“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 7/10.”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 7/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$\frac{d}{dq}(f(q)) = \frac{\partial}{\partial x}(U(x,\xi)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial x}(V(x,\xi)),$$

is likewise defined on each point,

$$q_n = x_n + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi_n, \quad \forall n \in \{1,2,3,\ldots\ldots\ldots,N\}.$$

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<u>PROOF:-</u>

From Definition DII-9, we recall that the first derivative of 'f', with respect to q, is denoted by the formula,

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q) - f(q)}{\delta q}\right],$$

such that the corresponding increment,

$$\delta q = \delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\delta\xi, \qquad (0 < |\delta q| < \eta)$$

and the function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]V(x,\xi)$$

$$\Longrightarrow f(q+\delta q) = f(x+\delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right](\xi+\delta\xi))$$

$$= U(x+\delta x, \xi+\delta\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right] V(x+\delta x, \xi+\delta\xi).$$

After making the appropriate algebraic substitutions, we further obtain

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0} \left[\frac{U(x+\delta x, \xi+\delta\xi) - U(x,\xi)}{\delta q} + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\left(\frac{V(x+\delta x, \xi+\delta\xi) - V(x,\xi)}{\delta q}\right)\right]$$

and hence at each point, $q = q_n$, this derivative is likewise expressed as

$$\frac{d}{dq}(f(q)) = \frac{d}{dq}(f(q))\Big|_{q=q_n}$$

$$= \lim_{\delta q \to 0} \left[\frac{U(x_n+\delta x, \xi_n+\delta\xi) - U(x_n,\xi_n)}{\delta q} + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\left(\frac{V(x_n+\delta x, \xi_n+\delta\xi) - V(x_n,\xi_n)}{\delta q}\right)\right].$$

From the calculus of functions of several real variables, we also perceive that the continuity of the partial derivatives, $\frac{\partial}{\partial x}(U(x,\xi))$, $\frac{\partial}{\partial x}(V(x,\xi))$, $\frac{\partial}{\partial \xi}(U(x,\xi))$, $\frac{\partial}{\partial \xi}(V(x,\xi))$, at each point, (x_n, ξ_n), will guarantee the existence of the formulae —

$$U(x_n+\delta x, \xi_n+\delta\xi) - U(x_n,\xi_n)$$

$$= \frac{\partial}{\partial x}(U(x_n,\xi_n))\delta x + \frac{\partial}{\partial \xi}(U(x_n,\xi_n))\delta\xi + \eta_1\sqrt{\delta x^2 + \delta\xi^2} \;;$$

$$V(x_n + \delta x, \xi_n + \delta\xi) - V(x_n, \xi_n)$$

$$= \frac{\partial}{\partial x}(V(x_n, \xi_n))\delta x + \frac{\partial}{\partial \xi}(V(x_n, \xi_n))\delta\xi + \eta_2\sqrt{\delta x^2 + \delta\xi^2} \ ,$$

such that the real numbers, $\eta_1, \eta_2 \to 0$ as $(\delta x, \delta\xi) \to (0,0)$. Subsequently, we may write

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{\frac{\partial}{\partial x}(U(x_n, \xi_n))\delta x + \frac{\partial}{\partial \xi}(U(x_n, \xi_n))\delta\xi + \eta_1\sqrt{\delta x^2 + \delta\xi^2}}{\delta q} + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\left(\frac{\frac{\partial}{\partial x}(V(x_n, \xi_n))\delta x + \frac{\partial}{\partial \xi}(V(x_n, \xi_n))\delta\xi + \eta_2\sqrt{\delta x^2 + \delta\xi^2}}{\delta q}\right)\right]$$

and since the existence of the first derivative, $\frac{d}{dq}(f(q))$, also implies the existence of the quaternion analogues of the Cauchy–Riemann equations, namely –

$$\left\{\begin{aligned} \frac{\partial}{\partial x}(U(x,\xi)) &= \frac{\partial}{\partial \xi}(V(x,\xi)) \\ \frac{\partial}{\partial \xi}(U(x,\xi)) &= -\frac{\partial}{\partial x}(V(x,\xi)) \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \frac{\partial}{\partial x}(U(x_n,\xi_n)) &= \frac{\partial}{\partial \xi}(V(x_n,\xi_n)) \\ \frac{\partial}{\partial \xi}(U(x_n,\xi_n)) &= -\frac{\partial}{\partial x}(V(x_n,\xi_n)) \end{aligned}\right\}, \ \forall q = q_n,$$

by virtue of Theorem TII–12, it is evident that

$$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{\frac{\partial}{\partial x}(U(x_n, \xi_n))\delta x - \frac{\partial}{\partial x}(V(x_n, \xi_n))\delta\xi + \eta_1\sqrt{\delta x^2 + \delta\xi^2}}{\delta q} + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\left(\frac{\frac{\partial}{\partial x}(V(x_n, \xi_n))\delta x + \frac{\partial}{\partial x}(U(x_n, \xi_n))\delta\xi + \eta_2\sqrt{\delta x^2 + \delta\xi^2}}{\delta q}\right)\right]$$

$$= \lim_{\delta q \to 0} \left[\begin{array}{l} \left(\frac{\partial}{\partial x}(U(x_n, \xi_n)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \frac{\partial}{\partial x}(V(x_n, \xi_n)) \right) \frac{\delta x}{\delta q} + \\ \left(\left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \frac{\partial}{\partial x}(U(x_n, \xi_n)) - \frac{\partial}{\partial x}(V(x_n, \xi_n)) \right) \frac{\delta \xi}{\delta q} + \\ \left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \eta_2 \right) \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \end{array} \right]$$

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$$= \lim_{\delta q \to 0} \left[\begin{array}{l} \left(\frac{\partial}{\partial x}(U(x_n, \xi_n)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \frac{\partial}{\partial x}(V(x_n, \xi_n)) \right) \frac{\delta x}{\delta q} + \\ \left(\frac{\partial}{\partial x}(U(x_n, \xi_n)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \frac{\partial}{\partial x}(V(x_n, \xi_n)) \right) \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \frac{\delta \xi}{\delta q} + \\ \left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \eta_2 \right) \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \end{array} \right]$$

$$= \lim_{\delta q \to 0} \left[\begin{array}{l} \left(\begin{array}{l} \frac{\partial}{\partial x}(U(x_n, \xi_n)) + \\ \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \frac{\partial}{\partial x}(V(x_n, \xi_n)) \end{array} \right) \left(\delta x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \delta \xi \right) \frac{1}{\delta q} + \\ \left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] \eta_2 \right) \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \end{array} \right]$$

$$= \lim_{\delta q \to 0} \left[\frac{\partial}{\partial x}(U(x_n, \xi_n)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] \frac{\partial}{\partial x}(V(x_n, \xi_n)) + \left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\eta_2\right) \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \right].$$

