}^2+\hat{y}^2)}{(y^2+\hat{x}^2+\hat{y}^2)(x^2+y^2+\hat{x}^2+\hat{y}^2)}$$

$$= \frac{x}{x^2+y^2+\hat{x}^2+\hat{y}^2} - \frac{(iy+j\hat{x}+k\hat{y})}{x^2+y^2+\hat{x}^2+\hat{y}^2}$$

$$= \frac{x - iy - j\hat{x} - k\hat{y}}{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}$$

$$= \bar{q}/|q|^2 = q^{-1}, \ \forall q \in dom(log) \subset \mathbb{H} - \{0\}, \text{ as required. } \underline{\underline{Q.E.D.}}$$

<u>Definition DII-7.</u>

Let the quaternion hypercomplex function,

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})$$

$$= \mathcal{U}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] \mathcal{V}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2}),$$

be defined on a domain, $dom(f) \subset \mathbb{H}$, such that its corresponding real and imaginary parts, namely –

$$u_1(x,y,\hat{x},\hat{y}) = \mathcal{U}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2}),$$

$$v_1(x,y,\hat{x},\hat{y}) = \frac{y\,\mathcal{V}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}},$$

$$u_2(x,y,\hat{x},\hat{y}) = \frac{\hat{x}\,\mathcal{V}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}},$$

$$v_2(x,y,\hat{x},\hat{y}) = \frac{\hat{y}\,\mathcal{V}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}.$$

Functions of this type are said to be *quasi-analytic*, if and only if the partial differential equations,

$$\frac{\partial}{\partial x}\left(\mathcal{U}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})\right) = \frac{\partial}{\partial(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}\left(\mathcal{V}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})\right) ,$$

$$\frac{\partial}{\partial(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}\left(\mathcal{U}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})\right) = -\frac{\partial}{\partial x}\left(\mathcal{V}(x, \sqrt{y^2+\hat{x}^2+\hat{y}^2})\right) ,$$

are likewise satisfied, $\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(f) \subset \mathbb{H}$. We shall accordingly refer to these equations as the *quaternion analogues of the Cauchy-Riemann equations* from complex variable analysis.

Definition DII-8.

Let there exist a quaternion number,

$$q = x + iy + j\hat{x} + k\hat{y}$$

$$= x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi ,$$

such that its corresponding imaginary parts, namely –

$$y = \lambda_1 \xi / \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} ,$$

$$\hat{x} = \lambda_2 \xi / \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} ,$$

$$\hat{y} = \lambda_3 \xi / \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} , \quad \forall \lambda_1, \lambda_2, \lambda_3, \xi \in \mathbb{R} .$$

The quaternion hypercomplex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right),$$

is said to be <u>quasi-complex</u>, if and only if it can be written in the form –

$$f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]V(x,\xi),$$

where $U(x,\xi)$ and $V(x,\xi)$ are real variable functions of x and ξ.

<u>Theorem TII–10.</u>

Let the quaternion hypercomplex function,

$$f(q) = \mathcal{U}(x, \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\mathcal{V}(x, \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}),$$

be defined on a domain, $\mathrm{dom}(f) \subset \mathbb{H}$. Subsequently, it may be proven that, if this function is <u>quasi-analytic</u>, then the corresponding function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right)$$

$$= \mathcal{U}(x, |\xi|) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\frac{\xi}{|\xi|}\mathcal{V}(x, |\xi|),$$

will likewise generate the set of partial differential equations,

$$\frac{\partial}{\partial x}\big(\mathcal{U}(x,|\xi|)\big) = \frac{\partial}{\partial|\xi|}\big(\mathcal{V}(x,|\xi|)\big),$$

$$\frac{\partial}{\partial|\xi|}\big(\mathcal{U}(x,|\xi|)\big) = -\frac{\partial}{\partial x}\big(\mathcal{V}(x,|\xi|)\big).$$

*   *   *

PROOF:-

From Definition DII-7, we recall that the quasi-analytic function,

$$f(q) = \mathcal{U}(x,\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\mathcal{V}(x,\sqrt{y^2+\hat{x}^2+\hat{y}^2}) \qquad (i),$$

yields the concomitant set of partial differential equations, namely -

$$\frac{\partial}{\partial x}\big(\mathcal{U}(x,\sqrt{y^2+\hat{x}^2+\hat{y}^2})\big) = \frac{\partial}{\partial(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}\big(\mathcal{V}(x,\sqrt{y^2+\hat{x}^2+\hat{y}^2})\big) \qquad (ii),$$

$$\frac{\partial}{\partial(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}\big(\mathcal{U}(x,\sqrt{y^2+\hat{x}^2+\hat{y}^2})\big) = -\frac{\partial}{\partial x}\big(\mathcal{V}(x,\sqrt{y^2+\hat{x}^2+\hat{y}^2})\big) \qquad (iii),$$

$\forall q = x+iy+j\hat{x}+k\hat{y} \in dom(f) \subset \mathbb{H}$.

