“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 9/10.”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 9/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$\therefore \int_{C} f(q)\,dq + \int_{-\Gamma_1} f(q)\,dq + \int_{-\Gamma_2} f(q)\,dq + \ldots\ldots\ldots\ldots + \int_{-\Gamma_M} f(q)\,dq = 0.$$

Finally, since the entire oriented boundary of R, namely —

$$\mathbf{B} = C \cup -\Gamma_1 \cup -\Gamma_2 \cup \ldots\ldots\ldots\ldots\ldots \cup -\Gamma_M,$$

as stated in the preamble to the proof of this theorem, we therefore let the definite integral,

$$\int_{\mathbf{B}} f(q)\,dq = \int_{C} f(q)\,dq + \int_{-\Gamma_1} f(q)\,dq + \int_{-\Gamma_2} f(q)\,dq + \ldots\ldots\ldots\ldots + \int_{-\Gamma_M} f(q)\,dq,$$

and hence the preceding integral sum can likewise be written in the form —

$\int_{\mathbf{B}} f(q)\,dq = 0$, as required. <u>Q.E.D.</u>

<u>Theorem TII-29.</u>

Let q_0 and q be any two points on a simply connected domain,

$$D \subseteq \Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle,$$

throughout which the quasi-complex quaternion function,

$$f(s) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic. Hence, it may be shown that, if C_1 and C_2 are two contours which connect q_0 to q and lie entirely within D, then the existence of the definite integral,

$$F(q) = \int_{q_0}^{q} f(s)\,ds,$$

on the domain, D, also implies that the derivative,

$$\frac{d}{dq}(F(q)) = f(q),$$

likewise exists throughout the same domain.

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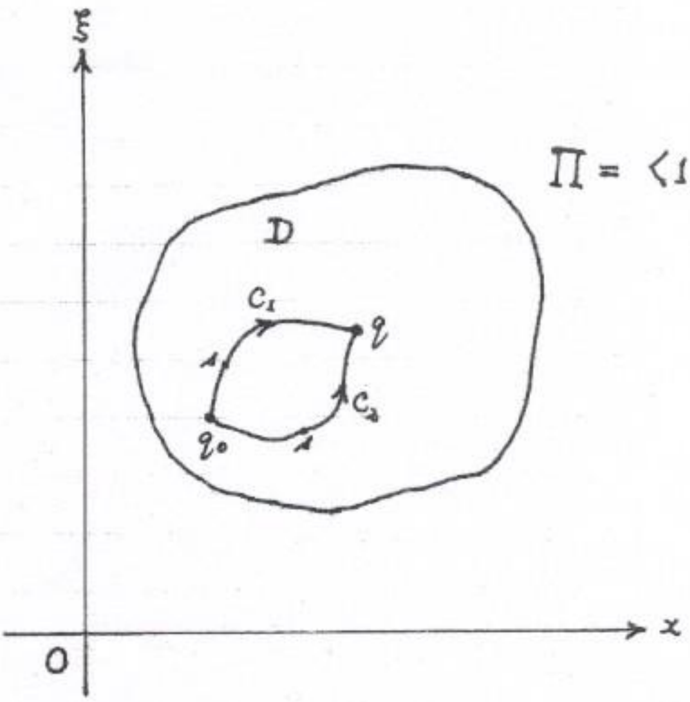
PROOF:-



$$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$$

Fig. 1.

From Fig. 1 above, we initially recognise that, since C_1 and C_2 are two arbitrary contours which both connect the points, q and q_0, and lie entirely within the simply connected domain, D, the union of these contours, $C = C_1 \cup -C_2$, accordingly represents a simple closed contour within this domain. Furthermore, if the quasi-complex function,

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$$f(s) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic throughout D, it is therefore analytic at every point on and interior to the simple closed contour, C, whereupon it follows that the definite integral,

$$\int_C f(s)\,ds = 0,$$

likewise exists by virtue of Theorem TII–25 and hence we may write

$$\int_C f(s)\,ds = \int_{C_1} f(s)\,ds + \int_{-C_2} f(s)\,ds$$

$$= \int_{C_1} f(s)\,ds - \int_{C_2} f(s)\,ds = 0,$$

in view of the fact that the parametric quaternion function, $s = s(t)$, represents any point on the simple closed contour, $C = C_1 \cup -C_2$. Subsequently, we let the definite integral,

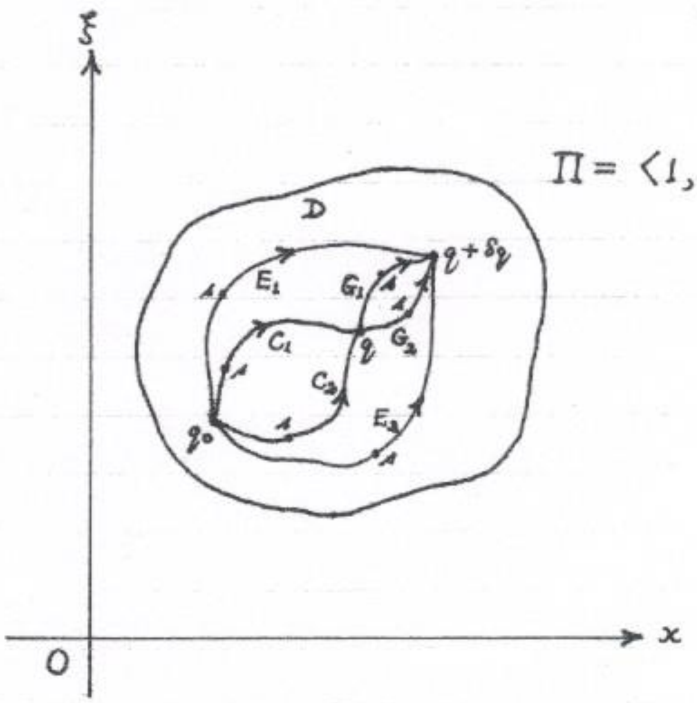
$$\int_{C_1} f(s)\,ds = \int_{C_2} f(s)\,ds = \int_{q_0}^{q} f(s)\,ds,$$

bearing in mind the criteria of Definition DII-23.