For the purposes of evaluating the above limit, we firstly observe that the modulus,

$$\left| \eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\eta_2 \right| = \sqrt{\eta_1^2 + \eta_2^2},$$

and since the modulus of the increment,

$$|\delta q| = \sqrt{\delta x^2 + \delta \xi^2},$$

it likewise follows that the modulus,

$$\left| \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \right| = 1,$$

and hence

$$\left| \left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\eta_2\right) \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \right| =$$

$$\left| \eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\eta_2 \right| \left| \frac{\sqrt{\delta x^2 + \delta \xi^2}}{\delta q} \right| = \sqrt{\eta_1^2 + \eta_2^2}.$$

Bearing in mind that the real numbers, $\eta_1, \eta_2 \to 0$ as $(\delta x, \delta\xi) \to (0,0)$, we accordingly perceive that the limit,

$$\lim_{\delta q \to 0}\left\|\left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\eta_2\right)\frac{\sqrt{\delta x^2 + \delta\xi^2}}{\delta q}\right\| = 0 = |0| \implies$$

$$\lim_{\delta q \to 0}\left[\left(\eta_1 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\eta_2\right)\frac{\sqrt{\delta x^2 + \delta\xi^2}}{\delta q}\right] = 0,$$

and, finally, the first derivative of $f(q)$, with respect to q,

$$\frac{d}{dq}(f(q)) = \frac{\partial}{\partial x}(U(x_n, \xi_n)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial x}(V(x_n, \xi_n))$$

$$= \frac{\partial}{\partial x}(U(x, \xi)) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\frac{\partial}{\partial x}(V(x, \xi)),$$

$$\forall q = q_n = x_n + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi_n,$$

such that $n \in \{1, 2, 3, \ldots\ldots\ldots, N\}$ and N is some arbitrary positive integer, as required. <u>Q.E.D.</u>

<u>Definition DII-10.</u>

A quasi-complex function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi)$$

$$= U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]V(x,\xi),$$

is said to be <u>analytic</u> at a point,

$$q_0 = x_0 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi_0,$$

if its corresponding first derivative, $\frac{d}{dq}(f(q))$, exists not only at q_0, but also at each point,

$$q = x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi,$$

in some δ-neighbourhood of q_0.

<u>Definition DII-11.</u>

A quasi-complex function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]V(x,\xi),$$

is said to be <u>entire</u> if it is <u>analytic</u>, $\forall q \in \left\{x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right]\xi \,\middle|\, x, \xi \in \mathbb{R}\right\}$.

Definition DII-12.

If a quasi-complex function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi)$$

$$= U(x, \xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] V(x, \xi),$$

fails to be analytic at a point,

$$q_0 = x_0 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_0,$$

but is analytic at every point,

$$q = x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi,$$

in some δ-neighbourhood of q_0, then q_0 is called a singular point of the quasi-complex function.

We conclude our discussion of this topic with the following remarks:-

(a) The partial derivative of $f(q)$ with respect to x,

$$\frac{\partial}{\partial x}(f(q)) = \frac{\partial}{\partial x}(u_1(x, y, \hat{x}, \hat{y})) + i\frac{\partial}{\partial x}(v_1(x, y, \hat{x}, \hat{y})) + j\frac{\partial}{\partial x}(u_2(x, y, \hat{x}, \hat{y})) +$$

$$h\frac{\partial}{\partial x}(v_2(x,y,\hat{x},\hat{y})),$$

which was derived by way of Theorem TII-2, is analogous to the analytic derivative,

$$f'(z) = \frac{d}{dz}(f(z)) = \frac{\partial}{\partial x}(f(z)) = \frac{\partial}{\partial x}(u(x,y)) + i\frac{\partial}{\partial x}(v(x,y)) \quad (2\text{-}3),$$

from complex variable analysis.

(b) The partial derivative of q^n with respect to x,

$$\frac{\partial}{\partial x}(q^n) = nq^{n-1},$$

which was derived by way of Theorem TII-5, is analogous to the analytic derivative,

$$\frac{d}{dz}(z^n) = \frac{\partial}{\partial x}(z^n) = nz^{n-1} \quad (2\text{-}4),$$

from complex variable analysis.

(c) The partial derivative of $\exp(q)$ with respect to x,

$$\frac{\partial}{\partial x}(\exp(q)) = \exp(q),$$

which was derived by way of Theorem TII-6, is analogous to the analytic derivative,

$$\frac{d}{dz}(\exp(z)) = \frac{\partial}{\partial x}(\exp(z)) = \exp(z) \quad (2\text{-}5),$$

from complex variable analysis.

(d) The partial derivatives of sin(q) and cos(q) with respect to x,

$\frac{\partial}{\partial x}(\sin(q)) = \cos(q)$;

$\frac{\partial}{\partial x}(\cos(q)) = -\sin(q)$,

which were derived by way of Theorem TII-7, are analogous to the analytic derivatives,

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$$\frac{d}{dz}(\sin(z)) = \frac{\partial}{\partial x}(\sin(z)) = \cos(z) \quad (2\text{-}6),$$

$$\frac{d}{dz}(\cos(z)) = \frac{\partial}{\partial x}(\cos(z)) = -\sin(z) \quad (2\text{-}7),$$

from complex variable analysis.

(e) The partial derivatives of sinh(q) and cosh(q) with respect to x,

$\frac{\partial}{\partial x}(\sinh(q)) = \cosh(q)$;

$\frac{\partial}{\partial x}(\cosh(q)) = \sinh(q)$,

which were derived by way of Theorem TII-8, are analogous to the analytic derivatives,

$$\frac{d}{dz}(\sinh(z)) = \frac{\partial}{\partial x}(\sinh(z)) = \cosh(z) \quad (2\text{-}8),$$

$$\frac{d}{dz}(\cosh(z)) = \frac{\partial}{\partial x}(\cosh(z)) = \sinh(z) \quad (2-9),$$

from complex variable analysis.

