"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 10/10."

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author's previous submission, namely -

"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.",

which has been published under the **'VIXRA' Mathematics subheading:-** ***'Functions and Analysis'***.

II. COPY OF AUTHOR'S ORIGINAL PAPER – PART 10/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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Definition DIII-3.

Let there exist an infinite series of quaternion numbers,

$$q_1 + q_2 + \dots\dots\dots + q_n + \dots\dots\dots,$$

whose partial sum,

$$S_N = \sum_{n=1}^{N} q_n,$$

converges to a limit, S, as $N \to \infty$, in other words,

$$\lim_{N\to\infty}(S_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = S.$$

Henceforth, we shall refer to the algebraic difference between the limit, S, and the partial sum, S_N, as the remainder,

$$R_N = S - S_N.$$

Theorem TIII-3.

Let there exist an infinite series of quaternion numbers,

$$q_1 + q_2 + \dots\dots\dots + q_n + \dots\dots\dots,$$

whose partial sum,

$$S_N = \sum_{n=1}^{N} q_n,$$

converges to a limit, S, as $N \rightarrow \infty$.

In the circumstances, it may prove that the existence of the resultant limit,

$$\lim_{N \rightarrow \infty} (S_N) = S,$$

also implies that the limit,

$$\lim_{N \rightarrow \infty} (R_N) = 0,$$

simultaneously exists.

* * *

PROOF:-

In accordance with Definition DIII-1, we deduce that the limit,

$$\lim_{N \rightarrow \infty} (S_N) = S,$$

exists if and only if, for each real number, $\epsilon > 0$, there exists a positive integer, N_0, such that the modular inequality,

$$|S_N - S| < \epsilon,$$

whenever $N > N_0$.

However, from Definition DIII-3, we also recall that the remainder,

$$R_N = S - S_N,$$

whence it immediately follows that the simultaneous occurrence of the inequalities,

$$|R_N - 0| = |R_N| = |S - S_N| = |S_N - S| < \epsilon \text{ and } N > N_0,$$

are precisely the very conditions deemed necessary for the existence of the limit,

$$\lim_{N \to \infty}(R_N) = 0,$$

as required. Q.E.D.

Theorem TIII-4.

Let there exist an infinite series of quaternion numbers,

$$q_1 + q_2 + \dots\dots + q_n + \dots\dots,$$

which further gives rise to the infinite series of real numbers,

$$|q_1| + |q_2| + \dots\dots + |q_n| + \dots\dots$$

Subsequently, it may be shown that, if the series, $\sum_{n=1}^{\infty}|q_n|$, converges, then the concomitant series, $\sum_{n=1}^{\infty} q_n$, likewise converges and is therefore said to be <u>absolutely convergent</u>.

* * *

<u>PROOF</u>:–

To initiate the proof of this theorem, we recall that the modulus,

$$|q_n| = |x_n + iy_n + j\hat{x}_n + k\hat{y}_n| = \sqrt{x_n^2 + y_n^2 + \hat{x}_n^2 + \hat{y}_n^2} \implies$$

$$|x_n|, |y_n|, |\hat{x}_n|, |\hat{y}_n| \leqslant |q_n|.$$

Consequently, if the series of real numbers,

$$\sum_{n=1}^{\infty}|q_n| = \sum_{n=1}^{\infty}\sqrt{x_n^2 + y_n^2 + \hat{x}_n^2 + \hat{y}_n^2},$$

converges, then, by virtue of the comparison theorem for real number series (<u>viz</u>. Appendix A2), it therefore follows that the four real number

series,

$$\sum_{n=1}^{\infty}|x_n|\,;\ \sum_{n=1}^{\infty}|y_n|\,;\ \sum_{n=1}^{\infty}|\hat{x}_n|\,;\ \sum_{n=1}^{\infty}|\hat{y}_n|,$$

likewise converge. Moreover, in view of the established theorems on real number series, we further deduce that the convergence of these particular series automatically guarantees the convergence of the real number series,

$$\sum_{n=1}^{\infty} x_n \,;\, \sum_{n=1}^{\infty} y_n \,;\, \sum_{n=1}^{\infty} \hat{x}_n \,;\, \sum_{n=1}^{\infty} \hat{y}_n \,,$$

such that we obtain the real variable limits,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} x_n\right) = \sum_{n=1}^{\infty} x_n = X,$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} y_n\right) = \sum_{n=1}^{\infty} y_n = Y,$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{x}_n\right) = \sum_{n=1}^{\infty} \hat{x}_n = \hat{X},$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{y}_n\right) = \sum_{n=1}^{\infty} \hat{y}_n = \hat{Y}.$$

