“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 2/10.”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 2/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$e^{(iY+j\hat{X}\pm k\sqrt{1-Y^2-\hat{X}^2})n\theta} = \left[e^{(iY+j\hat{X}\pm k\sqrt{1-Y^2-\hat{X}^2})\theta}\right]^n$$

$$= \cos(n\theta) + (iY + j\hat{X} \pm k\sqrt{1-Y^2-\hat{X}^2})\sin(n\theta) ,$$

$\forall n \in \{0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, \pm\infty\}$. Q.E.D.

With the results of the preceding Theorems TI-1, TI-2 and TI-3 firmly established, we are now in a position to formally define the quaternion analogue of Eq. (1-30).

Definition DI-9.

Let there exist an exponential quaternion hypercomplex function, $\exp(q)$, having a domain,

$$\mathrm{dom}(\exp) \subseteq \mathbb{H} .$$

Henceforth, we define $\exp(q)$ as being a single-valued function such that

$$\begin{aligned} \exp(q) &= \exp(x + iy + j\hat{x} + k\hat{y}) \\ &= \exp\left[x + \left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right] \\ &= e^x e^{(iy + j\hat{x} + k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})\sqrt{y^2+\hat{x}^2+\hat{y}^2}} , \end{aligned}$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\exp) \subseteq \mathbb{H}$.

From this definition we further deduce that, by writing the quaternion number,

$$q = x + iy \quad (\hat{x} = \hat{y} = 0) \qquad (1-40a);$$
$$q = x + j\hat{x} \quad (y = \hat{y} = 0) \qquad (1-40b);$$

$$q = x + k\hat{y} \quad (y = \hat{x} = 0) \qquad (1-40c),$$

our definitive formula for the exponential function, exp(q), is respectively reduced to

$$\begin{aligned} \exp(q) &= \exp(x + iy) \\ &= e^{x} e^{(iy/\sqrt{y^2})\sqrt{y^2}} \\ &= e^{x} e^{(iy/|y|)|y|} \\ &= e^{x} e^{iy} \qquad (1-41a); \end{aligned}$$

$$\begin{aligned} \exp(q) &= \exp(x + j\hat{x}) \\ &= e^{x} e^{(j\hat{x}/\sqrt{\hat{x}^2})\sqrt{\hat{x}^2}} \\ &= e^{x} e^{(j\hat{x}/|\hat{x}|)|\hat{x}|} \\ &= e^{x} e^{j\hat{x}} \qquad (1-41b); \end{aligned}$$

$$\begin{aligned} \exp(q) &= \exp(x + k\hat{y}) \\ &= e^{x} e^{(k\hat{y}/\sqrt{\hat{y}^2})\sqrt{\hat{y}^2}} \\ &= e^{x} e^{(k\hat{y}/|\hat{y}|)|\hat{y}|} \\ &= e^{x} e^{k\hat{y}} \qquad (1-41c), \end{aligned}$$

whereupon it is now evident that Eqs. (1 − 41a) to (1 − 41c) are logically compatible with Eq. (1 − 30), insofar as the quaternion basis elements, i, j and k, behave just like the imaginary number, $i = (0,1) \in \mathbb{C}$.

It is also instructive to note that the quaternion product,

$$
\begin{aligned}
e^{iy}e^{j\hat{x}}e^{k\hat{y}} &= (\cos y + i\sin y)(\cos\hat{x} + j\sin\hat{x})(\cos\hat{y} + k\sin\hat{y}) \\
&= [\cos y(\cos\hat{x} + j\sin\hat{x}) + i\sin y(\cos\hat{x} + j\sin\hat{x})](\cos\hat{y} + k\sin\hat{y}) \\
&= [\cos y\cos\hat{x} + j\cos y\sin\hat{x} + i\sin y\cos\hat{x} + ij\sin y\sin\hat{x}](\cos\hat{y} + k\sin\hat{y}) \\
&= \cos y\cos\hat{x}\cos\hat{y} + i\sin y\cos\hat{x}\cos\hat{y} + j\cos y\sin\hat{x}\cos\hat{y} + \\
&\quad k\sin y\sin\hat{x}\cos\hat{y} + k\cos y\cos\hat{x}\sin\hat{y} + ik\sin y\cos\hat{x}\sin\hat{y} + \\
&\quad jk\cos y\sin\hat{x}\sin\hat{y} + k^2\sin y\sin\hat{x}\sin\hat{y}
\end{aligned}
$$

$$
\begin{aligned}
&= \cos y\cos\hat{x}\cos\hat{y} + i\sin y\cos\hat{x}\cos\hat{y} + j\cos y\sin\hat{x}\cos\hat{y} + \\
&\quad k\sin y\sin\hat{x}\cos\hat{y} + k\cos y\cos\hat{x}\sin\hat{y} - j\sin y\cos\hat{x}\sin\hat{y} + \\
&\quad i\cos y\sin\hat{x}\sin\hat{y} - \sin y\sin\hat{x}\sin\hat{y}
\end{aligned}
$$

$$
\begin{aligned}
&= \cos y\cos\hat{x}\cos\hat{y} - \sin y\sin\hat{x}\sin\hat{y} + i(\sin y\cos\hat{x}\cos\hat{y} + \cos y\sin\hat{x}\sin\hat{y}) \\
&\quad + j(\cos y\sin\hat{x}\cos\hat{y} - \sin y\cos\hat{x}\sin\hat{y}) + \\
&\quad k(\sin y\sin\hat{x}\cos\hat{y} + \cos y\cos\hat{x}\sin\hat{y})
\end{aligned}
$$

$$
\neq \cos\left(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sin\left(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right)
$$

$$
= \exp\left(\left(iy + j\hat{x} + k\hat{y}/\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right)\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right)
$$

$$
= \exp(iy + j\hat{x} + k\hat{y}) \qquad (1-42),
$$

$\forall y, \hat{x}, \hat{y} \in \mathbb{R}$, and indeed this same argument also applies to the quaternion products, $e^{iy}e^{k\hat{y}}e^{j\hat{x}}$, $e^{j\hat{x}}e^{iy}e^{k\hat{y}}$, $e^{j\hat{x}}e^{k\hat{y}}e^{iy}$, $e^{k\hat{y}}e^{iy}e^{j\hat{x}}$ and $e^{k\hat{y}}e^{j\hat{x}}e^{iy}$. Since these products are not generally commutative in multiplication, the author subsequently did not utilise any of their properties when it came to enunciating Definition DI-9. His rationale for doing otherwise will be discussed later on in this section.