Now let us postulate the existence of an integral function,

$$F(q) = \int_{q_0}^{q} f(s)\,ds \implies F(q+\delta q) = \int_{q_0}^{q+\delta q} f(s)\,ds,$$

whose upper limit of integration, $s = q+\delta q$, is located somewhere in the neighborhood of q, as indicated in Fig. 2 below.



$$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \rangle$$

<u>N.B.</u> The contours, E_1, $-C_1$, and the smooth arc, $-G_1$, are joined contiguously so as to form a simple closed contour, $\mathcal{E} = E_1 \cup -G_1 \cup -C_1$, thus implying that the definite integral,

$$\int_{\mathcal{E}} f(s)\,ds = 0.^{\dagger}$$

Fig. 2.

Henceforth, it is evident that the difference function,

$$F(q+\delta q) - F(q) = \int_{q_0}^{q+\delta q} f(s)\,ds - \int_{q_0}^{q} f(s)\,ds .$$

Moreover, from Fig. 2, we likewise deduce that the definite integrals,

$$\int_{G_1} f(s)\,ds = \int_{G_2} f(s)\,ds = \int_{q}^{q+\delta q} f(s)\,ds \;;$$

$$\int_{E_1} f(s)\,ds = \int_{E_2} f(s)\,ds = \int_{q_0}^{q+\delta q} f(s)\,ds ,$$

by virtue of Theorem TII-25 and Definition DII-23, and since the integral sum,

$$\int_{E_1} f(s)\,ds + \int_{-G_1} f(s)\,ds + \int_{-C_1} f(s)\,ds = 0,^{\dagger}$$

it therefore follows that the definite integral,

$$\int_{-G_1} f(s)\,ds = -\left(\int_{E_1} f(s)\,ds + \int_{-C_1} f(s)\,ds \right)$$

$$\therefore\ -\int_{G_1} f(s)\,ds = -\left(\int_{E_1} f(s)\,ds - \int_{C_1} f(s)\,ds\right)$$

$$\therefore\ \int_{G_1} f(s)\,ds = \int_{E_1} f(s)\,ds - \int_{C_1} f(s)\,ds$$

$$\therefore\ {}_{q}\!\int^{q+\delta q} f(s)\,ds = {}_{q_0}\!\int^{q+\delta q} f(s)\,ds - {}_{q_0}\!\int^{q} f(s)\,ds$$

$$= F(q+\delta q) - F(q).$$

Similarly, we deduce that the function,

$$f(q) = f(q)\,\delta q/\delta q = (f(q)/\delta q)\,\delta q = (f(q)/\delta q)(q+\delta q - q)$$

$$= (f(q)/\delta q)\left[s\right]_{q}^{q+\delta q} = (f(q)/\delta q)\ {}_{q}\!\int^{q+\delta q} 1\,ds = (1/\delta q)\ {}_{q}\!\int^{q+\delta q} f(q)\,ds,$$

and hence we perceive that the new difference function,

$$\frac{F(q+\delta q) - F(q)}{\delta q} - f(q) = (1/\delta q)\ {}_{q}\!\int^{q+\delta q} f(s)\,ds - (1/\delta q)\ {}_{q}\!\int^{q+\delta q} f(q)\,ds$$

$$= (1/\delta q)\ {}_{q}\!\int^{q+\delta q} (f(s) - f(q))\,ds.$$

However, since the quaternion function, $f(s)$, is analytic throughout D, it is therefore analytic at every point, $s = s(t)$, located on the arbitrary smooth arc, G_1. Consequently, the first derivative of $f(s)$, with respect to 's',

$$\frac{d}{ds}(f(s)) = \left[\frac{d}{ds}\right]_{G_1}(f(s))$$

$$= \frac{d}{dt}[f(s(t))] / \frac{d}{dt}[s(t)],$$

and hence, if the real and imaginary parts of the parametric functions, $f(s(t))$ and $s(t)$, are differentiable at $t = T$, they are also continuous at $t = T$ and thus, by virtue of the previously established theorems on the limits and continuity of quaternion hypercomplex functions, we deduce that the function, $f(s) = f(s(t))$, is likewise continuous at some point, $s = s(t) = s(T) = q$, in other words —

$$\lim_{s \to q}[f(s)] = f(q) \Longrightarrow |f(s) - f(q)| < \epsilon, \text{ whenever}$$

$$|s - q| < \delta, \ \forall \delta, \epsilon > 0.$$

Clearly, the modulus,

$$\left|\frac{F(q+\delta q) - F(q)}{\delta q} - f(q)\right| = \left|(1/\delta q) \int_q^{q+\delta q} (f(s) - f(q))\, ds\right|$$

$$= |1/\delta q| \left| \int_q^{q+\delta q} (f(s) - f(q))\, ds \right|$$

$$= 1/|\delta q| \left| \int_T^{T+\delta T} (f(s(t)) - f(q)) \frac{d}{dt}[s(t)]\, dt \right| \quad (q = s(T) \ \& \ q + \delta q = s(T+\delta T))$$

$$\leq 1/|\delta q| \int_T^{T+\delta T} \left|(f(s(t)) - f(q)) \frac{d}{dt}[s(t)]\right| dt$$

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$$= 1/|\delta q| \int_{T}^{T+\delta T} |f(s(t)) - f(q)| |\tfrac{d}{dt}[s(t)]| \, dt.$$

But, since we had already established that both the upper limit of integration,

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$$s = q + \delta q \Longrightarrow s - q = \delta q \Longrightarrow |\delta q| = |s - q| < \delta,$$

and also the modulus,

$$|f(s) - f(q)| < \epsilon, \quad \forall \delta, \epsilon > 0,$$

it therefore follows that the definite integral,

$$\int_{T}^{T+\delta T} |f(s(t)) - f(q)| |\tfrac{d}{dt}[s(t)]| \, dt$$

$$< \int_{T}^{T+\delta T} \epsilon |\tfrac{d}{dt}[s(t)]| \, dt = \epsilon \int_{T}^{T+\delta T} |\tfrac{d}{dt}[s(t)]| \, dt = \epsilon L,$$

insofar as the definite integral,

$$L = \int_{T}^{T+\delta T} |\tfrac{d}{dt}[s(t)]| \, dt,$$

accordingly represents the length of the smooth arc, G_1, bounded by the limits, T and $T + \delta T$, having regard to the provisions of Definition DII-17. Moreover, we observe that our choice of smooth arc, G_1, between the limits, q and $q + \delta q$, is quite arbitrary (viz. Fig. 2) and hence, by letting this particular smooth arc be a rectilinear segment which connects these endpoints, we subsequently deduce that the definite integral,