Finally, let there exist a quaternion number,

$$S = X + iY + j\hat{X} + k\hat{Y},$$

whereupon, by virtue of the preceding Theorem TIII-2, we conclude that the limit,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = \sum_{n=1}^{\infty} q_n = S,$$

exists, in other words the series, $\sum_{n=1}^{\infty} q_n$, converges, as required. Q.E.D.

We conclude our discussion of this topic with the following remarks:-

(a) The sequence of partial sums,

$$\{S_N\} = \{\sum_{n=1}^{N} q_n\},$$

and its concomitant limit,

$$\lim_{N\to\infty}(S_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = \sum_{n=1}^{\infty} q_n = S,$$

which were defined by way of Definition DIII-2, are analogous to the sequences of partial sums,

$$\{S_N\} = \{\sum_{n=1}^{N} z_n\} \quad (3-3);$$

$$\{S_N\} = \{\sum_{n=1}^{N} x_n\} \quad (3-4),$$

as well as their associated limits,

$$\lim_{N\to\infty}(S_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} z_n\right) = \sum_{n=1}^{\infty} z_n = S \quad (3-5);$$

$$\lim_{N\to\infty}(S_N) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} x_n\right) = \sum_{n=1}^{\infty} x_n = S \quad (3-6),$$

from real and complex variable analysis respectively.

(b) The dependence of the limit,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) = \sum_{n=1}^{\infty} q_n = S,$$

upon the simultaneous existence of the real variable limits,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} x_n\right) = X;\ \lim_{N\to\infty}\left(\sum_{n=1}^{N} y_n\right) = Y;\ \lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{x}_n\right) = \hat{X};$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} \hat{y}_n\right) = \hat{Y},$$

which was proven by way of Theorem TIII-2, is likewise analogous to the complex valued limit [viz. Eq. (3-5)] being dependent upon the simultaneous existence of the real variable limits,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} x_n\right) = X \;\; \& \;\; \lim_{N\to\infty}\left(\sum_{n=1}^{N} y_n\right) = Y.$$

(c) The remainder,

$$R_N = S - S_N,$$

whose existence was defined by way of Definition DIII-3, is analogous to its counterparts from real and complex variable analysis respectively.

(d) The limit,

$$\lim_{N\to\infty}(R_N) = 0,$$

whose existence was proven by way of Theorem TIII-3, is analogous to its counterparts from real and complex variable analysis respectively.

(e) The dependence of the infinite series,

$$S = \sum_{n=1}^{\infty} q_n,$$

being convergent whenever the infinite series,

$$S^* = \sum_{n=1}^{\infty} |q_n|,$$

converges, which was proven by way of Theorem TIII-4, is analogous to the infinite series,

$$S = \sum_{n=1}^{\infty} z_n \quad \text{and} \quad S = \sum_{n=1}^{\infty} x_n ,$$

being convergent whenever the infinite series,

$$S^* = \sum_{n=1}^{\infty} |z_n| \quad \text{and} \quad S^* = \sum_{n=1}^{\infty} |x_n| ,$$

from real and complex variable analysis respectively converge.

3. The Series Expansions of Quaternion Hypercomplex Functions, restricted to Smooth Arcs embedded in q-Space, about a Non-singular Point.

In this part of Section III, we will

(a) enunciate one theorem pertaining to this particular topic

AND

(b) compare this theorem with its more familiar analogues from real and complex variable analysis.

Theorem TIII-5.