Our next task will be to derive a formula which explicitly describes the function, $\exp(q)$, in terms of its constituent real and imaginary parts and to that end we shall both state and prove the following theorem :-

Theorem TI-4.

Let there exist an exponential quaternion hypercomplex function, $\exp(q)$, having a domain,

$$\mathrm{dom}(\exp) \subseteq \mathbb{H}.$$

Subsequently, it may be proven that this function can be algebraically expressed as

$$\exp(q) = \exp(x + iy + j\hat{x} + k\hat{y})$$

$$= e^x\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{e^x y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{e^x \hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{e^x \hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(exp) \subseteq \mathbb{H}$.

* * *

<u>PROOF:-</u>

From Definition DI-9, we immediately recall that the exponential function, $exp(q)$, is given by

$$exp(q) = exp(x + iy + j\hat{x} + k\hat{y})$$
$$= e^{x} e^{(iy + j\hat{x} + k\hat{y} / \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}} ,$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(exp) \subseteq \mathbb{H}$.

Similarly, in accordance with Theorems TI-2 and TI-3, we deduce that the function,

$$e^{(iy + j\hat{x} + k\hat{y} / \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}} =$$

$$\cos(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] \sin(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) ,$$

whence it also follows, after algebraically expanding the above expression, that

$$exp(q) = exp(x + iy + j\hat{x} + k\hat{y})$$

$$= e^{x}\left[\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right]$$

$$= e^{x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{e^{x}y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{e^{x}\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{e^{x}\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

$\forall q = x+iy+j\hat{x}+k\hat{y} \in \mathrm{dom}(\exp) \subseteq \mathbb{H}$, as required. <u>Q.E.D.</u>

From Churchill et al. [1], we also perceive that the exponential complex function, $\exp(z)$, is characterised by the following algebraic properties, namely –

$(\exp(z))^n = \exp(nz), \ \forall z \in \mathbb{C}$ (1–43),

$\exp(z_1)\exp(z_2) = \exp(z_1+z_2), \ \forall z_1, z_2 \in \mathbb{C}$ (1–44),

$\exp(z+2\pi i) = \exp(z), \ \forall z \in \mathbb{C}$ (1–45),

$\overline{\exp(z)} = \exp(\bar{z}), \ \forall z \in \mathbb{C}$ (1–46).

Needless to say, the aim of our next five theorems is to demonstrate that suitable quaternion analogues of Eqs. (1–43) to (1–46) can likewise be formulated with the aid of the previously established results.

<u>Theorem TI–5.</u>

Let there exist a <u>single-valued</u> quaternion hypercomplex function,

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}),$$

having a domain, $dom(f) \subseteq \mathbb{H}$.

Hence, it may be shown that the composite function, $\exp \circ f$, possesses an algebraic structure given by the formula,

$$(\exp \circ f)(q) = \exp(f(q))$$
$$= e^{u_1(x,y,\hat{x},\hat{y})} e^{Q(x,y,\hat{x},\hat{y}) R(x,y,\hat{x},\hat{y})},$$

where the auxiliary functions,

$$Q(x,y,\hat{x},\hat{y}) = \frac{i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})}{\sqrt{[v_1(x,y,x,y)]^2 + [u_2(x,y,\hat{x},\hat{y})]^2 + [v_2(x,y,\hat{x},\hat{y})]^2}},$$

$$R(x,y,\hat{x},\hat{y}) = \sqrt{[v_1(x,y,\hat{x},\hat{y})]^2 + [u_2(x,y,\hat{x},\hat{y})]^2 + [v_2(x,y,\hat{x},\hat{y})]^2},$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(f) \subseteq \mathbb{H}.$$

* * *

<u>PROOF</u>:-

From Definition DI-9 we instantaneously recall that the exponential function, $\exp(q)$, is defined as

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$$\exp(q) = e^{x} e^{(iy + j\hat{x} + k\hat{y} / \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}},$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\exp) \subseteq \mathbb{H}$,

and hence our required result analogously follows from this definition in as much as we deduce, after making the appropriate algebraic substitutions, that the function,

$$(\exp \circ f)(q) = \exp(f(q))$$

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$$= \exp(u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}))$$

$$= e^{u_1(x,y,\hat{x},\hat{y})} e^{Q(x,y,\hat{x},\hat{y}) R(x,y,\hat{x},\hat{y})},$$

where the auxiliary functions,

$$Q(x,y,\hat{x},\hat{y}) = \frac{i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})}{\sqrt{[v_1(x,y,\hat{x},\hat{y})]^2 + [u_2(x,y,\hat{x},\hat{y})]^2 + [v_2(x,y,\hat{x},\hat{y})]^2}},$$

$$R(x,y,\hat{x},\hat{y}) = \sqrt{[v_1(x,y,\hat{x},\hat{y})]^2 + [u_2(x,y,\hat{x},\hat{y})]^2 + [v_2(x,y,\hat{x},\hat{y})]^2},$$

$\forall q = x + iy + j\hat{x} + ky \in \mathrm{dom}(f) \subseteq \mathbb{H}$. <u>Q.E.D.</u>

<u>Theorem TI-6.</u>

Let there exist an exponential function,

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$$\exp(q) = e^{x} e^{(iy + j\hat{x} + k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})\sqrt{y^2+\hat{x}^2+\hat{y}^2}},$$

having a domain, $dom(\exp) \subseteq \mathbb{H}$.