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$$L = \int_{T}^{T+\delta T} \left|\tfrac{d}{dt}[s(t)]\right| dt = |\delta q|,$$

whereupon the modulus,

$$\left|\frac{F(q+\delta q) - F(q)}{\delta q} - f(q)\right| \leqslant 1/|\delta q| \int_{T}^{T+\delta T} |f(s(t)) - f(q)|\left|\tfrac{d}{dt}[s(t)]\right| dt$$

$$< 1/|\delta q|(\epsilon|\delta q|) = \epsilon.$$

Finally, we perceive that the simultaneous occurrence of the inequalities,

$$\left|\frac{F(q+\delta q) - F(q)}{\delta q} - f(q)\right| < \epsilon \;\&\; |\delta q| < \delta, \quad \forall \delta, \epsilon > 0,$$

is a necessary condition for the existence of the limit,

$$\lim_{\delta q \to 0}\left[\frac{F(q+\delta q) - F(q)}{\delta q}\right] = \frac{d}{dq}(F(q)) = f(q), \text{ as required. } \underline{\underline{\text{Q.E.D.}}}$$

<u>Definition DII-24.</u>

Let there exist a simply connected domain,

$$D \subseteq \Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle,$$

throughout which the quasi-complex quaternion function,

$$f(s) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is _analytic_. A function, $F(q)$, defined on D, is said to be the _anti-derivative or indefinite integral of 'f', with respect to 'q'_, in other words –

$$F(q) = \int f(q)\,dq,$$

if and only if the derivative,

$$\frac{d}{dq}(F(q)) = f(q),$$

likewise exists throughout the same domain.

Theorem TII–30.

Let there exist a simply connected domain,

$$D \subseteq \Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle,$$

throughout which the quasi-complex quatenion function,

$$f(s) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic. Subsequently, it may be shown that the definite integral,

$$\int_{\alpha}^{\beta} f(q)\,dq = F(\beta) - F(\alpha) = \left[F(q)\right]_{\alpha}^{\beta},$$

likewise exists provided that the function, $f(q)$, is also analytic throughout the same domain.

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PROOF :-

From the preceding Theorem TII-29, we recall the existence of the integral function,

$$F(q) = \int_{q_0}^{q} f(s)\,ds,$$

on a simply connected domain, D, which implies that the quasi-complex function, $f(s)$, is also analytic throughout the same domain. Consequently, by performing a dummy substitution of variables, we perceive that the definite integrals,

$$F(Q) = \int_{q_0}^{Q} f(s)\,ds = \int_{q_0}^{Q} f(q)\,dq,$$

$$F(Q+\delta Q) = \int_{q_0}^{Q+\delta Q} f(s)\,ds = \int_{q_0}^{Q+\delta Q} f(q)\,dq,$$

likewise exist, if and only if the function, $f(q)$, is analytic throughout D. Furthermore, by setting the upper limits of integration, $Q = \alpha$ and $Q + \delta Q = \beta$, respectively, it also follows that

$$F(\alpha) = \int_{q_0}^{\alpha} f(q)\,dq \quad \& \quad F(\beta) = \int_{q_0}^{\beta} f(q)\,dq \implies$$

$$F(\beta) - F(\alpha) = \int_{q_0}^{\beta} f(q)\,dq - \int_{q_0}^{\alpha} f(q)\,dq .$$

However, since it has already been established from the preceding Theorem TII-29 that the integral difference,

$$\int_{q_0}^{q+\delta q} f(s)\,ds - \int_{q_0}^{q} f(s)\,ds = \int_{q}^{q+\delta q} f(s)\,ds \implies$$

$$\int_{q_0}^{Q+\delta Q} f(q)\,dq - \int_{q_0}^{Q} f(q)\,dq = \int_{Q}^{Q+\delta Q} f(q)\,dq ,$$

after performing another dummy substitution of variables, we therefore deduce that the integral difference,

$$\int_{q_0}^{\beta} f(q)\,dq - \int_{q_0}^{\alpha} f(q)\,dq = \int_{\alpha}^{\beta} f(q)\,dq ,$$

and thus the definite integral,

$$F(\beta) - F(\alpha) = \int_{\alpha}^{\beta} f(q)\,dq = \left[F(q)\right]_{\alpha}^{\beta} ,$$

$$\forall q \in D \subseteq \Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle ,$$

having regard to the standard notation for definite integrals from both real and complex variable analysis, as required. <u>Q.E.D.</u>

Theorem TII-31.

Let there exist a positively oriented simple closed contour,

$C \subset \Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$.

Furthermore, let the quasi-complex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = U(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] V(x,\xi),$$

be analytic at every point, $q \in \Pi$, within and on C.

Hence, it may be demonstrated that, if q_0 is some point interior to C, then the definite integral,

$$f(q_0) = -\frac{1}{2\pi}\left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\int_C \left(\frac{f(q)}{q - q_0}\right) dq,$$

likewise exists in terms of the conditions specified above. This particular result shall otherwise be referred to as the <u>quaternion analogue of the Cauchy Integral Formula</u> from complex variable analysis.

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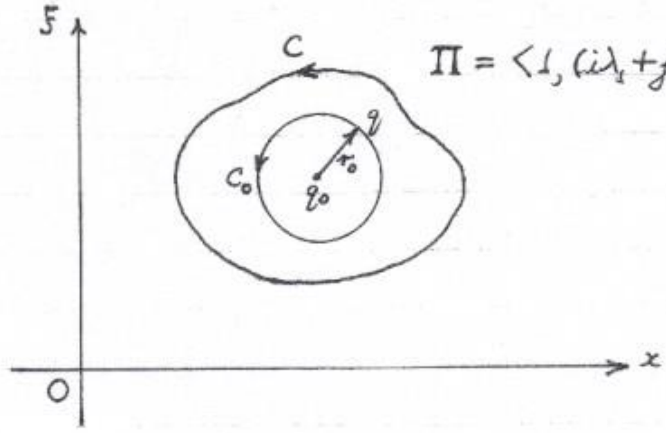
PROOF:-

To facilitate the proof of this theorem, we firstly construct the following diagram:-



$$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$$

Fig. 1.