Let there exist a quaternion hypercomplex function,

$$f(q) = u_1(x, y, \hat{x}, \hat{y}) + i v_1(x, y, \hat{x}, \hat{y}) + j u_2(x, y, \hat{x}, \hat{y}) + k v_2(x, y, \hat{x}, \hat{y}),$$

which is restricted to some arbitrary smooth arc, C, denoted by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b].$$

In the circumstances, it may be proven that, if

(a) $q(t_0)$ is a <u>non-singular point</u> located on the smooth arc, C, insofar as $t_0 \in (a,b)$

AND

(b) the real variable parametric functions, $u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))$, $v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))$, $u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))$ and $v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))$, may be individually represented as <u>convergent Taylor series expansions</u> about the point, t_0,

then the parametric function, $f(q(t))$, can likewise be represented as a <u>convergent Taylor series expansion</u> of the form,

$$f(q(t)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left[f(q(t))\right]\right]_{t=t_0} (t-t_0)^n/n!,$$

having a <u>radius of convergence</u>,

$$R = \min\{r_1, r_2, r_3, r_4, t_0-a, b-t_0\},$$

such that

(i) $r_1 \geq 0$ is the <u>radius of convergence</u> for the <u>Taylor series expansion</u> of $u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0;

(ii) $r_2 \geq 0$ is the <u>radius of convergence</u> for the <u>Taylor series expansion</u> of $v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0;

(iii) $r_3 \geq 0$ is the <u>radius of convergence</u> for the <u>Taylor series expansion</u> of $u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0;

(iv) $r_4 \geq 0$ is the <u>radius of convergence</u> for the <u>Taylor series expansion</u> of $v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0.

* * *

<u>PROOF</u>:–

Since, as previously stated in the preamble to this proof, the quaternion function,

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}),$$

is restricted to some arbitrary smooth arc, C, denoted by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \quad \forall t \in [a,b],$$

it therefore follows that its corresponding parametric function,

$$f(q(t)) = u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) + i\, v_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) + j\, u_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) + k\, v_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) .$$

Furthermore, by letting this function be differentiable in 't', we now wish to show that the nth order parametric derivative,

$$\frac{d^n}{dt^n}[f(q(t))] = \frac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + i\frac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + j\frac{d^n}{dt^n}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] + k\frac{d^n}{dt^n}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] ,$$

$\forall n \in \{0, 1, 2, \ldots\ldots\ldots, \infty\}$, upon utilising the principle of mathematical induction.

Hence, for $n = 0$, let the zeroth order parametric derivative,

$$\begin{aligned}\frac{d^0}{dt^0}[f(q(t))] &= f(q(t)) \\ &= \frac{d^0}{dt^0}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + i\frac{d^0}{dt^0}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + \\ &\quad j\frac{d^0}{dt^0}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] + k\frac{d^0}{dt^0}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] ,\end{aligned}$$

bearing in mind that the zeroth order real variable parametric derivatives,

$$\frac{d^0}{dt^0}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] = u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) ;$$

$$\frac{d^0}{dt^0}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] = v_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) ;$$

$$\frac{d^0}{dt^0}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] = u_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) ;$$

$$\frac{d^0}{dt^0}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] = v_2(x(t),y(t),\hat{x}(t),\hat{y}(t)).$$

For $n = 1$, we perceive from the previously established theorems on differentiating quaternion hypercomplex functions that the <u>first order parametric derivative</u>,

$$\frac{d}{dt}[f(q(t))] = \frac{d}{dt}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] + i\frac{d}{dt}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] +$$

$$j\frac{d}{dt}[u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] + k\frac{d}{dt}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))].$$

Subsequently, for $n = 2$, we deduce that the <u>second order parametric derivative</u>,

$$\frac{d^2}{dt^2}[f(q(t))] = \frac{d}{dt}\left(\frac{d}{dt}[f(q(t))]\right)$$

$$= \frac{d}{dt}\left(\frac{d}{dt}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right) + i\frac{d}{dt}\left(\frac{d}{dt}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right) +$$

$$j\frac{d}{dt}\left(\frac{d}{dt}[u_2(x(t),y(t),\hat{x}(t),y(t))]\right) + k\frac{d}{dt}\left(\frac{d}{dt}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right)$$