Subsequently, it may be proven that the formula,

$$\exp(nq) = (\exp(q))^{n}, \text{ where } n \in \{0, \pm 1, \pm 2, \pm 3, \ldots\ldots, \pm\infty\},$$

is likewise valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\exp) \subseteq \mathbb{H}$.

* * *

PROOF:-

From Theorem TI-5, we initially perceive that

$$\exp(nq) = \exp(nx + iny + jn\hat{x} + kn\hat{y})$$

$$= e^{nx} e^{Q.R},$$

where the auxiliary functions,

$$Q = \frac{iny + jn\hat{x} + kn\hat{y}}{\sqrt{n^2y^2 + n^2\hat{x}^2 + n^2\hat{y}^2}}$$

$$= \frac{iny + jn\hat{x} + kn\hat{y}}{|n|\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}$$

$$= \pm\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}},$$

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$$R = \sqrt{n^2y^2 + n^2\hat{x}^2 + n^2\hat{y}^2} = |n|\sqrt{y^2+\hat{x}^2+\hat{y}^2},$$

and hence we obtain, after making the appropriate substitutions,

$$\exp(nq) = e^{nx}e^{\pm(iy+j\hat{x}+k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})|n|\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$$

$$= e^{nx}e^{(iy+j\hat{x}+k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})n\sqrt{y^2+\hat{x}^2+\hat{y}^2}}.$$

Finally, in accordance with Theorems TI-2, TI-3 and Definition DI-9, it logically follows that

$$\exp(nq) = e^{nx}e^{(iy+j\hat{x}+k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})n\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$$

$$= (e^x)^n\left[e^{(iy+j\hat{x}+k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]^n$$

$$= \left[e^xe^{(iy+j\hat{x}+k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]^n,$$

wherever $n \in \{0, \pm1, \pm2, \pm3, \ldots\ldots\ldots, \pm\infty\}$, bearing in mind that the real variable exponential function, e^x, always commutes in multiplication with any quaternion number as indicated by Definition DI-8, and thus we conclude that the equation,

$$\exp(nq) = \left[e^xe^{(iy+j\hat{x}+k\hat{y}/\sqrt{y^2+\hat{x}^2+\hat{y}^2})\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]^n$$

$$= (\exp(q))^n,$$

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is also a valid result, $\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(exp) \subseteq \mathbb{H}$,
as required. Q.E.D.

Theorem TI-7.

Let there exist two quatenion hypercomplex functions, ϕ_1 and ϕ_2, having a common domain of defintion,

$$dom(\phi_1) = dom(\phi_2) \subseteq \mathbb{H}.$$

Now, if each of these functions is denoted respectively as

$$\phi_1(q) = \mathcal{U}_1(x,y,\hat{x},\hat{y}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\mathcal{V}_1(x,y,\hat{x},\hat{y}),$$

$$\phi_2(q) = \mathcal{U}_2(x,y,\hat{x},\hat{y}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\mathcal{V}_2(x,y,\hat{x},\hat{y}),$$

$$\forall \mathcal{U}_1(x,y,\hat{x},\hat{y}), \ldots\ldots\ldots, \mathcal{V}_2(x,y,\hat{x},\hat{y}) \in \mathbb{R},$$

then, in the circumstances, it may be shown that the formula,

$$exp(\phi_1(q) + \phi_2(q)) = exp(\phi_1(q))\, exp(\phi_2(q)),$$

is also valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\phi_1) = dom(\phi_2) \subseteq \mathbb{H}$.

* * *

PROOF:-

Firstly, by virtue of Theorem TI-5, we perceive that

$$\exp(\phi_1(q)+\phi_2(q)) = \exp\left[\mathcal{U}_1(x,y,\hat{x},\hat{y}) + \mathcal{U}_2(x,y,\hat{x},\hat{y}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right](\mathcal{V}_1(x,y,\hat{x},\hat{y})+\mathcal{V}_2(x,y,\hat{x},\hat{y}))\right]$$

$$= e^{[\mathcal{U}_1(x,y,\hat{x},\hat{y})+\mathcal{U}_2(x,y,\hat{x},\hat{y})]}\, e^{\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right](\mathcal{V}_1(x,y,\hat{x},\hat{y})+\mathcal{V}_2(x,y,\hat{x},\hat{y}))}$$

$$= e^{\mathcal{U}_1(x,y,\hat{x},\hat{y})}\, e^{\mathcal{U}_2(x,y,\hat{x},\hat{y})}\, e^{\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right](\mathcal{V}_1(x,y,\hat{x},\hat{y})+\mathcal{V}_2(x,y,\hat{x},\hat{y}))},$$

bearing in mind that the auxiliary functions corresponding to $\exp(\phi_1(q)+\phi_2(q))$,

$$Q = \pm\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] \text{ and } R = |\mathcal{V}_1(x,y,\hat{x},\hat{y}) + \mathcal{V}_2(x,y,\hat{x},\hat{y})|.$$

Now, from Theorem TI-3, we initially deduced that

$$e^{(iY+j\hat{X}\pm k\sqrt{1-Y^2-\hat{X}^2})\theta_1}\, e^{(iY+j\hat{X}\pm k\sqrt{1-Y^2-\hat{X}^2})\theta_2} =$$

$$e^{(iY+j\hat{X}\pm k\sqrt{1-Y^2-\hat{X}^2})(\theta_1+\theta_2)}, \quad \forall \theta_1, \theta_2 \in \mathbb{R},$$

whereupon, in a completely analogous manner to this result, it therefore follows that

$$e^{\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right](\mathcal{V}_1(x,y,\hat{x},\hat{y})+\mathcal{V}_2(x,y,\hat{x},\hat{y}))} =$$

$$e^{\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\mathcal{V}_1(x,y,\hat{x},\hat{y})}\, e^{\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\mathcal{V}_2(x,y,\hat{x},\hat{y})},$$

since

$$\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]^2 = -1 \,,$$

by virtue of Theorem TI−2, and hence we finally obtain from Theorem TI−5

$$\exp(\phi_1(q) + \phi_2(q)) =$$

$$e^{U_1(x,y,\hat{x},\hat{y})}\, e^{U_2(x,y,\hat{x},\hat{y})}\, e^{\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] V_1(x,y,\hat{x},\hat{y})}\, e^{\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] V_2(x,y,\hat{x},\hat{y})} \;\dagger$$

$$= e^{U_1(x,y,\hat{x},\hat{y}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] V_1(x,y,\hat{x},\hat{y})}\, e^{U_2(x,y,\hat{x},\hat{y}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] V_2(x,y,\hat{x},\hat{y})}$$

$= \exp(\phi_1(q)) \exp(\phi_2(q))$, as required. Q.E.D.