(f) The partial derivative of $\log(q)$, with respect to x,

$$\frac{\partial}{\partial x}(\log(q)) = q^{-1},$$

which was derived by way of Theorem TII-9, is analogous to the analytic derivative,

$$\frac{d}{dz}(\log(z)) = \frac{\partial}{\partial x}(\log(z)) = z^{-1} \quad (2-10),$$

from complex variable analysis.

(g) The remaining results enunciated via Definition DII-7, Definition DII-8, Theorem TII-10,..............., Definition DII-12, are analogous to those

definitions and theorems from complex variable analysis which relate to the Cauchy-Riemann equations, namely -

$$\frac{\partial}{\partial x}(u(x,y)) = \frac{\partial}{\partial y}(v(x,y)) \quad (2-11),$$

$$\frac{\partial}{\partial y}(u(x,y)) = -\frac{\partial}{\partial x}(v(x,y)) \quad (2-12),$$

as well as some characteristic properties of the analytic derivative, $f'(z)$ [viz. Eq.(2-3)].

(h) The results enunciated via Definition DII-5 and Theorems TII-2, TII-3 & TII-4 are analogous to similar definitions and theorems from real and complex variable analysis pertaining to partial differentiation.

3. Further Properties of the Indefinite and Definite Integrals of Quaternion Hypercomplex Functions.

In this part of Section II, we will

(a) enunciate various definitions and theorems whose purpose is to expand upon those techniques of integration initially developed by the author in his first paper [5]

AND

(b) compare these definitions and theorems with their more familiar analogues from real and complex variable analysis.

Definition DII-13.

Let there exist a multi-valued quaternion hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 \quad , \\ \vdots \\ [f(q)]_n \end{cases}$$

whose component <u>single-valued</u> functions are restricted to a smooth arc, C, thus defined by the equation,

$q(t) = x(t) + iy(t) + j\hat{z}(t) + k\hat{y}(t), \ \forall t \in [a,b].$

Consequently, the <u>indefinite integral</u> or <u>anti-derivative</u> of 'f', with respect to the real parameter, 't', is defined by the formula,

$$F(q(t)) = F^*(t) = \int f(q(t))\,dt = \begin{cases} \int [f(q(t))]_1\,dt \ , \\ \quad \vdots \\ \int [f(q(t))]_n\,dt \end{cases}$$

such that the corresponding derivative,

$$\frac{d}{dt}[F(q(t))] = \frac{d}{dt}[F^*(t)] = \frac{d}{dt}\left[\int f(q(t))\,dt\right] = \begin{cases} [f(q(t))]_1 \ . \\ \quad \vdots \\ [f(q(t))]_n \end{cases}$$

<u>Definition DII-14.</u>

Let there exist a <u>multi-valued</u> quaternion hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 \ , \\ \quad \vdots \\ [f(q)]_n \end{cases}$$

whose component single-valued functions are restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) \ , \ \forall t \in [a, b].$$

Consequently, the definite integral of 'f', defined with respect to the real parameter, 't', over the interval, $[a, b]$, is accordingly given by the formula,

$$F(q(b)) - F(q(a)) = F^*(b) - F^*(a)$$

$$= \int_a^b f(q(t))\,dt = \begin{cases} \int_a^b [f(q(t))]_1\,dt \\ \vdots \\ \int_a^b [f(q(t))]_n\,dt \end{cases} .$$

Definition DII-15.

Let there exist a multi-valued quaternion hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 \\ \vdots \\ [f(q)]_n \end{cases} ,$$

whose component single-valued functions are restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t) \ , \ \forall t \in [a, b].$$

Consequently, the indefinite integral of 'f', with respect to the quaternion hypercomplex variable, 'q', is defined by the formula,

$$\int f(q)(dq)_c = \begin{cases} \int [f(q)]_1 (dq)_c \,, \\ \vdots \\ \int [f(q)]_n (dq)_c \end{cases}$$

such that its constituent indefinite integrals,

$$\int [f(q)]_m (dq)_c = \int [f(q(t))]_m \frac{d}{dt}[q(t)]\, dt, \quad \forall m \in \{1, 2, \ldots\ldots\ldots, n\}.$$

Definition DII-16.

Let there exist a multi-valued quaternion hypercomplex function,

$$f(q) = \begin{cases} [f(q)]_1 \,, \\ \vdots \\ [f(q)]_n \end{cases}$$

whose component single-valued functions are restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{z}(t) + k\hat{y}(t), \quad \forall t \in [a, b].$$

Subsequently, the definite integral of 'f', defined with respect to the quaternion hypercomplex variable, 'q', over the interval, [a, b], is given by the formula,

$$\int_C f(q)\,dq = \begin{cases} \int_C [f(q)]_1\,dq\,, \\ \vdots \\ \int_C [f(q)]_n\,dq \end{cases}$$

such that its constituent definite integrals,

$$\int_C [f(q)]_m\,dq = \int_a^b [f(q(t))]_m \frac{d}{dt}[q(t)]\,dt, \quad \forall m \in \{1, 2, \ldots\ldots\ldots, n\}.$$

Theorem TII-14.

Let there exist two single-valued quaternion hypercomplex functions, $f(q)$ and $g(q)$, which are both restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + \hat{y}(t), \quad \forall t \in [a, b].$$

In the circumstances, we can prove that the indefinite integral, with respect to 't', of the sum of these functions, is equal to the sum of their respective indefinite integrals, in other words –

$$\int [f(q(t)) + g(q(t))]\,dt = \int f(q(t))\,dt + \int g(q(t))\,dt,$$

provided that the indefinite integrals,

$$\int f(q(t))\,dt \text{ and } \int g(q(t))\,dt,$$

are similarly defined, $\forall t \in [a,b]$.