$$= \frac{d^2}{dt^2}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] + i\frac{d^2}{dt^2}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] +$$

$$j\frac{d^2}{dt^2}[u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] + k\frac{d^2}{dt^2}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))],$$

and similarly, for $n = 3$, the <u>third order parametric derivative</u>,

$$\frac{d^3}{dt^3}[f(q(t))] = \frac{d}{dt}\left(\frac{d^2}{dt^2}[f(q(t))]\right)$$

$$= \frac{d}{dt}\left(\frac{d^2}{dt^2}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right) + i\frac{d}{dt}\left(\frac{d^2}{dt^2}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right) +$$

$$j\frac{d}{dt}\left(\frac{d^2}{dt^2}[u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right) + k\frac{d}{dt}\left(\frac{d^2}{dt^2}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right)$$

$$= \frac{d^3}{dt^3}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] + i\frac{d^3}{dt^3}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))] +$$

$$j\frac{d^3}{dt^3}[u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))] + k\frac{d^3}{dt^3}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))].$$

In view of these preceding arguments, we anticipate that, for any integer, $n \in \{0, 1, 2, \ldots\ldots, \infty\}$, the <u>nth order parametric derivative</u>,

$$\frac{d^n}{dt^n}[f(q(t))] = \frac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + i\frac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] +$$

$$j\frac{d^n}{dt^n}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] + k\frac{d^n}{dt^n}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] .$$

Now, in order to prove that this assumption is valid, we accordingly deduce that the <u>(n+1) th order parametric derivative</u>,

$$\frac{d^{(n+1)}}{dt^{(n+1)}}[f(q(t))] = \frac{d}{dt}\left(\frac{d^n}{dt^n}[f(q(t))]\right)$$

$$= \frac{d}{dt}\left(\frac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\right) + i\frac{d}{dt}\left(\frac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\right) +$$

$$j\frac{d}{dt}\left(\frac{d^n}{dt^n}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\right) + k\frac{d}{dt}\left(\frac{d^n}{dt^n}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\right)$$

$$= \frac{d^{(n+1)}}{dt^{(n+1)}}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] + i\frac{d^{(n+1)}}{dt^{(n+1)}}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))] +$$

$$j\frac{d^{(n+1)}}{dt^{(n+1)}}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] + k\frac{d^{(n+1)}}{dt^{(n+1)}}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))] ,$$

$\forall n \in \{0, 1, 2, \ldots\ldots, \infty\}$, as required.

Moreover, by setting $t = t_0 \in (a, b)$, insofar as $q(t_0)$ is a <u>non-singular point</u> located on the smooth arc, C, it therefore follows that the quaternion constant,

$$\left[\frac{d^n}{dt^n}[f(q(t))]\right]_{t=t_0} =$$

$$\left[\frac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\right]_{t=t_0} + i\left[\frac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\right]_{t=t_0} +$$

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$$j\left[\frac{d^n}{dt^n}\left[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))\right]\right]_{t=t_0} + k\left[\frac{d^n}{dt^n}\left[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))\right]\right]_{t=t_0} ,$$

$\forall n \in \{0, 1, 2, \ldots\ldots\ldots, \infty\}$.

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From the calculus of real variable functions, we recall that, if a function, $f(x)$, has 'N' continuous derivatives on the interval, I, and the number, $a \in I$, then

$$f(x) = \sum_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x)$$

$$= \sum_{n=0}^{N-1} \left[\frac{f^{(n)}(x)}{n!}\right]_{x=a} (x-a)^n + R_N(x)$$

$$= \sum_{n=0}^{N-1} \left[f^{(n)}(x)\right]_{x=a} (x-a)^n/n! + R_N(x)$$

$$= \sum_{n=0}^{N-1} \left[\frac{d^n}{dx^n}\left[f(x)\right]\right]_{x=a} (x-a)^n/n! + R_N(x),$$

where the <u>remainder function</u>,

$$R_N(x) = \frac{1}{(N-1)!}\int_a^x f^{(N)}(s)\,(x-s)^{N-1}\,ds.$$

Subsequently, if $R_N(x) \longrightarrow 0$ as $N \longrightarrow \infty$, then the function,

$$f(x) = \lim_{N\to\infty}\left[f(x)\right]$$

$$= \lim_{N\to\infty}\left[\sum_{n=0}^{N-1}\left[\frac{d^n}{dx^n}\left[f(x)\right]\right]_{x=a}(x-a)^n/n! + R_N(x)\right]$$

