“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 3/10.”

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I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”,

which has been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

II. COPY OF AUTHOR’S ORIGINAL PAPER – PART 3/10.

For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.

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$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \cos(x_2) \sinh(\lambda \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) ;$$

$$\sin(q_1 + q_2) = \sin(x_1 + x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)(\lambda + 1))$$

$$= \sin(x_1 + x_2) \cosh(|\lambda + 1| \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) +$$

$$\left(\frac{\lambda + 1}{|\lambda + 1|}\right)\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \cos(x_1 + x_2) \sinh(|\lambda + 1| \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2})$$

$$= \sin(x_1 + x_2) \cosh((\lambda + 1) \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) +$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \cos(x_1 + x_2) \sinh((\lambda + 1) \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) ;$$

$$\cos(q_1) = \cos(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= \cos(x_1) \cosh(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) - \left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \sin(x_1) \sinh(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) ;$$

$$\cos(q_2) = \cos(x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)\lambda)$$

$$= \cos(x_2) \cosh(|\lambda| \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) -$$

$$\left(\frac{\lambda}{|\lambda|}\right)\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \sin(x_2) \sinh(|\lambda| \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2})$$

$$= \cos(x_2) \cosh(\lambda \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}) -$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \sin(x_2) \sinh\left(\lambda \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) ;$$

$$\cos(q_1 + q_2) = \cos(x_1 + x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)(\lambda + 1))$$

$$= \cos(x_1 + x_2) \cosh\left(|\lambda + 1| \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) -$$

$$\left(\frac{\lambda + 1}{|\lambda + 1|}\right)\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \sin(x_1 + x_2) \sinh\left(|\lambda + 1| \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)$$

$$= \cos(x_1 + x_2) \cosh\left((\lambda + 1) \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) -$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \sin(x_1 + x_2) \sinh\left((\lambda + 1) \sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right).$$

(a) Upon utilising the previously established results, it is evident, by virtue of Theorem TI-10, that the trigonometric expression,

$$\sin(q_1)\cos(q_2) + \cos(q_1)\sin(q_2)$$

$$= \left[\sin(x_1)\cosh\left(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) + \left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \cos(x_1) \sinh\left(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)\right] \times$$

$$\left[\cos(x_2)\cosh\left(\lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) - \left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right) \sin(x_2) \sinh\left(\lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)\right]$$

$$+$$

$$\left[\cos(x_1)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) - \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sin(x_1)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\right] \times$$

$$\left[\sin(x_2)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cos(x_2)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\right]$$

$$= \sin(x_1)\cos(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) -$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sin(x_1)\sin(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cos(x_1)\cos(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\cos(x_1)\sin(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\cos(x_1)\sin(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cos(x_1)\cos(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) -$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sin(x_1)\sin(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\sin(x_1)\cos(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)$$

$$
\begin{aligned}
= {} & \sin(x_1)\cos(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \sin(x_1)\cos(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cos(x_1)\sin(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cos(x_1)\sin(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}
$$

$$+$$

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\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)
\left[\begin{aligned}
& -\sin(x_1)\sin(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) - \\
& \sin(x_1)\sin(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cos(x_1)\cos(x_2)\sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cos(x_1)\cos(x_2)\cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}\right]
$$

$$
= \sin(x_1)\cos(x_2)\left[\begin{aligned}
& \cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}\right] +
$$

$$
\cos(x_1)\sin(x_2)\left[\begin{aligned}
& \sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}\right] +
$$

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\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)
\left[\begin{aligned}
& -\sin(x_1)\sin(x_2)\left[\begin{aligned}
& \cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}\right] \\
& \qquad + \\
& \cos(x_1)\cos(x_2)\left[\begin{aligned}
& \sinh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cosh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cosh\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sinh\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}\right]
\end{aligned}\right]
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\begin{aligned}
= {} & \sin(x_1)\cos(x_2)\cosh\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cos(x_1)\sin(x_2)\cosh\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +
\end{aligned}
$$

$$
\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)
\left[\begin{aligned}
& -\sin(x_1)\sin(x_2)\sinh\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\
& \cos(x_1)\cos(x_2)\sinh\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)
\end{aligned}\right]
$$

$$= \left(\sin(x_1)\cos(x_2) + \cos(x_1)\sin(x_2)\right)\cosh\left((\lambda+1)\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) \; +$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\left(\cos(x_1)\cos(x_2) - \sin(x_1)\sin(x_2)\right)\sinh\left((\lambda+1)\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)$$

$$= \sin(x_1 + x_2)\cosh\left((\lambda+1)\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) \; +$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cos(x_1 + x_2)\sinh\left((\lambda+1)\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)$$

$$= \sin(q_1 + q_2),$$

since, from real variable analysis, it has already been established that the hyperbolic functions,

$$\sinh(s) = \frac{e^{s} - e^{-s}}{2} \text{ and } \cosh(s) = \frac{e^{s} + e^{-s}}{2}, \; \forall s \in \mathbb{R},$$

also give rise to the identities,

$$\sinh(s_1 + s_2) = \sinh(s_1)\cosh(s_2) + \cosh(s_1)\sinh(s_2),$$
$$\cosh(s_1 + s_2) = \cosh(s_1)\cosh(s_2) + \sinh(s_1)\sinh(s_2),$$
$$\sinh(-s) = -\sinh(s),$$
$$\cosh(-s) = \cosh(s),$$

as required. Q.E.D.