* * *

PROOF:-

We shall commence the proof of this theorem by firstly writing the functions, $f(q)$ and $g(q)$, in terms of their corresponding real and imaginary parts, namely -

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y}),$$

$$g(q) = r_1(x,y,\hat{x},\hat{y}) + is_1(x,y,\hat{x},\hat{y}) + jr_2(x,y,\hat{x},\hat{y}) + ks_2(x,y,\hat{x},\hat{y})$$

$$\therefore f(q) + g(q) = u_1(x,y,\hat{x},\hat{y}) + r_1(x,y,\hat{x},\hat{y}) + i(v_1(x,y,\hat{x},\hat{y}) + s_1(x,y,\hat{x},\hat{y})) + j(u_2(x,y,\hat{x},\hat{y}) + r_2(x,y,\hat{x},\hat{y})) + k(v_2(x,y,\hat{x},\hat{y}) + s_2(x,y,\hat{x},\hat{y})).$$

Furthermore, by restricting each of these functions to a smooth arc, C, thus denoted by the equation,

$$q = q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b],$$

it therefore follows that

$$f(q) = f(q(t)) = u_1(x(t),y(t),\hat{x}(t),\hat{y}(t)) + iv_1(x(t),y(t),\hat{x}(t),\hat{y}(t)) + ju_2(x(t),y(t),\hat{x}(t),\hat{y}(t)) + kv_2(x(t),y(t),\hat{x}(t),\hat{y}(t))$$

$$= u_1^*(t) + i v_1^*(t) + j u_2^*(t) + k v_2^*(t) \, ,$$

$$g(q) = g(q(t)) = r_1(x(t), y(t), \hat{z}(t), \hat{y}(t)) + i s_1(x(t), y(t), \hat{z}(t), \hat{y}(t)) + j r_2(x(t), y(t), \hat{z}(t), \hat{y}(t)) + k s_2(x(t), y(t), \hat{z}(t), \hat{y}(t))$$

$$= r_1^*(t) + i s_1^*(t) + j r_2^*(t) + k s_2^*(t) \, ,$$

$$\therefore f(q) + g(q) = f(q(t)) + g(q(t))$$

$$= u_1^*(t) + r_1^*(t) + i(v_1^*(t) + s_1^*(t)) + j(u_2^*(t) + r_2^*(t)) + k(v_2^*(t) + s_2^*(t)) \, .$$

From the previously established theorems on the integration of quaternion hypercomplex functions, we recall that the indefinite integrals of $f(q(t))$, $g(q(t))$ and $f(q(t)) + g(q(t))$, with respect to 't', are accordingly expressed as —

$$\int f(q(t))\, dt = \int u_1^*(t)\, dt + i \int v_1^*(t)\, dt + j \int u_2^*(t)\, dt + k \int v_2^*(t)\, dt \, ,$$

$$\int g(q(t))\, dt = \int r_1^*(t)\, dt + i \int s_1^*(t)\, dt + j \int r_2^*(t)\, dt + k \int s_2^*(t)\, dt \, ,$$

$$\int [f(q(t)) + g(q(t))]\, dt = \int (u_1^*(t) + r_1^*(t))\, dt + i \int (v_1^*(t) + s_1^*(t))\, dt + j \int (u_2^*(t) + r_2^*(t))\, dt + k \int (v_2^*(t) + s_2^*(t))\, dt \, ,$$

provided that the corresponding real and imaginary parts thereof are similarly defined, $\forall t \in [a, b]$. Finally, we deduce that the sum of these indefinite integrals,

$$\int f(q(t))\,dt + \int g(q(t))\,dt = \int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt + \int r_1^*(t)\,dt + i\int s_1^*(t)\,dt + j\int r_2^*(t)\,dt + k\int s_2^*(t)\,dt$$

$$= \int u_1^*(t)\,dt + \int r_1^*(t)\,dt + i\left(\int v_1^*(t)\,dt + \int s_1^*(t)\,dt\right) + j\left(\int u_2^*(t)\,dt + \int r_2^*(t)\,dt\right) + k\left(\int v_2^*(t)\,dt + \int s_2^*(t)\,dt\right)$$

$$= \int (u_1^*(t) + r_1^*(t))\,dt + i\int (v_1^*(t) + s_1^*(t))\,dt + j\int (u_2^*(t) + r_2^*(t))\,dt + k\int (v_2^*(t) + s_2^*(t))\,dt$$

$$= \int [f(q(t)) + g(q(t))]\,dt,$$

the indefinite integral, with respect to 't', of the sum of the functions, $f(q(t))$ and $g(q(t))$, as required. Q.E.D.

Theorem TII-15.

Let there exist two single-valued quaternion hypercomplex functions, $f(q)$ and $g(q)$, which are both restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

Subsequently, it may be shown that the definite integral, with respect to 't', of the sum of these functions is equal to the sum of their respective definite integrals, in other words -

$$\int_a^b [f(q(t)) + g(q(t))]\,dt = \int_a^b f(q(t))\,dt + \int_a^b g(q(t))\,dt,$$

provided that the definite integrals,

$$\int_a^b f(q(t))\,dt \text{ and } \int_a^b g(q(t))\,dt,$$

are similarly defined, $\forall t \in [a, b]$.

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PROOF:-

From the preceding Theorem TII-14, we recall that the functions,

$$\left.\begin{aligned} f(q) &= f(q(t)) = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t) \\ g(q) &= g(q(t)) = r_1^*(t) + is_1^*(t) + jr_2^*(t) + ks_2^*(t) \end{aligned}\right\} \Longrightarrow$$

$$\begin{aligned} f(q) + g(q) &= f(q(t)) + g(q(t)) \\ &= u_1^*(t) + r_1^*(t) + i(v_1^*(t) + s_1^*(t)) + j(u_2^*(t) + r_2^*(t)) + \\ &\quad k(v_2^*(t) + s_2^*(t)), \end{aligned}$$

and hence, in view of the previously established theorems on the integration of quaternion hypercomplex functions, it is evident that the definite integrals of $f(q(t))$, $g(q(t))$ and $f(q(t)) + g(q(t))$, with respect to 't', are accordingly expressed as —

$$\int_a^b f(q(t))\,dt = \int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt,$$

$$\int_a^b g(q(t))\,dt = \int_a^b r_1^*(t)\,dt + i\int_a^b s_1^*(t)\,dt + j\int_a^b r_2^*(t)\,dt + k\int_a^b s_2^*(t)\,dt,$$

$$\int_a^b [f(q(t)) + g(q(t))]\,dt = \int_a^b (u_1^*(t) + r_1^*(t))\,dt + i\int_a^b (v_1^*(t) + s_1^*(t))\,dt + j\int_a^b (u_2^*(t) + r_2^*(t))\,dt + k\int_a^b (v_2^*(t) + s_2^*(t))\,dt\,,$$