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$$= \lim_{N\to\infty}\Big[\sum_{n=0}^{N-1}\Big[\tfrac{d^n}{dx^n}[f(x)]\Big]_{x=a}(x-a)^n/n!\Big] + \lim_{N\to\infty}\big[R_N(x)\big]$$

$$= \sum_{n=0}^{\infty}\Big[\tfrac{d^n}{dx^n}[f(x)]\Big]_{x=a}(x-a)^n/n!\,,$$

which is its corresponding <u>Taylor series expansion</u> about the point, a.

Now let there exist four real variable functions, $u_1^*(t)$, $v_1^*(t)$, $u_2^*(t)$ and $v_2^*(t)$, which have 'N' continuous derivatives on the interval, (a, b). Since the number, $t_0 \in (a, b)$, then, in an analogous manner to the preceding formulae,

we deduce that

$$u_1^*(t) = \sum_{n=0}^{N-1}\Big[\tfrac{d^n}{dt^n}[u_1^*(t)]\Big]_{t=t_0}(t-t_0)^n/n! + A_N(t)\,;$$

$$v_1^*(t) = \sum_{n=0}^{N-1}\Big[\tfrac{d^n}{dt^n}[v_1^*(t)]\Big]_{t=t_0}(t-t_0)^n/n! + B_N(t)\,;$$

$$u_2^*(t) = \sum_{n=0}^{N-1}\Big[\tfrac{d^n}{dt^n}[u_2^*(t)]\Big]_{t=t_0}(t-t_0)^n/n! + C_N(t)\,;$$

$$v_2^*(t) = \sum_{n=0}^{N-1}\Big[\tfrac{d^n}{dt^n}[v_2^*(t)]\Big]_{t=t_0}(t-t_0)^n/n! + D_N(t)\,,$$

where the <u>remainder functions</u>,

$$A_N(t) = \frac{1}{(N-1)!}\int_{t_0}^{t} u_1^{*(N)}(s)(t-s)^{N-1}\,ds\;;$$

$$B_N(t) = \frac{1}{(N-1)!}\int_{t_0}^{t} v_1^{*(N)}(s)(t-s)^{N-1}\,ds\;;$$

$$C_N(t) = \frac{1}{(N-1)!}\int_{t_0}^{t} u_2^{*(N)}(s)(t-s)^{N-1}\,ds \;;$$

$$D_N(t) = \frac{1}{(N-1)!}\int_{t_0}^{t} v_2^{*(N)}(s)(t-s)^{N-1}\,ds \;.$$

Furthermore, if $A_N(t) \to 0$; $B_N(t) \to 0$; $C_N(t) \to 0$ and $D_N(t) \to 0$ as $N \to \infty$, then the functions,

$$u_1^*(t) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[u_1^*(t)]\right]_{t=t_0}(t-t_0)^n/n! \;;$$

$$v_1^*(t) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[v_1^*(t)]\right]_{t=t_0}(t-t_0)^n/n! \;;$$

$$u_2^*(t) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[u_2^*(t)]\right]_{t=t_0}(t-t_0)^n/n! \;;$$

$$v_2^*(t) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[v_2^*(t)]\right]_{t=t_0}(t-t_0)^n/n! \;.$$