† Once again, we are reminded that the functions, $e^{U_1(x,y,\hat{x},\hat{y})}, e^{U_2(x,y,\hat{x},\hat{y})} \in \mathbb{R}$, and subsequently commute in multiplication with any other quaternion function.

Theorem TI−8.

Let there exist an exponential function,

$$\exp(q) = \exp(x + iy + j\hat{x} + k\hat{y}) \,,$$

having a domain,

$$\mathrm{dom}(\exp) \subseteq \mathbb{H} \,.$$

Subsequently, it may be shown that the formula,

$$\exp\left(q + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi\right) = \exp(q),$$

is also valid, $\forall n \in \{0, \pm 1, \pm 2, \pm 3, \ldots\ldots\ldots, \pm\infty\}$.

* * *

PROOF :-

It is evident that any quaternion variable, q, may be written as

$$q = x + iy + j\hat{x} + k\hat{y}$$

$$= x + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}$$

and therefore, in accordance with Theorem TI-7, we further deduce that

$$\exp\left(q + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi\right) = \exp\left[\begin{array}{l} x + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] \sqrt{y^2 + \hat{x}^2 + \hat{y}^2} \\ + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi \end{array}\right]$$

$$= \exp\left(x + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] \sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\right) \exp\left(\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi\right)$$

$$= \exp(q) \exp\left(\left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] 2n\pi\right)$$

$$= \exp(q)\left(\cos(2n\pi) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] \sin(2n\pi)\right)$$

$$= \exp(q)\left(1 + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] \cdot 0\right)$$

$= \exp(q)\cdot 1$

$= \exp(q)$, $\forall n \in \{0, \pm 1, \pm 2, \pm 3, \ldots\ldots, \pm\infty\}$, as required. <u>Q.E.D.</u>

<u>Theorem TI-9.</u>

Let there exist an exponential function,

$$\exp(q) = \exp(x + iy + j\hat{x} + k\hat{y}),$$

having a domain,

$$\mathrm{dom}(\exp) \subseteq \mathbb{H}.$$

Henceforth, it may be proven that the <u>conjugate</u> of this function,

$$\overline{\exp(q)} = \exp(\overline{q}),\ \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\exp) \subseteq \mathbb{H}.$$

*　　　　*　　　　*

<u>PROOF</u>:-

From Theorem TI-4, we recall that the exponential function, $\exp(q)$, is algebraically denoted by the formula,

$$\exp(q) = \exp(x + iy + j\hat{x} + k\hat{y})$$

$$= e^x \cos(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) + i\left[\frac{e^x y \sin(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2})}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] +$$

$$j\left[\frac{e^{x}\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]+k\left[\frac{e^{x}\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] \quad \text{(i)},$$

whereupon we surmise that its corresponding conjugate,

$$\overline{\exp(q)} = e^{x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{e^{x}y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{e^{x}\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]-k\left[\frac{e^{x}\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right].$$

In an analogous manner to Eq. (i), we likewise deduce that

$$\exp(\bar{q}) = \exp(\overline{x+iy+j\hat{x}+k\hat{y}})$$

$$= \exp(x-iy-j\hat{x}-k\hat{y})$$

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$$= e^{x}\cos\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right) + i\left[\frac{e^{x}(-y)\sin\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right] +$$

$$j\left[\frac{e^{x}(-\hat{x})\sin\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right]+k\left[\frac{e^{x}(-\hat{y})\sin\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right]$$

$$= e^{x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{e^{x}y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{e^{x}\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]-k\left[\frac{e^{x}\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$= \overline{\exp(q)}$, $\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\exp) \subseteq \mathbb{H}$,

as required. Q.E.D.

We conclude our discussion of the exponential quaternion hypercomplex function with the following remarks:-

(a) The results of the preceding Theorems TI-6, TI-8 and TI-9 are completely analogous with Eqs. (1-43), (1-45) and (1-46).

(b) The algebraic properties enunciated in Theorem TI-7, however, are comparable to but not wholly analogous with Eq. (1-44) in view of the restrictions placed on the quaternion functions, $\phi_1(q)$ and $\phi_2(q)$, via the formula,

$$\exp(\phi_1(q) + \phi_2(q)) = \exp(\phi_1(q))\exp(\phi_2(q)) \qquad (1-47).$$

Further examples of this type of behaviour will be revealed in the remaining portions of this section.

4. The Trigonometric Functions.

From complex variable analysis, we recall that the trigonometric functions, namely – $\sin(z)$, $\cos(z)$, $\tan(z)$, $\mathrm{cosec}(z)$, $\sec(z)$ and $\cot(z)$, are ultimately derived from the exponential functions, e^{iz} and e^{-iz}. Indeed, Churchill et al. [1] provide us with suitable derivations of these functions as well as their concomitant algebraic properties. Hence, we will demonstrate in what follows that quaternion analogues of $\sin(z)$, $\cos(z)$, etc., likewise exist and possess properties which are similar to their complex valued counterparts. Before proceeding to derive the definitive formulae for $\sin(q)$ and $\cos(q)$, we shall firstly enunciate the following preliminary theorem :–

Theorem TI–10.