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provided that the corresponding real and imaginary parts thereof are similarly defined, $\forall t \in [a, b]$. Finally, we deduce that the sum of these definite integrals,

$$\int_a^b f(q(t))\,dt + \int_a^b g(q(t))\,dt = \int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt + \int_a^b r_1^*(t)\,dt + i\int_a^b s_1^*(t)\,dt + j\int_a^b r_2^*(t)\,dt + k\int_a^b s_2^*(t)\,dt$$

$$= \int_a^b u_1^*(t)\,dt + \int_a^b r_1^*(t)\,dt + i\left(\int_a^b v_1^*(t)\,dt + \int_a^b s_1^*(t)\,dt\right) + j\left(\int_a^b u_2^*(t)\,dt + \int_a^b r_2^*(t)\,dt\right) + k\left(\int_a^b v_2^*(t)\,dt + \int_a^b s_2^*(t)\,dt\right)$$

$$= \int_a^b (u_1^*(t) + r_1^*(t))\,dt + i\int_a^b (v_1^*(t) + s_1^*(t))\,dt + j\int_a^b (u_2^*(t) + r_2^*(t))\,dt + k\int_a^b (v_2^*(t) + s_2^*(t))\,dt$$

$$= \int_a^b [f(q(t)) + g(q(t))]\,dt,$$

the definite integral, with respect to 't', of the sum of the functions, $f(q(t))$ and $g(q(t))$, as required. <u>Q.E.D</u>.

Theorem TII–16.

Let there exist a single-valued quaternion hypercomplex function, $f(q)$, which is accordingly restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \ \forall t \in [a, b].$$

In the circumstances, it may be shown that, for any quaternion constant, q_0, the integral formulae,

(a) $q_0 \int f(q(t))\, dt = \int q_0 f(q(t))\, dt$,

(b) $\left(\int f(q(t))\, dt\right) q_0 = \int f(q(t))\, q_0\, dt$,

are always valid, provided that the indefinite integral,

$$\int f(q(t))\, dt,$$

is similarly defined, $\forall t \in [a, b]$.

* * *

PROOF:–

Let the function,

$$f(q) = f(q(t)) = u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t) ,$$

be integrable, with respect to 't', over the closed interval, $[a, b]$, such that we obtain the indefinite integral,

$$\int f(q(t))\,dt = \int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt,$$

whose corresponding real and imaginary parts are likewise defined over the same interval. Furthermore, let there also exist an arbitrary quaternion constant,

$$q_0 = c_1 + ic_2 + jc_3 + kc_4, \quad \forall c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

(a) Henceforth, we deduce that the integral formula,

$$q_0\int f(q(t))\,dt = (c_1 + ic_2 + jc_3 + kc_4)\int f(q(t))\,dt$$

$$= (c_1 + ic_2 + jc_3 + kc_4)\left(\int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt\right)$$

$$= c_1\int u_1^*(t)\,dt - c_2\int v_1^*(t)\,dt - c_3\int u_2^*(t)\,dt - c_4\int v_2^*(t)\,dt +$$

$$i\left(c_1\int v_1^*(t)\,dt + c_2\int u_1^*(t)\,dt + c_3\int v_2^*(t)\,dt - c_4\int u_2^*(t)\,dt\right) +$$

$$j\left(c_1\int u_2^*(t)\,dt - c_2\int v_2^*(t)\,dt + c_3\int u_1^*(t)\,dt + c_4\int v_1^*(t)\,dt\right) +$$

$$k\left(c_1\int v_2^*(t)\,dt + c_2\int u_2^*(t)\,dt - c_3\int v_1^*(t)\,dt + c_4\int u_1^*(t)\,dt\right)$$

$$= \int\left(c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t)\right)dt +$$

$$i\int\left(c_1 v_1^*(t) + c_2 u_1^*(t) + c_3 v_2^*(t) - c_4 u_2^*(t)\right)dt +$$

$$j\int\left(c_1 u_2^*(t) - c_2 v_2^*(t) + c_3 u_1^*(t) + c_4 v_1^*(t)\right) dt +$$

$$k\int\left(c_1 v_2^*(t) + c_2 u_2^*(t) - c_3 v_1^*(t) + c_4 u_1^*(t)\right) dt.$$

Furthermore, from the established definition of quaternion multiplication, it likewise follows that the product function,

$$q_0 f(q(t)) = (c_1 + ic_2 + jc_3 + kc_4)(u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t))$$

$$= c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t) +$$

$$i(c_1 v_1^*(t) + c_2 u_1^*(t) + c_3 v_2^*(t) - c_4 u_2^*(t)) +$$

$$j(c_1 u_2^*(t) - c_2 v_2^*(t) + c_3 u_1^*(t) + c_4 v_1^*(t)) +$$

$$k(c_1 v_2^*(t) + c_2 u_2^*(t) - c_3 v_1^*(t) + c_4 u_1^*(t)) \quad ,$$

from which we obtain the corresponding indefinite integral,

$$\int q_0 f(q(t))\, dt = \int (c_1 + ic_2 + jc_3 + kc_4)(u_1^*(t) + iv_1^*(t) + ju_2^*(t) + kv_2^*(t))\, dt$$

$$= \int\left(c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t)\right) dt +$$

$$i\int\left(c_1 v_1^*(t) + c_2 u_1^*(t) + c_3 v_2^*(t) - c_4 u_2^*(t)\right) dt +$$

$$j\int\left(c_1 u_2^*(t) - c_2 v_2^*(t) + c_3 u_1^*(t) + c_4 v_1^*(t)\right) dt +$$

$$k\int\left(c_1 v_2^*(t) + c_2 u_2^*(t) - c_3 v_1^*(t) + c_4 u_1^*(t)\right) dt$$

$$= (c_1 + ic_2 + jc_3 + kc_4)\left(\int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt\right)$$

$$= q_0 \int f(q(t))\,dt,$$ as required. <u>Q.E.D.</u>

(b) Similarly, we deduce that the integral formula,

$$\left(\int f(q(t))\,dt\right)q_0 = \left(\int f(q(t))\,dt\right)(c_1 + ic_2 + jc_3 + kc_4)$$

$$= \left(\int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt\right)(c_1 + ic_2 + jc_3 + kc_4)$$

$$= c_1\int u_1^*(t)\,dt - c_2\int v_1^*(t)\,dt - c_3\int u_2^*(t)\,dt - c_4\int v_2^*(t)\,dt +$$

$$i\left(c_2\int u_1^*(t)\,dt + c_1\int v_1^*(t)\,dt + c_4\int u_2^*(t)\,dt - c_3\int v_2^*(t)\,dt\right) +$$

$$j\left(c_3\int u_1^*(t)\,dt - c_4\int v_1^*(t)\,dt + c_1\int u_2^*(t)\,dt + c_2\int v_2^*(t)\,dt\right) +$$

$$k\left(c_4\int u_1^*(t)\,dt + c_3\int v_1^*(t)\,dt - c_2\int u_2^*(t)\,dt + c_1\int v_2^*(t)\,dt\right)$$

$$= \int\left(c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t)\right) dt +$$

$$i\int\left(c_2 u_1^*(t) + c_1 v_1^*(t) + c_4 u_2^*(t) - c_3 v_2^*(t)\right) dt +$$

$$j\int\left(c_3 u_1^*(t) - c_4 v_1^*(t) + c_1 u_2^*(t) + c_2 v_2^*(t)\right) dt +$$

$$k\int\left(c_4 u_1^*(t) + c_3 v_1^*(t) - c_2 u_2^*(t) + c_1 v_2^*(t)\right) dt .$$