From the above diagram, it is evident that the positively oriented simple closed contour, C_0, is a circle about the point, q_0, which is accordingly represented by the formula,

$$|q - q_0| = r_0 ,$$

and whose radius, r_0, is sufficiently small such that C_0 is interior to C. Since the quasi-complex function, $f(q)$, is analytic at every point, $q \in \Pi$, within and on C, it therefore follows that the function, $f(q)/(q-q_0)$, must also be analytic at every point, $q \in \Pi$, within and on C, except for the singular point, q_0. Subsequently, in view of the criteria specified in Theorem TII-28, we deduce that the definite integral,

$$\int_B \left(\frac{f(q)}{q-q_0}\right) dq = 0 \qquad (B = C \cup -C_0)$$

$$\therefore \int_C \left(\frac{f(q)}{q-q_0}\right) dq + \int_{-C_0} \left(\frac{f(q)}{q-q_0}\right) dq = 0$$

$$\therefore \int_C \left(\frac{f(q)}{q-q_0}\right) dq - \int_{C_0} \left(\frac{f(q)}{q-q_0}\right) dq = 0$$

$$\therefore \int_C \left(\frac{f(q)}{q-q_0}\right) dq = \int_{C_0} \left(\frac{f(q)}{q-q_0}\right) dq$$

$$= \int_{C_0} \left(\frac{f(q_0) + f(q) - f(q_0)}{q-q_0}\right) dq$$

$$= \int_{C_0} \left(\frac{f(q_0)}{q-q_0} + \frac{f(q) - f(q_0)}{q-q_0}\right) dq$$

$$= \int_{C_0} \left(\frac{f(q_0)}{q-q_0}\right) dq + \int_{C_0} \left(\frac{f(q) - f(q_0)}{q-q_0}\right) dq$$

$$= f(q_0) \int_{C_0} \left(\frac{1}{q-q_0}\right) dq + \int_{C_0} \left(\frac{f(q) - f(q_0)}{q-q_0}\right) dq .$$

From the geometric construction depicted in Fig. 2, it immediately follows that the parametric quaternion function,

$$q = q(\theta) = x_0 + r_0 \cos(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] (y_0 + r_0 \sin(\theta))$$

$$= x_0 + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] y_0 + r_0 \cos(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] r_0 \sin(\theta)$$

$$= q_0 + r_0\cos(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] r_0\sin(\theta) \implies$$

$$q(\theta) - q_0 = r_0\cos(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] r_0\sin(\theta),$$

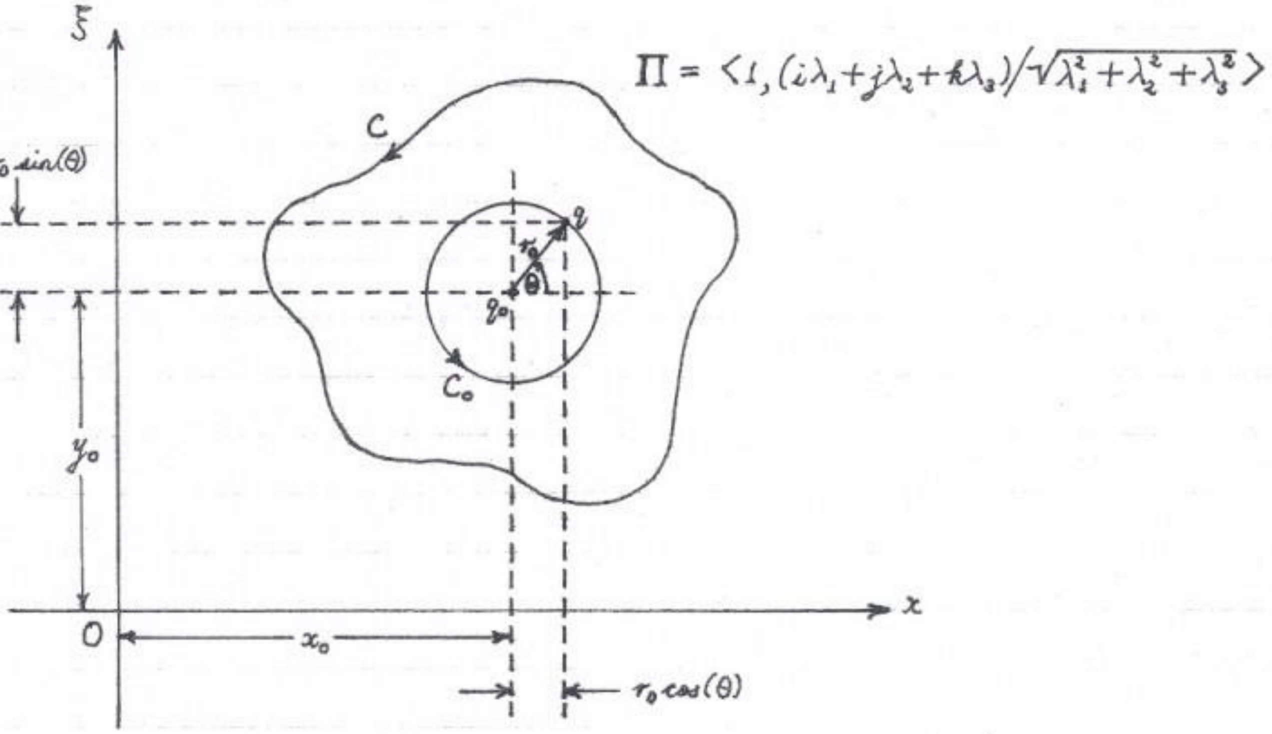


Fig. 2.

and hence the parametric first derivative of this function,

$$\frac{d}{d\theta}[q(\theta)] = -r_0\sin(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] r_0\cos(\theta)$$

$$= \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]^2 r_0 \sin(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] r_0 \cos(\theta)$$

$$= \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\left(r_0 \cos(\theta) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] r_0 \sin(\theta)\right)$$

$$= \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right](q(\theta) - q_0).$$

As a further consequence of the preceding arguments, we similarly perceive that the definite integral,

$$\int_{C_0}\left(\frac{1}{q - q_0}\right) dq = \int_0^{2\pi} \frac{d}{d\theta}[q(\theta)] / (q(\theta) - q_0)\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right](q(\theta) - q_0) / (q(\theta) - q_0)\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] d\theta$$

$$= \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] 2\pi,$$

and hence the definite integral,

$$\int_C \left(\frac{f(q)}{q - q_0}\right) dq = f(q_0)\left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] 2\pi + \int_{C_0}\left(\frac{f(q) - f(q_0)}{q - q_0}\right) dq.$$

Moreover, since the function, $f(q)$, is analytic at every point interior to and on the simple closed contour, C, it is therefore both analytic and continuous at q_0, the centre of C_0, such that the inequality,

$$|f(q) - f(q_0)| < \eta ,$$

exists whenever $|q - q_0| < \delta, \quad \forall \delta, \eta > 0$.