Let there exist two quaternion numbers, $U_1 + QV_1$ and $U_2 + QV_2$, such that

(i) $U_1, U_2, V_1, V_2 \in \mathbb{R}$;

(ii) $Q = iY + j\hat{X} \pm k\sqrt{1 - Y^2 - \hat{X}^2} \Longrightarrow \begin{cases} Q^2 = -1; \\ \hat{X} \in [-\sqrt{1-Y^2}, \sqrt{1-Y^2}]; \\ Y \in [-1, 1]; \\ 1 - Y^2 - \hat{X}^2 \geq 0. \end{cases}$

In the circumstances, we may subsequently prove that the following formulae for the products and quotients of such numbers are always valid, that is to say –

(a) $(U_1 + QV_1)(U_2 + QV_2) = (U_2 + QV_2)(U_1 + QV_1)$,

(b) $\dfrac{U_1 + QV_1}{U_2 + QV_2} = \dfrac{(U_1 + QV_1)(\overline{U_2 + QV_2})}{|U_2 + QV_2|^2} = \dfrac{(\overline{U_2 + QV_2})(U_1 + QV_1)}{|U_2 + QV_2|^2}$.

*　　　　*　　　　*

PROOF:-

(a) By virtue of the established laws of multiplication and distribution for quaternion numbers (viz. Reference [5], Section VI), we deduce that the quaternion product,

$$
\begin{aligned}
(U_1 + QV_1)(U_2 + QV_2) &= U_1(U_2 + QV_2) + QV_1(U_2 + QV_2) \\
&= U_1U_2 + U_1QV_2 + QV_1U_2 + QV_1QV_2 \\
&= U_1U_2 + QU_1V_2 + QV_1U_2 + Q^2V_1V_2 \quad \dagger \\
&= U_1U_2 + Q(U_1V_2 + V_1U_2) - V_1V_2 \\
&= U_1U_2 - V_1V_2 + Q(U_1V_2 + V_1U_2).
\end{aligned}
$$

Similarly, for the quaternion product, $(U_2 + QV_2)(U_1 + QV_1)$, we obtain

$$
\begin{aligned}
(U_2 + QV_2)(U_1 + QV_1) &= U_2(U_1 + QV_1) + QV_2(U_1 + QV_1) \\
&= U_2U_1 + U_2QV_1 + QV_2U_1 + QV_2QV_1 \\
&= U_1U_2 + QV_1U_2 + QU_1V_2 + Q^2V_1V_2 \\
&= U_1U_2 + Q(V_1U_2 + U_1V_2) - V_1V_2 \\
&= U_1U_2 - V_1V_2 + Q(U_1V_2 + V_1U_2) \\
&= (U_1 + QV_1)(U_2 + QV_2),
\end{aligned}
$$

as required. Q.E.D.

† Once again, we are reminded that the real values, U_1, V_1, U_2 and V_2, readily form commutative products with Q, since it has already been established that any real number commutes in multiplication with any quaternion number.

(b) In accordance with the established definitions and theorems on the quotients of quaternion numbers (viz. Reference [5], Section VI), we perceive that

$$\frac{U_1 + QV_1}{U_2 + QV_2} = \begin{cases} \dfrac{(U_1 + QV_1)(\overline{U_2 + QV_2})}{|U_2 + QV_2|^2} & (i). \\[2ex] \dfrac{(\overline{U_2 + QV_2})(U_1 + QV_1)}{|U_2 + QV_2|^2} \end{cases}$$

Furthermore, by virtue of the established definitions and theorems on the <u>conjugates</u> of quaternion numbers (<u>viz.</u> Reference [5], Section VI), it logically follows that

$$\begin{aligned} \overline{U_2 + QV_2} &= \overline{U}_2 + \overline{QV_2} \\ &= U_2 + \overline{Q}V_2 , \end{aligned}$$

since $U_2, V_2 \in \mathbb{R} \Longrightarrow \begin{cases} \overline{U}_2 = U_2 , \ \overline{V}_2 = V_2 ; \\ \overline{QV_2} = \overline{V}_2 \overline{Q} = V_2 \overline{Q} = \overline{Q} V_2 , \end{cases}$

and also $Q\overline{Q} = |Q|^2 = |Q^2| = |-1| = 1 \Longrightarrow \overline{Q} = -Q$.

Finally, from part <u>(a)</u> of this theorem, we likewise deduce that

$$\begin{aligned} (U_1 + QV_1)(\overline{U_2 + QV_2}) &= (U_1 + QV_1)(U_2 + \overline{Q}V_2) \\ &= (U_1 + QV_1)(U_2 - QV_2) \\ &= (U_2 - QV_2)(U_1 + QV_1) \\ &= (U_2 + \overline{Q}V_2)(U_1 + QV_1) \\ &= (\overline{U_2 + QV_2})(U_1 + QV_1) \end{aligned}$$

and hence the quotient represented by Eq. <u>(i)</u>,

$$\frac{U_1+QV_1}{U_2+QV_2} = \frac{(U_1+QV_1)(\overline{U_2+QV_2})}{|U_2+QV_2|^2} = \frac{(\overline{U_2+QV_2})(U_1+QV_1)}{|U_2+QV_2|^2} ,$$

as required. Q.E.D.

Since the complex valued functions, $\sin(z)$ and $\cos(z)$, are respectively defined as

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \quad (1-48);$$

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \cos(x)\cosh(y) - i\sin(x)\sinh(y) \qquad (1-49),$$

we can likewise formulate quaternion analogues of Eqs. (1-48) and (1-49) by means of the next definition and theorem:-

Definition DI-10.

Let there exist two trigonometric quaternion hypercomplex functions, $\sin(q)$ and $\cos(q)$, each having as their respective domains, $dom(\sin)$ and $dom(\cos)$, such that

$$dom(\sin),\ dom(\cos) \subseteq \mathbb{H}.$$

Henceforth, we shall define the sine function,

$$\sin(q) = \frac{\exp(I(y,\hat{x},\hat{y})q) - \exp(-I(y,\hat{x},\hat{y})q)}{2I(y,\hat{x},\hat{y})} ,$$

and the cosine function,

$$\cos(q) = \frac{\exp(I(y,\hat{x},\hat{y})q) + \exp(-I(y,\hat{x},\hat{y})q)}{2},$$

where the auxiliary function,

$$I(y,\hat{x},\hat{y}) = \frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}},$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}$.