Finally, from the established definition of quaternion multiplication, it likewise follows that the product function,

$$f(q(t))\,q_0 = (u_1^*(t) + i v_1^*(t) + j u_2^*(t) + k v_2^*(t))(c_1 + ic_2 + jc_3 + kc_4)$$

$$= c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t) +$$

$$i\big(c_2 u_1^*(t) + c_1 v_1^*(t) + c_4 u_2^*(t) - c_3 v_2^*(t)\big) +$$

$$j\big(c_3 u_1^*(t) - c_4 v_1^*(t) + c_1 u_2^*(t) + c_2 v_2^*(t)\big) +$$

$$k\big(c_4 u_1^*(t) + c_3 v_1^*(t) - c_2 u_2^*(t) + c_1 v_2^*(t)\big) \quad ,$$

whence we obtain the corresponding indefinite integral,

$$\int f(q(t))\,q_0\,dt = \int (u_1^*(t) + i v_1^*(t) + j u_2^*(t) + k v_2^*(t))(c_1 + ic_2 + jc_3 + kc_4)\,dt$$

$$= \int \big(c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t)\big)\,dt +$$

$$i\int \big(c_2 u_1^*(t) + c_1 v_1^*(t) + c_4 u_2^*(t) - c_3 v_2^*(t)\big)\,dt +$$

$$j\int \big(c_3 u_1^*(t) - c_4 v_1^*(t) + c_1 u_2^*(t) + c_2 v_2^*(t)\big)\,dt +$$

$$k\int \big(c_4 u_1^*(t) + c_3 v_1^*(t) - c_2 u_2^*(t) + c_1 v_2^*(t)\big)\,dt$$

$$= \left(\int u_1^*(t)\,dt + i\int v_1^*(t)\,dt + j\int u_2^*(t)\,dt + k\int v_2^*(t)\,dt\right)(c_1 + ic_2 + jc_3 + kc_4)$$

$= \left(\int f(q(t))\,dt\right) q_0$, as required. <u>Q.E.D.</u>

Theorem TII-17.

Let there exist a *single-valued* quaternion hypercomplex function, $f(q)$, which is accordingly restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a, b].$$

Subsequently, it may be shown that, for any quaternion constant, q_0, the integral formulae,

(a) $q_0 \int_a^b f(q(t))\, dt = \int_a^b q_0 f(q(t))\, dt$,

(b) $\left(\int_a^b f(q(t))\, dt\right) q_0 = \int_a^b f(q(t))\, q_0\, dt$,

are always valid, provided that the definite integral,

$$\int_a^b f(q(t))\, dt,$$

is similarly defined, $\forall t \in [a, b]$.

* * *

PROOF:-

Let the function,

$$f(q) = f(q(t)) = u_1^*(t) + i v_1^*(t) + j u_2^*(t) + k v_2^*(t),$$

be integrable, with respect to 't', over the closed interval, $[a,b]$, such that we obtain the definite integral,

$$\int_a^b f(q(t))\,dt = \int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt,$$

whose corresponding real and imaginary parts are likewise defined over the same interval. Furthermore, let there also exist an arbitrary quaternion constant,

$$q_0 = c_1 + ic_2 + jc_3 + kc_4\,,\quad \forall c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

(a) From the preceding Theorem TII-16, we recall that the indefinite integral,

$$\begin{aligned}\int q_0 f(q(t))\,dt = {} & \int\left(c_1u_1^*(t) - c_2v_1^*(t) - c_3u_2^*(t) - c_4v_2^*(t)\right)dt \; + \\ & i\int\left(c_1v_1^*(t) + c_2u_1^*(t) + c_3v_2^*(t) - c_4u_2^*(t)\right)dt \; + \\ & j\int\left(c_1u_2^*(t) - c_2v_2^*(t) + c_3u_1^*(t) + c_4v_1^*(t)\right)dt \; + \\ & k\int\left(c_1v_2^*(t) + c_2u_2^*(t) - c_3v_1^*(t) + c_4u_1^*(t)\right)dt \quad ,\end{aligned}$$

whence it is evident that the corresponding definite integral,

$$\begin{aligned}\int_a^b q_0 f(q(t))\,dt = {} & \int_a^b\left(c_1u_1^*(t) - c_2v_1^*(t) - c_3u_2^*(t) - c_4v_2^*(t)\right)dt \; + \\ & i\int_a^b\left(c_1v_1^*(t) + c_2u_1^*(t) + c_3v_2^*(t) - c_4u_2^*(t)\right)dt \; + \\ & j\int_a^b\left(c_1u_2^*(t) - c_2v_2^*(t) + c_3u_1^*(t) + c_4v_1^*(t)\right)dt \; +\end{aligned}$$

$$k\int_a^b \left(c_1 v_2^*(t) + c_2 u_2^*(t) - c_3 v_1^*(t) + c_4 u_1^*(t)\right) dt$$

$$= c_1\int_a^b u_1^*(t)\,dt - c_2\int_a^b v_1^*(t)\,dt - c_3\int_a^b u_2^*(t)\,dt - c_4\int_a^b v_2^*(t)\,dt \; +$$

$$i\left(c_1\int_a^b v_1^*(t)\,dt + c_2\int_a^b u_1^*(t)\,dt + c_3\int_a^b v_2^*(t)\,dt - c_4\int_a^b u_2^*(t)\,dt\right) +$$

$$j\left(c_1\int_a^b u_2^*(t)\,dt - c_2\int_a^b v_2^*(t)\,dt + c_3\int_a^b u_1^*(t)\,dt + c_4\int_a^b v_1^*(t)\,dt\right) +$$

$$k\left(c_1\int_a^b v_2^*(t)\,dt + c_2\int_a^b u_2^*(t)\,dt - c_3\int_a^b v_1^*(t)\,dt + c_4\int_a^b u_1^*(t)\,dt\right)$$

$$= (c_1 + ic_2 + jc_3 + kc_4)\left(\int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt\right)$$

$$= q_0\int_a^b f(q(t))\,dt,$$ as required. Q.E.D.