Subsequently, we now let the magnitude of the radius, r_0, be sufficiently small in order to obtain the result,

$$|q(\theta) - q_0| = r_0 < \delta \implies |f(q(\theta)) - f(q_0)| < \eta ,$$

whereupon we deduce that the modulus,

$$\left| \int_{C_0} \left(\frac{f(q) - f(q_0)}{q - q_0} \right) dq \right| = \left| \int_0^{2\pi} \left(\frac{f(q(\theta)) - f(q_0)}{q(\theta) - q_0} \right) \frac{d}{d\theta}[q(\theta)] \, d\theta \right|$$

$$\leqslant \int_0^{2\pi} \left| \left(\frac{f(q(\theta)) - f(q_0)}{q(\theta) - q_0} \right) \frac{d}{d\theta}[q(\theta)] \right| d\theta$$

$$= \int_0^{2\pi} \frac{|f(q(\theta)) - f(q_0)|}{|q(\theta) - q_0|} . \left|\frac{d}{d\theta}[q(\theta)]\right| d\theta$$

$$= \int_0^{2\pi} \frac{|f(q(\theta)) - f(q_0)|}{|q(\theta) - q_0|} . \left| \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{-\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right] (q(\theta) - q_0) \right| d\theta$$

$$= \int_0^{2\pi} \frac{|f(q(\theta)) - f(q_0)|}{|q(\theta) - q_0|} . |q(\theta) - q_0| \, d\theta$$

$$= \int_0^{2\pi} |f(q(\theta)) - f(q_0)|\, d\theta \; < \int_0^{2\pi} \eta\, d\theta \; = \; 2\pi\eta.$$

Observe that the simultaneous occurrence of the inequalities,

$$|q(\theta) - q_0| = r_0 < \delta \;; \quad \left| \int_{C_0} \left(\frac{f(q) - f(q_0)}{q - q_0} \right) dq \right| \; < 2\pi\eta \;, \; \forall \delta, \eta > 0,$$

is the very condition required for the existence of the limit,

$$\lim_{q(\theta) \to q_0} \left[\int_{C_0} \left(\frac{f(q) - f(q_0)}{q - q_0} \right) dq \right] \; = \; 0.$$

However, since the definite integral,

$$\int_{C_0} \left(\frac{f(q) - f(q_0)}{q - q_0} \right) dq \; = Q \,,$$

where Q is some arbitrary quaternion constant, we conclude that such a limit exists if and only if

$$\int_{C_0} \left(\frac{f(q) - f(q_0)}{q - q_0} \right) dq \; = \; Q \; = \; 0,$$

and hence, after making the appropriate algebraic substitution, it likewise follows that the definite integral,

$$\int_C \left(\frac{f(q)}{q-q_0}\right) dq = f(q_0)\left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right] 2\pi \implies$$

$$f(q_0) = -\frac{1}{2\pi}\left[\frac{i\lambda_1 + j\lambda_2 + \lambda_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\right] \int_C \left(\frac{f(q)}{q-q_0}\right) dq, \text{ as required. } \underline{\underline{Q.E.D.}}$$

We conclude our discussion of this topic with the following remarks:-

[1] The integral formula,

$$\int [f(q(t)) + g(q(t))]\, dt = \int f(q(t))\, dt + \int g(q(t))\, dt$$

which was derived by way of Theorem TII-14, is analogous to the integral formula,

$$\int [f(z(t)) + g(z(t))]\, dt = \int f(z(t))\, dt + \int g(z(t))\, dt \quad (2\text{-}13),$$

from complex variable analysis.

[2] The integral formula,

$$\int_a^b [f(q(t)) + g(q(t))]\, dt = \int_a^b f(q(t))\, dt + \int_a^b g(q(t))\, dt,$$

which was derived by way of Theorem TII-15, is analogous to the integral formula,

$$\int_a^b [f(z(t)) + g(z(t))]\, dt = \int_a^b f(z(t))\, dt + \int_a^b g(z(t))\, dt \quad (2\text{-}14),$$

from complex variable analysis.

[3] The integral formulae,

$$q_0 \int f(q(t))\,dt = \int q_0 f(q(t))\,dt \;;$$

$$\left(\int f(q(t))\,dt\right) q_0 = \int f(q(t))\, q_0\,dt ,$$

which were derived by way of Theorem TII–16, are analogous to the integral formula,

$$z_0 \int f(z(t))\,dt = \int z_0 f(z(t))\,dt = \left(\int f(z(t))\,dt\right) z_0 =$$

$$\int f(z(t))\, z_0\,dt \qquad (2\text{–}15),$$

from complex variable analysis.

[4] The integral formulae,

$$q_0 \int_a^b f(q(t))\,dt = \int_a^b q_0 f(q(t))\,dt \;;$$

$$\left(\int_a^b f(q(t))\,dt\right) q_0 = \int_a^b f(q(t))\, q_0\,dt ,$$

which were derived by way of Theorem TII–17, are analogous to the integral formula,

$$z_0 \int_a^b f(z(t))\,dt = \int_a^b z_0 f(z(t))\,dt = \left(\int_a^b f(z(t))\,dt\right) z_0 =$$

$$\int_a^b f(z(t))z_0 \, dt \qquad (2\text{-}16),$$

from complex variable analysis.

[5] The modular inequality,

$$\left|\int_a^b f(q(t))\,dt\right| \leq \int_a^b |f(q(t))|\,dt,$$

which was derived by way of Theorem TII-18, is analogous to the modular inequality,

$$\left|\int_a^b f(z(t))\,dt\right| \leq \int_a^b |f(z(t))|\,dt \qquad (2\text{-}17),$$

from complex variable analysis.