From the above equations, it is evident that the resultant infinite series are the <u>Taylor series expansions</u> corresponding to each function about the point, t_0. By rewriting these functions respectively as

$$u_1^*(t) = u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) \;;$$

$$v_1^*(t) = v_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) \;;$$

$$u_2^*(t) = u_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) \;;$$

$v_2^*(t) = v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$,

we likewise perceive, after making the appropriate algebraic substitutions, that

$$u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) = \sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n!\ ;$$

$$v_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) = \sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n!\ ;$$

$$u_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) = \sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n!\ ;$$

$$v_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) = \sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n!\ ,$$

and hence the parametric function,

$$\begin{aligned} f(q(t)) = &\sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \ + \\ &i\sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \ + \\ &j\sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \ + \\ &k\sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \end{aligned}$$

$$\begin{aligned} = &\sum_{n=0}^{\infty} \Big[\tfrac{d^n}{dt^n}[u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \ + \\ &\sum_{n=0}^{\infty} i\Big[\tfrac{d^n}{dt^n}[v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \ + \\ &\sum_{n=0}^{\infty} j\Big[\tfrac{d^n}{dt^n}[u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))]\Big]_{t=t_0} (t-t_0)^n/n! \ + \end{aligned}$$

$$\sum_{n=0}^{\infty} k \left[\frac{d^n}{dt^n}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0}(t-t_0)^n/n!$$

$$= \sum_{n=0}^{\infty} \left\{ \begin{array}{l} \left[\frac{d^n}{dt^n}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0}(t-t_0)^n/n! + \\ i\left[\frac{d^n}{dt^n}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0}(t-t_0)^n/n! + \\ j\left[\frac{d^n}{dt^n}[u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0}(t-t_0)^n/n! + \\ k\left[\frac{d^n}{dt^n}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0}(t-t_0)^n/n! \end{array} \right\}^{\dagger}$$

$$= \sum_{n=0}^{\infty} \left\{ \begin{array}{l} \left[\frac{d^n}{dt^n}[u_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0} + \\ i\left[\frac{d^n}{dt^n}[v_1(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0} + \\ j\left[\frac{d^n}{dt^n}[u_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0} + \\ k\left[\frac{d^n}{dt^n}[v_2(x(t),y(t),\hat{x}(t),\hat{y}(t))]\right]_{t=t_0} \end{array} \right\}(t-t_0)^n/n!$$

$$= \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}[f(q(t))]\right]_{t=t_0}(t-t_0)^n/n! \,,$$

which is its corresponding <u>Taylor series expansion</u> about the point, t_0.

From the calculus of real variable functions, we recall that a <u>radius of convergence, r,</u> is commonly associated with any power series of the form,

† It should also be noted that this statement likewise complies with Theorem TIII-2.

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k \text{ and } \sum_{k=0}^{\infty} a_k x^k ,$$

regardless of how the constant coefficients, a_k, have been generated. Clearly, this particular category of infinite series also includes Taylor series expansions corresponding to any differentiable function in 'x'. Hence, we note the following properties which arise from these considerations, namely -

[P1] $r_1 \geq 0$ is the radius of convergence for the Taylor series expansion of $u_1(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0;

[P2] $r_2 \geq 0$ is the radius of convergence for the Taylor series expansion of $v_1(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0;

[P3] $r_3 \geq 0$ is the radius of convergence for the Taylor series expansion of $u_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0;

[P4] $r_4 \geq 0$ is the radius of convergence for the Taylor series expansion of $v_2(x(t), y(t), \hat{x}(t), \hat{y}(t))$ about the point, t_0.

Since the interval,

$$(a, b) = (a, t_0] \cup [t_0, b),$$

we likewise note the following property in relation to its constituent sub-intervals, i.e.

[P5] $t_0 - a > 0$ and $b - t_0 > 0$ are the respective lengths of the sub-intervals, $(a, t_0]$ and $[t_0, b)$, upon which the above functions are differentiable in 't'.

Finally, in order to satisfy all of the properties [P1] $\longrightarrow$ [P5] simultaneously, we conclude that the parametric function, $f(q(t))$, can be represented as a convergent Taylor series expansion of the form,

$$f(q(t)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left[f(q(t))\right]\right]_{t=t_0} (t-t_0)^n/n!\,,$$

whose associated radius of convergence,

$R = \min\{r_1, r_2, r_3, r_4, t_0 - a, b - t_0\}$, as required. Q.E.D.