Furthermore, by setting the variables,

$$\left.\begin{aligned} y &= \lambda_1\xi/\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \\ \hat{x} &= \lambda_2\xi/\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \\ \hat{y} &= \lambda_3\xi/\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \end{aligned}\right\} \Longrightarrow \begin{aligned} &\sqrt{\xi^2} = |\xi| = \sqrt{y^2+\hat{x}^2+\hat{y}^2}, \\ &\forall \lambda_1,\lambda_2,\lambda_3,\xi \in \mathbb{R}, \end{aligned}$$

it therefore follows, after making the appropriate algebraic substitutions, that Eq. (i) reduces to the form,

$$f(q) = f(x+iy+j\hat{x}+k\hat{y}) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi\right)$$

$$= \mathcal{U}(x,|\xi|) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\xi}{|\xi|}\mathcal{V}(x,|\xi|) \qquad \text{(iv)},$$

and similarly Eqs. (ii) and (iii) are reduced to

$$\frac{\partial}{\partial x}(\mathcal{U}(x,|\xi|)) = \frac{\partial}{\partial|\xi|}(\mathcal{V}(x,|\xi|)) \qquad \text{(v)},$$

$$\frac{\partial}{\partial|\xi|}(\mathcal{U}(x,|\xi|)) = -\frac{\partial}{\partial x}(\mathcal{V}(x,|\xi|)) \qquad \text{(vi)}.$$

In summary, we observe that Eq. (i) reduces to Eq. (iv), which simultaneously generates Eqs. (v) and (vi), as required. Q.E.D.

Theorem TII–11.

Let there exist a single-valued *quasi-complex* function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi\right) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]V(x,\xi),$$

which is restricted to a smooth arc, C, thus denoted by the equation –

$$q(t) = x(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi(t), \quad \forall t \in [a, b].$$

In the circumstances, it may be shown that the parametric first derivative of $f(q)$, with respect to q, having been restricted to a smooth arc, C, is a single-valued function, namely—

$$\left[\frac{d}{dq}\right]_C (f(q)) = \frac{\frac{d}{dt}(U(x(t), \xi(t))) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\frac{d}{dt}(V(x(t), \xi(t)))}{\frac{d}{dt}(x(t)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\frac{d}{dt}(\xi(t))},$$

provided that the functions, $f(q)$ and q, are likewise differentiable in 't', $\forall t \in (a, b)$.

* * *

PROOF :-

From the previously established theorems on the analytical properties of quaternion hypercomplex functions, we recall that the first derivative of a function, $f(q)$, with respect to q, thus restricted to an arc, C, embedded in q-space, is accordingly denoted by the formula,

$$\left[\frac{d}{dq}\right]_C (f(q)) = \begin{cases} \dfrac{\frac{d}{dt}[f(q(t))]\,\overline{\frac{d}{dt}[q(t)]}}{\left|\frac{d}{dt}[q(t)]\right|^2}, \\[2ex] \dfrac{\overline{\frac{d}{dt}[q(t)]}\,\frac{d}{dt}[f(q(t))]}{\left|\frac{d}{dt}[q(t)]\right|^2} \end{cases}$$

provided that the functions, $f(q)$ and q, are differentiable in 't', $\forall t \in (a,b)$. Furthermore, by restricting the quasi-complex functions,

$$q = x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi ,$$

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]V(x,\xi),$$

to such an arc, $\mathcal{C}$, we likewise deduce that

$$q = q(t) = x(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi(t) \Longrightarrow$$

$$\overline{q(t)} = x(t) - \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi(t) ,$$

$$f(q) = f(q(t)) = U(x(t),\xi(t)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]V(x(t),\xi(t)), \ \forall t \in [a,b],$$

and hence the above differential formula for $\left[\frac{d}{dq}\right]_c(f(q))$ now reduces to

$$\left[\frac{d}{dq}\right]_c(f(q)) = \begin{cases} \dfrac{\left(\begin{array}{l}\frac{d}{dt}[U(x(t),\xi(t)) + Q^* V(x(t),\xi(t))] \times \\ \frac{d}{dt}[x(t) - Q^*\xi(t)]\end{array}\right)}{\left|\frac{d}{dt}[x(t) + Q^*\xi(t)]\right|^2} \\ \\ \dfrac{\left(\begin{array}{l}\frac{d}{dt}[x(t) - Q^*\xi(t)] \times \\ \frac{d}{dt}[U(x(t),\xi(t)) + Q^* V(x(t),\xi(t))]\end{array}\right)}{\left|\frac{d}{dt}[x(t) + Q^*\xi(t)]\right|^2} \end{cases}$$