[6] The integral formula,

$$L_a^b = \int_a^b \left|\tfrac{d}{dt}[q(t)]\right|\,dt,$$

which was defined by way of Definition DII-17, is analogous to the integral formula,

$$L_a^b = \int_a^b \left|\tfrac{d}{dt}[z(t)]\right|\,dt \qquad (2\text{-}18),$$

from complex variable analysis.

[7] The integral formula,

$$\int_{-C} f(q)\,dq = \int_{-b}^{-a} f(q(-t))\left(-\frac{d}{dt}[q(-t)]\right)dt,$$

such that C denotes a <u>smooth arc</u>, which was defined by way of Definition DII-18, is analogous to the integral formula,

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\left(-\frac{d}{dt}[z(-t)]\right)dt \qquad (2-19),$$

from complex variable analysis.

[8] The integral formulae,

$$\int [f(q) + g(q)](dq)_C = \int f(q)(dq)_C + \int g(q)(dq)_C\,;$$

$$\int_C [f(q) + g(q)]\,dq = \int_C f(q)\,dq + \int_C g(q)\,dq,$$

such that C denotes a <u>smooth arc</u>, which were derived by way of Theorem TII-19, are analogous to the integral formulae,

$$\int [f(z(t)) + g(z(t))]\frac{d}{dt}[z(t)]\,dt = \int f(z(t))\frac{d}{dt}[z(t)]\,dt + \int g(z(t))\frac{d}{dt}[z(t)]\,dt \qquad (2-20);$$

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz \qquad (2-21),$$

respectively from complex variable analysis.

[9] The integral formulae,

$$q_0 \int f(q)\,(dq)_c = \int q_0 f(q)\,(dq)_c\ ;$$

$$q_0 \int_C f(q)\,dq = \int_C q_0 f(q)\,dq\ ,$$

such that C denotes a <u>smooth arc</u>, which were derived by way of Theorem TII-20, are analogous to the integral formulae,

$$z_0 \int f(z(t))\,\tfrac{d}{dt}[z(t)]\,dt = \int z_0 f(z(t))\,\tfrac{d}{dt}[z(t)]\,dt \qquad (2\text{-}22);$$

$$z_0 \int_C f(z)\,dz = \int_C z_0 f(z)\,dz \qquad (2\text{-}23),$$

respectively from complex variable analysis.

[10] The integral formula,

$$\int_C f(q)\,dq = \sum_{n=1}^{N} \int_{K_n} f(q)\,dq\ ,$$

such that C denotes a <u>contour</u> and $K_n\ (n = 1, 2, \ldots\ldots\ldots, N)$ denotes each of its constituent <u>smooth arcs</u>, which was defined by way of Definition DII-19, is analogous to the integral formula,

$$\int_C f(z)\,dz = \sum_{n=1}^{N} \int_{K_n} f(z)\,dz \qquad (2\text{-}24),$$

from complex variable analysis.

[11] The integral formulae,

$$\int_C [f(q) + g(q)]\, dq = \int_C f(q)\, dq + \int_C g(q)\, dq \ ;$$

$$q_0 \int_C f(q)\, dq = \int_C q_0 f(q)\, dq ,$$

such that C denotes a contour, which were derived by way of Theorem TII-21, are analogous to the integral formulae, (2-21) and (2-23), respectively from complex variable analysis.

[12] The modular inequality,

$$\left| \int_C f(q)\, dq \right| \leqslant \int_a^b \left| f(q(t)) \tfrac{d}{dt}[q(t)] \right| dt ,$$

such that C denotes a smooth arc, which was derived by way of Theorem TII-22, is analogous to the modular inequality,

$$\left| \int_C f(z)\, dz \right| \leqslant \int_a^b \left| f(z(t)) \tfrac{d}{dt}[z(t)] \right| dt \qquad (2-25),$$

from complex variable analysis.

[13] The integral formula,

$$\int_C dq = 0 ,$$

such that C denotes a simple closed contour, which was derived by way of Theorem TII-23, is analogous to the integral formula,

$$\int_C dz = 0 \qquad (2-26),$$

from complex variable analysis.

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[14] The integral formula,

$$\int_C q\,dq = 0,$$

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such that C denotes a <u>simple closed contour</u>, which was derived by way of Theorem TII-24, is analogous to the integral formula,

$$\int_C z\,dz = 0 \qquad (2-27),$$

from complex variable analysis.

[15] The integral formula,

$$\int_C f(q)\,dq = 0,$$

such that C denotes a <u>simple closed contour</u>, which was derived by way of Theorem TII-25, is analogous to the integral formula,

$$\int_C f(z)\,dz = 0 \qquad (2-28),$$

from complex variable analysis.

[16] The integral formula,

$$\int_C f(q)\,dq = \int_\alpha^\beta f(q)\,dq,$$

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such that C denotes a <u>contour</u>, which was defined by way of Definition DII-23, is analogous to the integral formula,

$$\int_C f(z)\,dz = \int_\alpha^\beta f(z)\,dz \qquad (2\text{-}29),$$

from complex variable analysis.

[17] The integral formula,

$$\int_a^b F(t)\,dt = \int_{-b}^{-a} F(-t)\,dt,$$

which was derived by way of Theorem TII-26, is also valid wherever

the function, $F(t) \in \mathbb{R}$ or $F(t) = u(t) + i\,v(t) \in \mathbb{C}$.

[18] The integral formula,

$$\int_{-C} f(q)\,dq = -\int_C f(q)\,dq,$$

such that C denotes a <u>contour</u>, which was derived by way of Theorem TII-27, is analogous to the integral formula,

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz \qquad (2\text{-}30),$$

from complex variable analysis.

[19] The integral formula,

$$\int_B f(q)\,dq = 0,$$

such that $B = C \cup -\Gamma_1 \cup -\Gamma_2 \cup \ldots\ldots\ldots \cup -\Gamma_M$ denotes the entire oriented boundary of a region, R; C denotes a positively oriented simple closed contour and $\{\Gamma_\mu \mid \mu = 1, 2, 3, \ldots\ldots\ldots, M\}$ denotes a set of non-intersecting positively oriented simple closed contours, which was derived by way of Theorem TII-28, is analogous to the integral formula,

$$\int_B f(z)\,dz = 0 \qquad (2\text{-}31),$$

from complex variable analysis.