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Theorem TI-11.

Let there exist two trigonometric quaternion hypercomplex functions, $\sin(q)$ and $\cos(q)$, whose respective domains,

$\mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}$.

Subsequently, it may be proven that these functions can be algebraically expressed as

$$\text{(a)}\ \sin(q) = \sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cos(x)\, y \sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cos(x)\,\hat{x}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cos(x)\,\hat{y}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

(b) $\cos(q) = \cos(x)\cosh(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) - i\left[\frac{\sin(x)\,y\,\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$

$$j\left[\frac{\sin(x)\,\hat{x}\,\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\sin(x)\,\hat{y}\,\sinh(\sqrt{y^2+\hat{x}^2+\hat{y}^2})}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subseteq \mathbb{H}$.

* * *

PROOF:-

In accordance with Theorems TI-2, TI-3, TI-7 and Definition DI-10, we initially deduce that the exponential functions,

$$\exp(I(y,\hat{x},\hat{y})q) \doteq \exp\left[\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)(x+iy+j\hat{x}+k\hat{y})\right]$$

$$= \exp\left[\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\left(x+\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right]$$

$$= \exp\left[-\sqrt{y^2+\hat{x}^2+\hat{y}^2} + \left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)x\right]$$

$$= e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\left[\cos(x) + \left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(x)\right]$$

$$= e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\cos(x) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\sin(x);$$

$$\exp(-I(y,\hat{x},\hat{y})q) = \exp\left[-\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)(x+iy+j\hat{x}+k\hat{y})\right]$$

$$= \exp\left[-\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\left(x+\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right]$$

$$= \exp\left[\sqrt{y^2+\hat{x}^2+\hat{y}^2} - \left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)x\right]$$

$$= \exp\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\exp\left[-\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)x\right]$$

$$= e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\left[\cos(-x)+\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(-x)\right]$$

$$= e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\left[\cos(x)-\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(x)\right]$$

$$= e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\cos(x)-\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\sin(x) .$$

Hence, after making the appropriate algebraic substitutions and bearing in mind the provisions of Theorem TI-10, we further obtain the formulae -

(a) $\sin(q) = \dfrac{\exp(I(y,\hat{x},\hat{y})q) - \exp(-I(y,\hat{x},\hat{y})q)}{2I(y,\hat{x},\hat{y})}$

****

$$= -\frac{I(y,\hat{x},\hat{y})\left[\exp(I(y,\hat{x},\hat{y})q) - \exp(-I(y,\hat{x},\hat{y})q)\right]}{2}$$

$$= \frac{-I(y,\hat{x},\hat{y})\exp(I(y,\hat{x},\hat{y})q) + I(y,\hat{x},\hat{y})\exp(-I(y,\hat{x},\hat{y})q)}{2}$$

$$= -\tfrac{1}{2}I(y,\hat{x},\hat{y})\exp(I(y,\hat{x},\hat{y})q) + \tfrac{1}{2}I(y,\hat{x},\hat{y})\exp(-I(y,\hat{x},\hat{y})q)$$

$$= -\tfrac{1}{2}\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\left[e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\cos(x) + \left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\sin(x)\right] +$$

$$\tfrac{1}{2}\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\left[e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\cos(x) - \left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\sin(x)\right]$$

$$= -\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\tfrac{1}{2}\cos(x)e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}} + \tfrac{1}{2}\sin(x)e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}} +$$

$$\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\tfrac{1}{2}\cos(x)e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} + \tfrac{1}{2}\sin(x)e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$$

$$= \sin(x)\left(\frac{e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} + e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{2}\right) +$$

$$\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\cos(x)\left(\frac{e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} - e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{2}\right)$$

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$$= \sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cos(x)\,y\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cos(x)\,\hat{x}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cos(x)\,\hat{y}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

insofar as the real variable <u>hyperbolic functions</u>,

$$\sinh(s) = \frac{e^s - e^{-s}}{2} \text{ and } \cosh(s) = \frac{e^s + e^{-s}}{2},$$

such that

$s = \sqrt{y^2+\hat{x}^2+\hat{y}^2}$, as required. <u>Q.E.D</u> ;

(b) $\cos(q) = \left(\exp(I(y,\hat{x},\hat{y})q) + \exp(-I(y,\hat{x},\hat{y})q)\right)/2$

$$= \tfrac{1}{2}\left[e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\cos(x) + \left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\sin(x)\right] +$$

$$\tfrac{1}{2}\left[e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\cos(x) - \left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\sin(x)\right]$$

$$= \cos(x)\left(\frac{e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} + e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{2}\right) +$$

$$\left(\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(x)\left(\frac{e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}} - e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{2}\right)$$

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$$= \cos(x)\left(\frac{e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} + e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{2}\right) -$$

$$\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(x)\left(\frac{e^{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} - e^{-\sqrt{y^2+\hat{x}^2+\hat{y}^2}}}{2}\right)$$

$$= \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\sin(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\sin(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\sin(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

as required, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}$. <u>Q.E.D.</u>

In complex variable analysis, the four trigonometric functions, $\tan(z)$, $\sec(z)$, $\mathrm{cosec}(z)$ and $\cot(z)$, are defined by the following formulae :-

$\tan(z) = \sin(z)/\cos(z) \quad (\cos(z) \neq 0) \qquad (1-50);$

$\sec(z) = 1/\cos(z) \quad (\cos(z) \neq 0) \qquad (1-51);$

$\mathrm{cosec}(z) = 1/\sin(z) \quad (\sin(z) \neq 0) \qquad (1-52);$

$\cot(z) = \cos(z)/\sin(z) \quad (\sin(z) \neq 0) \qquad (1-53).$

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Now, the quaternion analogues of Eqs. (1−50), (1−51), (1−52) and (1−53) can be constructed simply by replacing the letter 'z' with the letter 'q' in each case, as is evident from our next definition, namely –

Definition DI−11.