(b) Similarly, from the preceding Theorem TII-16, we recall that the indefinite integral,

$$\int f(q(t))q_0\,dt = \int\left(c_1 u_1^*(t) - c_2 v_1^*(t) - c_3 u_2^*(t) - c_4 v_2^*(t)\right) dt \; +$$

$$i\int\left(c_2 u_1^*(t) + c_1 v_1^*(t) + c_4 u_2^*(t) - c_3 v_2^*(t)\right) dt \; +$$

$$j\int\left(c_3 u_1^*(t) - c_4 v_1^*(t) + c_1 u_2^*(t) + c_2 v_2^*(t)\right) dt \; +$$

$$k\int\left(c_4 u_1^*(t) + c_3 v_1^*(t) - c_2 u_2^*(t) + c_1 v_2^*(t)\right) dt \quad ,$$

whence it is evident that the corresponding definite integral,

$$\int_a^b f(q(t))\,q_0\,dt = \int_a^b (c_1u_1^*(t) - c_2v_1^*(t) - c_3u_2^*(t) - c_4v_2^*(t))\,dt \;+$$
$$i\int_a^b (c_2u_1^*(t) + c_1v_1^*(t) + c_4u_2^*(t) - c_3v_2^*(t))\,dt \;+$$
$$j\int_a^b (c_3u_1^*(t) - c_4v_1^*(t) + c_1u_2^*(t) + c_2v_2^*(t))\,dt \;+$$
$$k\int_a^b (c_4u_1^*(t) + c_3v_1^*(t) - c_2u_2^*(t) + c_1v_2^*(t))\,dt$$

$$= c_1\int_a^b u_1^*(t)\,dt - c_2\int_a^b v_1^*(t)\,dt - c_3\int_a^b u_2^*(t)\,dt - c_4\int_a^b v_2^*(t)\,dt \;+$$
$$i\left(c_2\int_a^b u_1^*(t)\,dt + c_1\int_a^b v_1^*(t)\,dt + c_4\int_a^b u_2^*(t)\,dt - c_3\int_a^b v_2^*(t)\,dt\right) +$$
$$j\left(c_3\int_a^b u_1^*(t)\,dt - c_4\int_a^b v_1^*(t)\,dt + c_1\int_a^b u_2^*(t)\,dt + c_2\int_a^b v_2^*(t)\,dt\right) +$$
$$k\left(c_4\int_a^b u_1^*(t)\,dt + c_3\int_a^b v_1^*(t)\,dt - c_2\int_a^b u_2^*(t)\,dt + c_1\int_a^b v_2^*(t)\,dt\right)$$

$$= \left(\int_a^b u_1^*(t)\,dt + i\int_a^b v_1^*(t)\,dt + j\int_a^b u_2^*(t)\,dt + k\int_a^b v_2^*(t)\,dt\right)(c_1 + ic_2 + jc_3 + kc_4)$$

$$= \left(\int_a^b f(q(t))\,dt\right)q_0,$$ as required. Q.E.D.

Theorem TII–18.

Let there exist a *single-valued* quaternion hypercomplex function, $f(q)$, which is accordingly restricted to a smooth arc, C, thus defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

Subsequently, it may be proven that the modular inequality,

$$\left|\int_a^{b_p} f(q(t))\,dt\right| \leq \int_a^{b_p} |f(q(t))|\,dt ,$$

is always valid, provided that the definite integral,

$$\int_a^{b_p} f(q(t))\,dt ,$$

is similarly defined, $\forall t \in [a,b]$.

* * *

PROOF:-

We firstly postulate the existence of a quaternion number,

$$q_0 = x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0 ,$$

such that the definite integral,

$$\int_a^{b_p} f(q(t))\,dt = q_0 .$$

Now, from the previously established definitions and theorems on the exponential and logarithmic functions, we deduce that the quaternion number,

$$\begin{aligned} q_0 &= \exp(\ln(q_0)) \\ &= \exp(\ln(x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0)) \end{aligned}$$

$$= \exp\left[\log\left(\sqrt{x_0^2+y_0^2+\hat{x}_0^2+\hat{y}_0^2}\right) + \left[\frac{iy_0+j\hat{x}_0+k\hat{y}_0}{\sqrt{y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right](\Theta_0+2n\pi)\right]$$

$$= \exp\left(\log\left(\sqrt{x_0^2+y_0^2+\hat{x}_0^2+\hat{y}_0^2}\right)\right)\exp\left(\left[\frac{iy_0+j\hat{x}_0+k\hat{y}_0}{\sqrt{y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right](\Theta_0+2n\pi)\right)$$

$$= \sqrt{x_0^2+y_0^2+\hat{x}_0^2+\hat{y}_0^2}\,\exp\left(\left[\frac{iy_0+j\hat{x}_0+k\hat{y}_0}{\sqrt{y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right]\Theta_0\right)$$

$$= r_0\exp\left(\left[\frac{iy_0+j\hat{x}_0+k\hat{y}_0}{\sqrt{y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right]\Theta_0\right),$$

whereupon the real constants,

$$r_0 = \sqrt{x_0^2+y_0^2+\hat{x}_0^2+\hat{y}_0^2},$$

$$\Theta_0 = \cos^{-1}\left[\frac{x_0}{\sqrt{x_0^2+y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right] \in [0,\pi],$$

and the integer, $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm\infty\}$.

Hence, we may write

$$r_0\exp\left(\left[\frac{iy_0+j\hat{x}_0+k\hat{y}_0}{\sqrt{y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right]\Theta_0\right) = \int_a^b f(q(t))\,dt \quad +$$

$$\therefore r_0 = \exp\left(-\left[\frac{iy_0+j\hat{x}_0+k\hat{y}_0}{\sqrt{y_0^2+\hat{x}_0^2+\hat{y}_0^2}}\right]\Theta_0\right)\int_a^b f(q(t))\,dt$$

$$\therefore r_0 = \int_a^b \exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t))\, dt.$$

† Note that the quaternion product on the L.H.S of this equation is <u>commutative in multiplication</u> since $r_0 \in \mathbb{R}$.