[20] The definite integral,

$$F(q) = \int_{q_0}^{q} f(s)\,ds \Longrightarrow \frac{d}{dq}(F(q)) = f(q),$$

which was derived by way of Theorem TII-29, is analogous to the definite integral,

$$F(z) = \int_{z_0}^{z} f(s)\,ds \Longrightarrow \frac{d}{dz}(F(z)) = f(z) \qquad (2\text{-}32),$$

from complex variable analysis.

[21] The anti-derivative (ie. indefinite integral),

$$F(q) = \int f(q)\,dq,$$

which was defined by way of Definition DII-24, is analogous to the anti-derivative (i.e. indefinite integral),

$$F(z) = \int f(z)\,dz \qquad (2\text{-}33),$$

from complex variable analysis.

[22] The definite integral,

$$\int_{\alpha}^{\beta} f(q)\,dq = F(\beta) - F(\alpha) = \left[F(q)\right]_{\alpha}^{\beta},$$

which was derived by way of Theorem TII-30, is analogous to the definite integral,

$$\int_{\alpha}^{\beta} f(z)\,dz = F(\beta) - F(\alpha) = \left[F(z)\right]_{\alpha}^{\beta} \qquad (2\text{-}34),$$

from complex variable analysis.

[23] The definite integral,

$$f(q_0) = -\frac{1}{2\pi}\left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\int_C \left(\frac{f(q)}{q - q_0}\right) dq \;,$$

such that C denotes a positively oriented simple closed contour, which was derived by way of Theorem TII-31, is analogous to the definite integral,

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$$f(z_0) = -\frac{i}{2\pi}\int_C \left(\frac{f(z)}{z-z_0}\right) dz \qquad (2-35),$$

from complex variable analysis.

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III. Series Expansions of Quaternion Hypercomplex Functions

Churchill et al. [1] have provided a comprehensive insight into various definitions and theorems pertaining to sequences and series of complex numbers as well as the series expansions of analytic complex valued functions. Hence, the purpose of this section is to elucidate the following concepts, namely –

(a) sequences of quaternion numbers;
(b) series of quaternion numbers;
(c) series expansions of quaternion hypercomplex functions, restricted to smooth arcs embedded in q-space, about a non-singular point.

1. Definition and Subsequent Properties of Sequences of Quaternion Numbers.

In this part of Section III, we will

(a) enunciate one definition and a theorem pertaining to this particular topic

AND

(b) compare the above mentioned definition and theorem with their more familiar analogues from real and complex variable analysis.

Definition DIII-1.

An infinite sequence of quaternion numbers,

$\{q_1, q_2, \ldots\ldots\ldots, q_n, \ldots\ldots\}$,

converges to a limit, q, as $n \to \infty$, in other words –

$\lim_{n \to \infty}(q_n) = q$,

if and only if, for each real number, $\epsilon > 0$, there exists a positive integer, n_0, such that the modular inequality,

$|q_n - q| < \epsilon$,

likewise exists, whenever $n > n_0$.

Theorem TIII-1.

Let there exist an infinite sequence of quaternion numbers,

$q_n = x_n + iy_n + j\hat{x}_n + k\hat{y}_n, \ \forall n \in \{1, 2, 3, \ldots\ldots\ldots, \infty\} = \mathbb{N}$.

Furthermore, let there also exist some arbitrary quaternion number,

$$q = x + iy + j\hat{x} + k\hat{y}.$$

Subsequently, it may be proven that the limit,

$$\lim_{n\to\infty}(q_n) = q,$$

exists, if and only if the real variable limits,

$$\lim_{n\to\infty}(x_n) = x \ , \lim_{n\to\infty}(y_n) = y \ ,$$

$$\lim_{n\to\infty}(\hat{x}_n) = \hat{x} \ , \lim_{n\to\infty}(\hat{y}_n) = \hat{y} \ ,$$

simultaneously exist.

* * *

PROOF:-

From Definition DIII-1, we recall that the existence of the limit,

$$\lim_{n\to\infty}(q_n) = q,$$

is dependant upon the proviso that, for each real number, $\epsilon > 0$, there exists a positive integer, n_0, such that the modular inequality,

$$|q_n - q| < \epsilon,$$

likewise exists, whenever $n > n_0$. However, by writing the quaternion numbers, q_n and q, as

$$q_n = x_n + iy_n + j\hat{x}_n + k\hat{y}_n\,,$$

$$q = x + iy + j\hat{x} + k\hat{y},$$

it automatically follows, after making the appropriate algebraic substitutions, that

$$|(x_n + iy_n + j\hat{x}_n + k\hat{y}_n) - (x + iy + j\hat{x} + k\hat{y})| < \epsilon$$

$$\therefore |(x_n - x) + i(y_n - y) + j(\hat{x}_n - \hat{x}) + k(\hat{y}_n - \hat{y})| < \epsilon \Longrightarrow$$

$$\left.\begin{matrix} |x_n - x| \\ |y_n - y| \\ |\hat{x}_n - \hat{x}| \\ |\hat{y}_n - y| \end{matrix}\right\} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2 + (\hat{x}_n - \hat{x})^2 + (\hat{y}_n - \hat{y})^2}$$

$$= |(x_n - x) + i(y_n - y) + j(\hat{x}_n - \hat{x}) + k(\hat{y}_n - y)| < \epsilon,$$

whenever $n > n_0$, which are precisely the conditions required for the

existence of the limits,

$$\lim_{n\to\infty}(x_n) = x,\ \lim_{n\to\infty}(y_n) = y\,,$$

$$\lim_{n\to\infty}(\hat{x}_n) = \hat{x},\ \lim_{n\to\infty}(\hat{y}_n) = \hat{y}\,.$$

Conversely, the existence of these same limits also implies that, for each real number, $\epsilon > 0$, there exist four positive integers, n_1, n_2, n_3, n_4, such that

(a) $|x_n - x| < \epsilon/4$, whenever $n > n_1$,

(b) $|y_n - y| < \epsilon/4$, whenever $n > n_2$,

(c) $|\hat{x}_n - \hat{x}| < \epsilon/4$, whenever $n > n_3$,

(d) $|\hat{y}_n - \hat{y}| < \epsilon/4$, whenever $n > n_4$.