We conclude our discussion of this topic with the following remarks:-

(a) The convergent Taylor series expansion for the parametric function, $f(q(t))$, which was derived by way of Theorem TIII-5, is analogous to the Taylor series expansion,

$$f(z(t)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left[f(z(t))\right]\right]_{t=t_0} (t-t_0)^n/n! \qquad (3-7),$$

corresponding to the complex valued parametric function,

$$f(z(t)) = u(x(t), y(t)) + i\,v(x(t), y(t)) \qquad (3-8).$$

(b) Since this function is restricted to some arbitrary smooth arc, C, denoted by the equation,

$$z(t) = x(t) + i\,y(t), \quad \forall t \in [a,b] \qquad (3-9),$$

we subsequently note that

(i) $z(t_0)$ is a non-singular point located on the smooth arc, C, insofar as $t_0 \in (a, b)$;

(ii) the real variable parametric functions, $u(x(t), y(t))$ and $v(x(t), y(t))$, must be individually represented as convergent Taylor series expansions about the point, t_0;

(iii) there exists a radius of convergence associated with Eq. (3-7), namely –

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$$R = \min\{r_1, r_2, t_0 - a, b - t_0\} \qquad (3-10),$$

such that $r_1 \geq 0$ and $r_2 \geq 0$ are the respective radii of convergence for the Taylor series expansions of $u(x(t), y(t))$ and $v(x(t), y(t))$ about the point, t_0.

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IV. Appraisal of Results with a View to Further Theoretical Development

In dealing with the subject matter presented throughout the preceding sections of this dissertation, the author has found it necessary to invoke a variety of concepts and techniques which, needless to say, have their origins in the analysis of real and complex variable functions. Granted the origin of these particular concepts and techniques, it is perhaps not surprising that one should accordingly venture to extend this methodology into the realm of quaternions and their corresponding functions – indeed, it could even be said that this dissertation has largely been an exercise with that very objective in mind.

Whilst not wishing to detract from the overall diversity of topics having been elucidated thus far, the author shall nevertheless draw the reader's attention to a number of unresolved problems which therefore warrant further investigation. We briefly outline these problems as follows:–

(1) From Section I we recall that the multi-valued <u>'nth' degree homogeneous polynomial</u> quaternion function,

$$P_n(q) = \begin{cases} a_0 + a_1 q + \dots\dots\dots + a_n q^n \\ \vdots \\ a_0 + q a_1 + \dots\dots\dots + q^n a_n \end{cases} = \begin{cases} P_{n1}(q) \\ \vdots \\ P_{nN_n}(q) \end{cases},$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(P_n) \subseteq \mathbb{H},$$

comprises <u>N_n branch functions</u>, where N_n is a positive integer whose value has yet to be determined. This being the case, we are immediately confronted with two separate problems, namely –

(a) Can we derive a general formula for N_n?

AND

(b) Can we likewise derive a formula or a set of formulae which generates the individual branch functions of any such polynomial function, $P_n(q)$?

(2) From Section I we also recall that the first degree polynomial function,

$$P_1(q) = \begin{cases} a_0 + a_1 q \, , \\ a_0 + q a_1 \end{cases}$$

implies the existence of the function,

$$(P_1(q))^2 = \begin{cases} a_0^2 + a_0 a_1 q + a_1 q a_0 + a_1 q a_1 q \, , \\ a_0^2 + a_0 q a_1 + q a_1 a_0 + q a_1 q a_1 \end{cases}$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(P_1) \subseteq \mathbb{H},$$

which is clearly not a homogeneous polynomial function in terms of Definition DI-7. Subsequently, we deduce that, for any $m \in \{2, 3, 4, \ldots \ldots, \infty\}$, the function, $(P_n(q))^m$, likewise cannot be a homogeneous polynomial function. Hence, the question naturally arises – can we derive a formula or a set of formulae which establishes inhomogeneous polynomial functions as being a distinct class in their own right?

(3) From Section III we recall that the parametric function,

$$f(q(t)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}[f(q(t))]\right]_{t=t_0} (t - t_0)/n! \, , \qquad \forall t \in [a, b],$$

that is to say a convergent Taylor series expansion which satisfies the criteria stipulated in Theorem TIII-5. Granted the restrictions imposed by this particular theorem, one may be tempted to ask if it is possible to represent any differentiable quaternion function, $f(q)$, as a generic convergent Taylor series expansion which is similar in form to those arising from real and complex variable analysis.