Let there exist four *trigonometric* quaternion hypercomplex functions, which shall be respectively denoted as tan(q), sec(q), cosec(q) and cot(q). Henceforth, we define

(a) the *tangent* function,

$$\tan(q) = \frac{\sin(q)}{\cos(q)} \qquad (\cos(q) \neq 0),$$

(b) the *secant* function,

$$\sec(q) = \frac{1}{\cos(q)} \qquad (\cos(q) \neq 0),$$

(c) the *cosecant* function,

$$\operatorname{cosec}(q) = \frac{1}{\sin(q)} \qquad (\sin(q) \neq 0),$$

(d) the *cotangent* function,

$$\cot(q) = \frac{\cos(q)}{\sin(q)} \qquad (\sin(q) \neq 0),$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subset \mathbb{H}$.

From Churchill et al. [1], we observe that the complex valued trigonometric functions also give rise to the following characteristic identities :-

$\sin(z + 2\pi) = \sin(z)$ (1-54);

$\sin(z + \pi) = -\sin(z)$ (1-55);

$\cos(z + 2\pi) = \cos(z)$ (1-56);

$\cos(z + \pi) = -\cos(z)$ (1-57);

$\sin^2(z) + \cos^2(z) = 1$ (1-58);

$\tan^2(z) + 1 = \sec^2(z)$ (1-59);

$\cot^2(z) + 1 = \operatorname{cosec}^2(z)$ (1-60);

$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$ (1-61);

$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$ (1-62);

$\sin(-z) = -\sin(z)$ (1-63);

$\cos(-z) = \cos(z)$ (1-64);

$\sin(\frac{\pi}{2} - z) = \cos(z)$ (1-65);

$\sin(2z) = 2\sin(z)\cos(z)$ (1-66);

$$\cos(2z) = \cos^2(z) - \sin^2(z) \quad (1-67).$$

Subsequently, we will demonstrate by means of the next four theorems that quaternion analogues of these identities likewise exist.

Theorem TI-12.

Let there exist two trigonometric quaternion hypercomplex functions, $\sin(q)$ and $\cos(q)$, whose respective domains,

$$\mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}.$$

Henceforth, it may be proven that these functions possess a periodic nature insofar as the formulae,

(a) $\sin(q+2\pi) = \sin(q)$,
(b) $\sin(q+\pi) = -\sin(q)$,
(c) $\cos(q+2\pi) = \cos(q)$,
(d) $\cos(q+\pi) = -\cos(q)$,

are valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}$.

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PROOF:-

We firstly note the validity of the following real variable trigonometric formulae:-

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(i) $\sin(x+2\pi) = \sin(x)\cos(2\pi) + \cos(x)\sin(2\pi)$
$$= \sin(x)\cdot 1 + \cos(x)\cdot 0$$
$$= \sin(x) \ ,$$

(ii) $\sin(x+\pi) = \sin(x)\cos(\pi) + \cos(x)\sin(\pi)$
$$= \sin(x)\cdot -1 + \cos(x)\cdot 0$$
$$= -\sin(x) \ ,$$

(iii) $\cos(x+2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi)$
$$= \cos(x)\cdot 1 - \sin(x)\cdot 0$$
$$= \cos(x) \ ,$$

(iv) $\cos(x+\pi) = \cos(x)\cos(\pi) - \sin(x)\sin(\pi)$
$$= \cos(x)\cdot -1 - \sin(x)\cdot 0$$
$$= -\cos(x) \ .$$

Now, in accordance with Theorem TI−11, we likewise deduce that

(a) $\sin(q+2\pi) = \sin(x + iy + j\hat{x} + k\hat{y} + 2\pi)$
$$= \sin(x + 2\pi + iy + j\hat{x} + k\hat{y})$$

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$$= \sin(x+2\pi)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cos(x+2\pi)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cos(x+2\pi)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cos(x+2\pi)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

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$$= \sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cos(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cos(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cos(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$= \sin(q)$, as required. Q.E.D ;

(b) $\sin(q+\pi) = \sin(x+iy+j\hat{x}+k\hat{y}+\pi)$
$\qquad = \sin(x+\pi+iy+j\hat{x}+k\hat{y})$

$$= \sin(x+\pi)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cos(x+\pi)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cos(x+\pi)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cos(x+\pi)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= -\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{-\cos(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{-\cos(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{-\cos(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= -\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\cos(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\cos(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\cos(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

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$$= -\left[\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cos(x)\,y\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + \right.$$

$$\left. j\left[\frac{\cos(x)\,\hat{x}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cos(x)\,\hat{y}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\right]$$

$= -\sin(q)$, as required. Q.E.D ;

(c) $\cos(q+2\pi) = \cos(x+iy+j\hat{x}+k\hat{y}+2\pi)$

$\qquad = \cos(x+2\pi+iy+j\hat{x}+k\hat{y})$

$$= \cos(x+2\pi)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\sin(x+2\pi)\,y\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\sin(x+2\pi)\,\hat{x}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\sin(x+2\pi)\,\hat{y}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\sin(x)\,y\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\sin(x)\,\hat{x}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\sin(x)\,\hat{y}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$= \cos(q)$, as required. Q.E.D ;

(d) $\cos(q+\pi) = \cos(x+iy+j\hat{x}+k\hat{y}+\pi)$

$\qquad = \cos(x+\pi+iy+j\hat{x}+k\hat{y})$

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$$= \cos(x+\pi)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\sin(x+\pi)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\sin(x+\pi)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\sin(x+\pi)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

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$$= -\cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{-\sin(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{-\sin(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{-\sin(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= -\cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\sin(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\sin(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\sin(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= -\left[\begin{array}{l}\cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\dfrac{\sin(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - \\ j\left[\dfrac{\sin(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\dfrac{\sin(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\end{array}\right]$$

$= -\cos(q)$,

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subseteq \mathbb{H}$, as required. <u>Q.E.D.</u>

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Theorem TI−13.