Hence, by letting the integer,

$$n_0 = \max\{n_1, n_2, n_3, n_4\},$$

we likewise perceive that

$$\left.\begin{array}{l} |x_n - x| \\ |y_n - y| \\ |\hat{x}_n - \hat{x}| \\ |\hat{y}_n - \hat{y}| \end{array}\right\} < \epsilon/4, \text{ whenever } n > n_0,$$

and since the modulus,

$$|x_n + iy_n + j\hat{x}_n + k\hat{y}_n - (x + iy + j\hat{x} + k\hat{y})| =$$

$$|(x_n - x) + i(y_n - y) + j(\hat{x}_n - \hat{x}) + k(\hat{y}_n - \hat{y})| \leqslant$$

$$|x_n - x| + |y_n - y| + |\hat{x}_n - \hat{x}| + |\hat{y}_n - \hat{y}| < \epsilon,$$

whenever $n > n_0$, by virtue of the previously established theorems on quaternion modular inequalities, we thus conclude that the limit,

$$\lim_{n\to\infty}(q_n) = q,$$

simultaneously exists in view of the conditions stipulated above, as required. Q.E.D.

We conclude our discussion of this topic with the following remarks:–

(a) The infinite sequence of quaternion numbers, $\{q_1, q_2, \ldots\ldots, q_n, \ldots\ldots\}$, and its concomitant limit,

$$\lim_{n\to\infty}(q_n) = q,$$

which were defined by way of Definition DIII−1, are analogous to the sequences, $\{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\}$ and $\{z_1, z_2, \ldots\ldots, z_n, \ldots\ldots\}$, as well as their associated limits,

$$\lim_{n\to\infty}(x_n) = x \quad (3-1);$$

$$\lim_{n\to\infty}(z_n) = z \quad (3-2),$$

from real and complex variable analysis respectively.

(b) The dependence of the limit,

$$\lim_{n\to\infty}(q_n) = q,$$

upon the simultaneous existence of the real variable limits,

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$$\lim_{n\to\infty}(x_n) = x;\ \lim_{n\to\infty}(y_n) = y;\ \lim_{n\to\infty}(\hat{x}_n) = \hat{x};\ \lim_{n\to\infty}(\hat{y}_n) = \hat{y},$$

which was proven by way of Theorem TIII-1, is likewise analogous to the complex valued limit [viz. Eq. (3-2)] being dependent upon the simultaneous existence of the real variable limits,

$$\lim_{n\to\infty}(x_n) = x \ \ \& \ \lim_{n\to\infty}(y_n) = y.$$

2. Definition and Subsequent Properties of Series of Quaternion Numbers.

In this part of Section III, we will

(a) enunciate two definitions and three theorems pertaining to this particular topic

AND

(b) compare the above mentioned definitions and theorems with their more familiar analogues from real and complex variable analysis.

Definition DIII-2.

An infinite series of quaternion numbers,

$q_1 + q_2 + \dots\dots\dots + q_n + \dots\dots\dots,$

is said to converge to a limit, S, called the <u>sum</u> of the series, if and only if the sequence of <u>partial sums</u>,

$$\{S_N\} = \{\sum_{n=1}^{N} q_n\},$$

also converges to S, that is to say

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$$\sum_{n=1}^{\infty} q_n = \lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = \lim_{N\to\infty}\left(S_N\right) = S.$$

<u>Theorem TIII-2</u>.

Let there exist an infinite sequence of quaternion numbers,

$$q_n = x_n + iy_n + j\hat{x}_n + k\hat{y}_n, \ \forall n \in \{1, 2, 3, \dots\dots\dots, \infty\} = \mathbb{N}.$$

Furthermore, let there also exist some arbitrary quaternion number,

$$S = X + iY + j\hat{X} + k\hat{Y}.$$

Subsequently, it may be proven that the limit,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = \sum_{n=1}^{\infty} q_n = S,$$

exists, if and only if the real variable limits,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} x_n\right) = X \ , \ \lim_{N\to\infty}\left(\sum_{n=1}^{N} y_n\right) = Y,$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{x}_n\right) = \hat{X} \ , \ \lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{y}_n\right) = \hat{Y},$$

simultaneously exist.

*　　　*　　　*

<u>PROOF</u>:–

From Definition DIII–2, we firstly recall that the partial sum,

$$S_N = \sum_{n=1}^{N} q_n$$
$$= \sum_{n=1}^{N} (x_n + iy_n + j\hat{x}_n + k\hat{y}_n)$$

$$= \sum_{n=1}^{N} x_n + \sum_{n=1}^{N} iy_n + \sum_{n=1}^{N} j\hat{x}_n + \sum_{n=1}^{N} k\hat{y}_n$$
$$= \sum_{n=1}^{N} x_n + i\sum_{n=1}^{N} y_n + j\sum_{n=1}^{N} \hat{x}_n + k\sum_{n=1}^{N} \hat{y}_n$$
$$= X_N + iY_N + j\hat{X}_N + k\hat{Y}_N,$$

such that $X_N = \sum_{n=1}^{N} x_n$; $Y_N = \sum_{n=1}^{N} y_n$; $\hat{X}_N = \sum_{n=1}^{N} \hat{x}_n$; $\hat{Y}_N = \sum_{n=1}^{N} \hat{y}_n$,

$\forall n \in \{1, 2, 3, \ldots\ldots\ldots, \infty\} = \mathbb{N}$.

However, since we also postulated the existence of some arbitrary quaternion number,

$$S = X + iY + j\hat{X} + k\hat{Y},$$

then, in accordance with the preceding Theorem TIII-1, it automatically follows that the limit,

$$\lim_{N\to\infty}(S_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = \sum_{n=1}^{\infty} q_n = S,$$

likewise exists, if and only if the real variable limits,

$$\lim_{N\to\infty}(X_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} x_n\right) = X\ ;$$

$$\lim_{N\to\infty}(Y_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} y_n\right) = Y\ ;$$

$$\lim_{N\to\infty}(\hat{X}_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{x}_n\right) = \hat{X}\ ;$$

$$\lim_{N\to\infty}(\hat{Y}_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{y}_n\right) = \hat{Y},$$

simultaneously exist, as required. Q.E.D.

To be continued via the author's next submission, namely -

*"**<u>A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 10/10</u>.**"*

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