$$= \left\{ \begin{array}{l} \dfrac{\left(\begin{array}{l} \left[\frac{d}{dt}(U(x(t),\xi(t))) + Q^* \frac{d}{dt}(V(x(t),\xi(t)))\right] \times \\ \left[\frac{d}{dt}(x(t)) - Q^* \frac{d}{dt}(\xi(t))\right] \end{array} \right)}{\left|\frac{d}{dt}(x(t)) + Q^* \frac{d}{dt}(\xi(t))\right|^2}, \\ \\ \dfrac{\left(\begin{array}{l} \left[\frac{d}{dt}(x(t)) - Q^* \frac{d}{dt}(\xi(t))\right] \times \\ \left[\frac{d}{dt}(U(x(t),\xi(t))) + Q^* \frac{d}{dt}(V(x(t),\xi(t)))\right] \end{array} \right)}{\left|\frac{d}{dt}(x(t)) + Q^* \frac{d}{dt}(\xi(t))\right|^2} \end{array} \right.$$

where the qustenion constant,

$$Q^* = \frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, \quad \forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

However, in view of the criteria specified in Theorem TI-10, it is also evident that the quasi-complex parametric derivative,

$$\left[\frac{d}{dq}\right]_c (f(q)) = \frac{\left(\begin{array}{l} \left[\frac{d}{dt}(U(x(t),\xi(t))) + Q^* \frac{d}{dt}(V(x(t),\xi(t)))\right] \times \\ \left[\frac{d}{dt}(x(t)) - Q^* \frac{d}{dt}(\xi(t))\right] \end{array} \right)}{\left|\frac{d}{dt}(x(t)) + Q^* \frac{d}{dt}(\xi(t))\right|^2}$$

$$= \frac{\left(\begin{array}{l} \left[\frac{d}{dt}(x(t)) - Q^* \frac{d}{dt}(\xi(t))\right] \times \\ \left[\frac{d}{dt}(U(x(t),\xi(t))) + Q^* \frac{d}{dt}(V(x(t),\xi(t)))\right] \end{array} \right)}{\left|\frac{d}{dt}(x(t)) + Q^* \frac{d}{dt}(\xi(t))\right|^2}$$

$$= \frac{\frac{d}{dt}(U(x(t),\xi(t))) + Q^* \frac{d}{dt}(V(x(t),\xi(t)))}{\frac{d}{dt}(x(t)) + Q^* \frac{d}{dt}(\xi(t))}$$

$$= \frac{\frac{d}{dt}(U(x(t), \xi(t))) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\frac{d}{dt}(V(x(t), \xi(t)))}{\frac{d}{dt}(x(t)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\frac{d}{dt}(\xi(t))},$$

$\forall t \in (a,b)$, as required. Q.E.D.

Definition DII-9.

Let there exist a quasi-complex quaternion hypercomplex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]V(x,\xi),$$

$\forall \lambda_1, \lambda_2, \lambda_3, U(x,\xi), V(x,\xi) \in \mathbb{R}$.

Subsequently, we define the first derivative of 'f', with respect to $q = x + [(i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}]\xi$, by the formula –

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q) - f(q)}{\delta q}\right],$$

provided that such a limit exists.

Theorem TII-12.

Let there exist a quasi-complex quaternion hypercomplex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]V(x,\xi),$$

$\forall \lambda_1, \lambda_2, \lambda_3, U(x,\xi), V(x,\xi) \in \mathbb{R}$.

In the circumstances, it may be proven that the first derivative of 'f', with respect to $q = x + [(i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}]\xi \in dom(f)$,

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q) - f(q)}{\delta q}\right],$$

likewise generates the set of partial differential equations,

$$\frac{\partial}{\partial x}(U(x,\xi)) = \frac{\partial}{\partial \xi}(V(x,\xi)),$$

$$\frac{\partial}{\partial \xi}(U(x,\xi)) = -\frac{\partial}{\partial x}(V(x,\xi)).$$

We shall refer to these equations as the <u>quaternion analogues of the Cauchy-Riemann equations</u> from complex variable analysis.