Furthermore, by setting the function,

$$\exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t)) = \phi(q(t))$$

$$= Re(\phi(q(t))) + i\, Im_i(\phi(q(t))) + j\, Im_j(\phi(q(t))) + k\, Im_k(\phi(q(t))),$$

it therefore follows that the definite integral,

$$r_0 = \int_a^b Re(\phi(q(t)))\, dt + i\int_a^b Im_i(\phi(q(t)))\, dt + j\int_a^b Im_j(\phi(q(t)))\, dt + k\int_a^b Im_k(\phi(q(t)))\, dt,$$

and, since $r_0 \in \mathbb{R}$, then

$$Im_i(\phi(q(t))) = Im_j(\phi(q(t))) = Im_k(\phi(q(t))) = 0$$

$$\Longrightarrow r_0 = \int_a^b Re(\phi(q(t)))\, dt$$

$$= \int_a^b Re\left[\exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t))\right] dt.$$

From the calculus of functions of a single real variable, we note that the inequality,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx ,$$

is always valid whenever $f(x) \leq g(x)$, $\forall x \in [a, b]$.

Since the real part,

$$Re\left[\exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t))\right]$$

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$$\leq \left|\exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t))\right|$$

$$= \left|\exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right)\right| \left|f(q(t))\right|$$

$$= \left|f(q(t))\right| ,$$

we perceive in an analogous manner to the preceding integral inequality that

$$r_0 = \int_a^b \exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t))\,dt$$

$$= \int_a^b Re\left[\exp\left(-\left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}}\right]\Theta_0\right) f(q(t))\right] dt$$

$$\leq \int_a^b |f(q(t))|\,dt\ , \quad \forall t \in [a,b].$$

Finally, in view of the fact that

$$r_0 \geq 0 \Longrightarrow r_0 = |r_0| = \left| \int_a^b \exp\left(- \left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}} \right] \Theta_0 \right) f(q(t))\,dt \right|$$

$$= \left| \exp\left(- \left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}} \right] \Theta_0 \right) \int_a^b f(q(t))\,dt \right|$$

$$= \left| \exp\left(- \left[\frac{iy_0 + j\hat{x}_0 + k\hat{y}_0}{\sqrt{y_0^2 + \hat{x}_0^2 + \hat{y}_0^2}} \right] \Theta_0 \right) \right| \left| \int_a^b f(q(t))\,dt \right|$$

$$= \left| \int_a^b f(q(t))\,dt \right| \ ,$$

we deduce that the inequality,

$$\left| \int_a^b f(q(t))\,dt \right| \leq \int_a^b |f(q(t))|\,dt\ ,$$

similarly exists, $\forall t \in [a,b]$, as required. Q.E.D.

Definition DII-17.

Let there exist a smooth arc, $\mathcal{C}$, which is defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t)\ , \quad \forall t \in [a,b].$$

We accordingly express the length, L_a^b, over the interval, $[a,b]$, of such an arc by the integral formula –

$$L_a^b = \int_a^b \left|\frac{d}{dt}[q(t)]\right| dt.$$

Definition DII-18.

Let there exist a smooth arc, C, which is defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

Let there also exist another smooth arc, $-C$, whose configuration is such that

(a) the definitive equation for this arc is given by

$$q(-t) = x(-t) + iy(-t) + j\hat{x}(-t) + k\hat{y}(-t), \ \forall t \in [-b,-a]$$

AND

(b) the definite integral,

$$\int_{-C} f(q)\,dq = \int_{-b}^{-a} f(q(-t))\left(-\frac{d}{dt}[q(-t)]\right) dt,$$

provided that the quaternion hypercomplex function, $f(q(-t))(-\frac{d}{dt}[q(-t)])$, is likewise integrable with respect to the real parameter 't'.

Theorem TII-19.

Let there exist two single-valued quaternion hypercomplex functions, $f(q)$ and $g(q)$, which are accordingly restricted to a smooth arc, C, thus defined by the equation,

$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t)$, $\forall t \in [a, b]$.

In the circumstances, it may be proven that the integral formulae,

(a) $\int [f(q) + g(q)](dq)_C = \int f(q)(dq)_C + \int g(q)(dq)_C$,

(b) $\int_C [f(q) + g(q)]\, dq = \int_C f(q)\, dq + \int_C g(q)\, dq$,

are always valid, provided that the functions, $f(q(t))\frac{d}{dt}[q(t)]$ and $g(q(t))\frac{d}{dt}[q(t)]$, are likewise integrable with respect to the real parameter 't'.

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PROOF:-

We initially recall from the previously established definitions on quaternion integrals that

(i) the indefinite integral,

$$\int \phi(q)(dq)_C = \int \phi(q(t))\frac{d}{dt}[q(t)]\, dt$$

AND

(ii) the definite integral,

$$\int_C \phi(q)\,dq = \int_a^b \phi(q(t))\frac{d}{dt}[q(t)]\,dt,$$

provided that the function, $\phi(q) = \phi(q(t))$, is integrable with respect to 't' over the interval, $[a,b]$.

(a) Hence, in an analogous manner to Eq.(i), we perceive that the indefinite integral,

$$\begin{aligned}\int[f(q)+g(q)](dq)_c &= \int(f+g)(q)(dq)_c \\ &= \int(f+g)(q(t))\frac{d}{dt}[q(t)]\,dt \\ &= \int[f(q(t))+g(q(t))]\frac{d}{dt}[q(t)]\,dt \\ &= \int\left[f(q(t))\frac{d}{dt}[q(t)]+g(q(t))\frac{d}{dt}[q(t)]\right]dt \\ &= \int f(q(t))\frac{d}{dt}[q(t)]\,dt+\int g(q(t))\frac{d}{dt}[q(t)]\,dt,\end{aligned}$$

bearing in mind the provisions of Theorem TII-14. However, since the indefinite integrals,

$$\int f(q(t))\frac{d}{dt}[q(t)]\,dt = \int f(q)(dq)_c,$$

$$\int g(q(t))\frac{d}{dt}[q(t)]\,dt = \int g(q)(dq)_c,$$

by virtue of Eq.(i), it automatically follows that the integral formula,

$$\int[f(q)+g(q)](dq)_c = \int f(q)(dq)_c + \int g(q)(dq)_c,$$

is likewise valid, $\forall t \in [a, b]$, as required. Q.E.D.

(b) Similarly, in an analogous manner to Eq. (ii), we perceive that the definite integral, To be continued via the author's next submission, namely -

"__A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 8/10__."

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