Let there exist two trigonometric quaternion hypercomplex functions, sin(q) and cos(q), each having as their respective domains,

dom(sin), dom(cos) $\subseteq \mathbb{H}$.

Subsequently, we may prove that the following trigonometric identities, namely −

(a) $\sin^2(q) + \cos^2(q) = 1$,
(b) $\tan^2(q) + 1 = \sec^2(q)$,

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(c) $\cot^2(q) + 1 = \operatorname{cosec}^2(q)$,

are always valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}$.

* * *

PROOF:−

From Theorem TI−11, we recall that the functions,

(i) $\sin(q) = \sin(x)\cosh(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) + \left(\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)\cos(x)\sinh(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2})$;

(ii) $\cos(q) = \cos(x)\cosh(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) - \left(\dfrac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right)\sin(x)\sinh(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2})$,

$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sin), \mathrm{dom}(\cos) \subseteq \mathbb{H}$.

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(a) Henceforth, in terms of Theorem TI−10, we further deduce that

$$\sin^2(q) + \cos^2(q) = (\sin(q))^2 + (\cos(q))^2$$

$$= \left[\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right]^2 +$$

$$\left[\cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right)\sin(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right]^2$$

$$= \sin^2(x)\cosh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \cos^2(x)\sinh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) 2\sin(x)\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\cos^2(x)\cosh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \sin^2(x)\sinh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) -$$

$$\left(\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right) 2\sin(x)\cos(x)\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \left(\sin^2(x) + \cos^2(x)\right)\cosh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \left(\cos^2(x) + \sin^2(x)\right)\sinh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \cosh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \sinh^2\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) = 1,$$

since, from real variable analysis, it has already been established that the hyperbolic functions,

$$\sinh(s) = \frac{e^s - e^{-s}}{2} \text{ and } \cosh(s) = \frac{e^s + e^{-s}}{2}, \ \forall s \in \mathbb{R},$$

also give rise to the identity,

$\cosh^2(s) - \sinh^2(s) = 1$, as required. Q.E.D.

(b) Granted the existence of identity (a) above, we likewise perceive in the light of both Theorem TI-10 and Definition DI-11 that

$$(\sin^2(q) + \cos^2(q))\left(\frac{1}{\cos(q)}\right)^2 = \left(\frac{1}{\cos(q)}\right)^2(\sin^2(q) + \cos^2(q)) = \left(\frac{1}{\cos(q)}\right)^2$$

$$\therefore \frac{(\sin(q))^2}{(\cos(q))^2} + \frac{(\cos(q))^2}{(\cos(q))^2} = \frac{1}{(\cos(q))^2}$$

$$\therefore \left(\frac{\sin(q)}{\cos(q)}\right)^2 + 1 = \left(\frac{1}{\cos(q)}\right)^2$$

$$\therefore (\tan(q))^2 + 1 = (\sec(q))^2$$

$\therefore \tan^2(q) + 1 = \sec^2(q)$, as required. Q.E.D.

(c) Similarly, by virtue of both Theorem TI-10 and Definition DI-11, it is also evident that

$$(\sin^2(q) + \cos^2(q))\left(\frac{1}{\sin(q)}\right)^2 = \left(\frac{1}{\sin(q)}\right)^2(\sin^2(q) + \cos^2(q)) = \left(\frac{1}{\sin(q)}\right)^2$$

$$\therefore \frac{(\sin(q))^2}{(\sin(q))^2} + \frac{(\cos(q))^2}{(\sin(q))^2} = \frac{1}{(\sin(q))^2}$$

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$$\therefore 1 + \left(\frac{\cos(q)}{\sin(q)}\right)^2 = \left(\frac{1}{\sin(q)}\right)^2$$

$$\therefore 1 + (\cot(q))^2 = (\operatorname{cosec}(q))^2$$

$$\therefore 1 + \cot^2(q) = \operatorname{cosec}^2(q),$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subseteq \mathbb{H}$, as required. Q.E.D.

Theorem TI-14.

Let there exist two quaternions,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$
$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2.$$

Hence, we may prove that the following trigonometric identities, namely –

(a) $\sin(q_1 + q_2) = \sin(q_1)\cos(q_2) + \cos(q_1)\sin(q_2)$,
(b) $\cos(q_1 + q_2) = \cos(q_1)\cos(q_2) - \sin(q_1)\sin(q_2)$,

are always valid, whenever

$$y_2 = \lambda y_1;\ \hat{x}_2 = \lambda\hat{x}_1;\ \hat{y}_2 = \lambda\hat{y}_1,\ \forall x_1, x_2, y_1, \hat{x}_1, \hat{y}_1, \lambda \in \mathbb{R}.$$

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PROOF:-

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By initially writing

$$y_2 = \lambda y_1 \,;\, \hat{x}_2 = \lambda \hat{x}_1 \,;\, \hat{y}_2 = \lambda \hat{y}_1 \,, \quad \forall \lambda \in \mathbb{R} \,,$$

it therefore follows that the quaternion,

$$\begin{aligned} q_2 &= x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2 \\ &= x_2 + i\lambda y_1 + j\lambda \hat{x}_1 + k\hat{y}_1 \\ &= x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)\lambda \;, \end{aligned}$$

and hence the quaternion sum,

$$\begin{aligned} q_1 + q_2 &= x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 + \\ &\quad\; x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)\lambda \\ &= x_1 + x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)(\lambda + 1) \,. \end{aligned}$$

Thus, in accordance with Theorem TI-11, we further deduce that

$$\sin(q_1) = \sin(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= \sin(x_1)\cosh\left(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) + \left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cos(x_1)\sinh\left(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) \,;$$

$$\sin(q_2) = \sin(x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)\lambda)$$

$$= \sin(x_2)\cosh\left(|\lambda|\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) +$$

$$\left(\frac{\lambda}{|\lambda|}\right)\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cos(x_2)\sinh\left(|\lambda|\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1}\right)$$

$$= \sin(x_2)\cosh\left(\lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) +$$

To be continued via the author's next submission, namely -

> *"**A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 3/10**."*

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