* * *

Assuredly, this list of unresolved problems is by no means exhaustive - rather its intended purpose is to show that we have now established a logical precedent whereupon one might carry out further research with a view to extending our knowledge of the analytical properties of quaternion hyper-complex functions.

V. APPENDICES

A1. Derivation of the Real Variable Function,

$$\mathcal{V}(x,y,\hat{x},\hat{y}) = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] = \Theta \in [0,\pi].$$

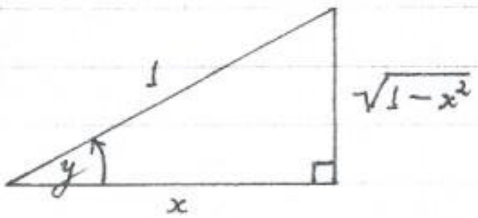
From Sales & Einar Hille [2], we recall that the inverse cosine function, $\cos^{-1}(x)$, is simply the inverse of the cosine function,

$y = \cos(x)$,

such that $x \in [0,\pi]$ and $y \in [-1,1]$, in other words –

$y = \cos^{-1}(x) \Longrightarrow x = \cos(y)$ (i),

such that $y \in [0,\pi]$ and $x \in [-1,1]$. Subsequently, we can interpret this particular result in terms of the following geometrical construction :–



Here, we note that the length of the opposite side with respect to the angle, y, namely –

$$\sqrt{1-x^2} \geqslant 0$$
$$\therefore 1-x^2 \geqslant 0$$
$$\therefore -x^2 \geqslant -1$$
$$\therefore x^2 \leqslant 1$$
$$\therefore -1 \leqslant x \leqslant 1 \Longrightarrow x \in [-1,1]$$, as previously stated.

From Theorem TI-20, we recall that the cosine function,

$$\cos(\vartheta(x,y,\hat{x},\hat{y})) = \frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}},$$

whereupon it follows that the inverse cosine function,

$$\vartheta(x,y,\hat{x},\hat{y}) = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right].$$

In an analogous manner to Eq. (i), we require that

$$\vartheta(x,y,\hat{x},\hat{y}) = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] = \Theta \in [0,\pi],$$

and hence the absolute value,

$$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}$$

$$\therefore -\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2} \leq x \leq \sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}$$

$$\therefore -1 \leq \frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}} \leq 1 \implies \frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}} \in [-1,1],$$

thereby satisfying the initial criteria for this particular function's domain and range.

A2. The Convergence of the Real Number Series,

$$\sum_{n=1}^{\infty}|x_n|;\ \sum_{n=1}^{\infty}|y_n|;\ \sum_{n=1}^{\infty}|\hat{x}_n|;\ \sum_{n=1}^{\infty}|\hat{y}_n|.$$

From Salas & Einar Hille [2], we recall that the Basic Comparison Theorem, which states that any series, $\sum a_k$, with nonnegative terms, a_k, converges if there exists a convergent series, $\sum b_k$, with nonnegative terms, b_k, such that $a_k \leq b_k$, for all sufficiently large values of k. With refrence to Theorem TIII-4, we note the inequality,

$$0 \leq \begin{cases} |x_n| \\ |y_n| \\ |\hat{x}_n| \\ |\hat{y}_n| \end{cases} \leq \sqrt{x_n^2+y_n^2+\hat{x}_n^2+\hat{y}_n^2} = |q_n|,\ \forall n \in \{1,2,3,\ldots\ldots,\infty\}.$$

Subsequently, if the real number series,

$$\sum_{n=1}^{\infty}|q_n| = \sum_{n=1}^{\infty}\sqrt{x_n^2+y_n^2+\hat{x}_n^2+\hat{y}_n^2},$$

converges, then, by virtue of the above stated basic comparison theorem, it therefore follows that the four real number series,

$$\sum_{n=1}^{\infty}|x_n|;\ \sum_{n=1}^{\infty}|y_n|;\ \sum_{n=1}^{\infty}|\hat{x}_n|;\ \sum_{n=1}^{\infty}|\hat{y}_n|,$$

must also converge.

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