* * *

<u>PROOF</u>:-

Let there exist a quaternion number,

$$q = x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi \implies \delta q = \delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\delta\xi,$$

such that we obtain the corresponding function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi)$$

$$= U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]V(x,\xi) \Longrightarrow$$

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$$f(q+\delta q) = f(x + \delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right](\xi + \delta\xi))$$

$$= U(x+\delta x, \xi+\delta\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]V(x+\delta x, \xi+\delta\xi),$$

$\forall \lambda_1, \lambda_2, \lambda_3, U(x,\xi), V(x,\xi) \in \mathbb{R}$.

Furthermore, we deduce from Definition DII-9 that, if the first derivative of 'f', with respect to $q = x + [(i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}]\xi \in \text{dom}(f)$, namely–

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q) - f(q)}{\delta q}\right],$$

exists, then its value must always be uniquely determined, regardless of the particular values we might assign to the quaternion increment, δq. Hence, by setting

$$\delta q = \delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\delta\xi = \delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\cdot 0 = \delta x$$

$$(\delta\xi = 0),$$

it is evident that the derivative,

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q)-f(q)}{\delta q}\right] = \lim_{\delta x \to 0}\left[\frac{f(q+\delta x)-f(q)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{U(x+\delta x,\xi)-U(x,\xi)}{\delta x} + \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\left(\frac{V(x+\delta x,\xi)-V(x,\xi)}{\delta x}\right)\right]$$

$$= \frac{\partial}{\partial x}(U(x,\xi)) + \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial x}(V(x,\xi)) \qquad \underline{(i)}.$$

Similarly, by setting

$$\delta q = \delta x + \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\delta\xi = 0 + \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\delta\xi$$

$$= \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\delta\xi \qquad (\delta x = 0),$$

it likewise follows that the derivative,

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q)-f(q)}{\delta q}\right]$$

$$= \lim_{\delta\xi \to 0}\left[\frac{U(x,\xi+\delta\xi)-U(x,\xi) + \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right](V(x,\xi+\delta\xi)-V(x,\xi))}{\left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\delta\xi}\right],$$

since $\delta q = \left[\dfrac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_2^2}}\right]\delta\xi \longrightarrow 0 \Longrightarrow \delta\xi \longrightarrow 0$,

$$\therefore \frac{d}{dq}(f(q)) = \lim_{\delta\xi \to 0}\left[\frac{V(x,\xi+\delta\xi)-V(x,\xi)}{\delta\xi} - \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\left(\frac{U(x,\xi+\delta\xi)-U(x,\xi)}{\delta\xi}\right)\right]$$

$$= \frac{\partial}{\partial\xi}(V(x,\xi)) - \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial\xi}(U(x,\xi)) \qquad \underline{(ii)}.$$

Finally, in view of our requirement that this derivative be uniquely determined, we can combine Eqs. (i) and (ii) into a single differential formula, namely –

$$\frac{d}{dq}(f(q)) = \frac{\partial}{\partial x}(U(x,\xi)) + \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial x}(V(x,\xi))$$

$$= \frac{\partial}{\partial\xi}(V(x,\xi)) - \left[\frac{i\lambda_1+j\lambda_2+k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial\xi}(U(x,\xi)),$$

which, by virtue of its corresponding real and imaginary parts, further yields the set of partial differential equations,

$$\frac{\partial}{\partial x}(U(x,\xi)) = \frac{\partial}{\partial\xi}(V(x,\xi)),$$

$$\frac{\partial}{\partial\xi}(U(x,\xi)) = -\frac{\partial}{\partial x}(V(x,\xi)),$$ as required. $\underline{\underline{Q.E.D.}}$

Theorem TII-13.

Let the quasi-complex function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi)$$

$$= U(x, \xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]V(x, \xi),$$

$\forall \lambda_1, \lambda_2, \lambda_3, U(x, \xi), V(x, \xi) \in \mathbb{R}$,

be defined throughout the η-neighbourhood of each point,

$$q_n = x_n + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n, \ \forall n \in \{1, 2, 3, \ldots\ldots\ldots\ldots, N\},$$

where N is some arbitrary positive integer.

Subsequently, it may be proven that if the partial derivatives, $\frac{\partial}{\partial x}(U(x,\xi))$, $\frac{\partial}{\partial x}(V(x,\xi))$, $\frac{\partial}{\partial \xi}(U(x,\xi))$, $\frac{\partial}{\partial \xi}(V(x,\xi))$, exist in that neighbourhood and, furthermore,

(a) are continuous at each point, (x_n, ξ_n), AND
(b) satisfy the quaternion analogues of the Cauchy-Riemann equations at each point, (x_n, ξ_n),

then the first derivative of $f(q)$, with respect to q, To be continued via the author's next submission, namely -

*"**A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 7/10.**"*

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