(b) From Theorem T1-13, we initially recognise that

$$\cos^2(q_1+q_2) = 1 - \sin^2(q_1+q_2),$$

bearing in mind our previous criteria, namely –

$$\left.\begin{array}{l} q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 \\ q_2 = x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1 \end{array}\right\}, \ \forall x_1, x_2, y_1, \hat{x}_1, \hat{y}_1, \lambda \in \mathbb{R}.$$

Furthemore, in accordance with both part (a) of this theorem and Theorem TI–10, we likewise deduce that

$$\cos^2(q_1+q_2) = 1 - (\sin(q_1)\cos(q_2) + \cos(q_1)\sin(q_2))^2$$

$$= 1 - (\sin^2(q_1)\cos^2(q_2) + 2\sin(q_1)\cos(q_1)\cos(q_2)\sin(q_2) + \cos^2(q_1)\sin^2(q_2))$$

$$= 1 - \sin^2(q_1)\cos^2(q_2) - 2\sin(q_1)\cos(q_1)\cos(q_2)\sin(q_2) - \cos^2(q_1)\sin^2(q_2)$$

$$= 1 - (1-\cos^2(q_1))\cos^2(q_2) - 2\sin(q_1)\cos(q_1)\cos(q_2)\sin(q_2) - (1-\sin^2(q_1))\sin^2(q_2)$$

$$= 1 - (\cos^2(q_2) - \cos^2(q_1)\cos^2(q_2)) - 2\sin(q_1)\cos(q_1)\cos(q_2)\sin(q_2) - (\sin^2(q_2) - \sin^2(q_1)\sin^2(q_2))$$

$$= 1 - \cos^2(q_2) + \cos^2(q_1)\cos^2(q_2) - 2\sin(q_1)\cos(q_1)\cos(q_2)\sin(q_2) - \sin^2(q_2) + \sin^2(q_1)\sin^2(q_2)$$

$$= 1 - (\cos^2(q_2) + \sin^2(q_2)) + \cos^2(q_1)\cos^2(q_2) - 2\sin(q_1)\cos(q_1)\sin(q_2)\cos(q_2) + \sin^2(q_1)\sin^2(q_2)$$

$$= 1 - 1 + \cos^2(q_1)\cos^2(q_2) - 2\sin(q_1)\cos(q_1)\sin(q_2)\cos(q_2) + \sin^2(q_1)\sin^2(q_2)$$

$$= \cos^2(q_1)\cos^2(q_2) - 2\sin(q_1)\cos(q_1)\sin(q_2)\cos(q_2) + \sin^2(q_1)\sin^2(q_2)$$

$$= (\cos(q_1)\cos(q_2) - \sin(q_1)\sin(q_2))^2$$

$$\therefore \cos(q_1+q_2) = \pm(\cos(q_1)\cos(q_2) - \sin(q_1)\sin(q_2)).$$

Finally, by writing

$$q_1 = x_1 \text{ and } q_2 = x_2 \ (y_1 = \hat{x}_1 = \hat{y}_1 = 0),$$

we deduce from the established real variable trigonometric formulae for compound angles that the above formula reduces to

$$\begin{aligned}\cos(q_1+q_2) &= \cos(x_1+x_2)\\ &= \cos(x_1)\cos(x_2) - \sin(x_1)\sin(x_2)\\ &\neq -[\cos(x_1)\cos(x_2) - \sin(x_1)\sin(x_2)],\ \forall x_1, x_2 \in \mathbb{R},\end{aligned}$$

and hence we obtain in the more general setting,

$$\cos(q_1+q_2) = \cos(q_1)\cos(q_2) - \sin(q_1)\sin(q_2) \quad (y_2 = \lambda y_1 ; \hat{x}_2 = \lambda\hat{x}_1 ; \hat{y}_2 = \lambda\hat{y}_1),$$

$\forall x_1, x_2, y_1, \hat{x}_1, \hat{y}_1, \lambda \in \mathbb{R}$, as required. Q.E.D.

Theorem TI-15.

Let there exist two trigonometric quaternion hypercomplex functions, $\sin(q)$ and $\cos(q)$, each having as their respective domains,

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$dom(\sin)$, $dom(\cos) \subseteq \mathbb{H}$.

Subsequently, we may prove that the following trigonometric identities, namely –

(a) $\sin(-q) = -\sin(q)$,

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(b) $\cos(-q) = \cos(q)$,
(c) $\sin(\frac{\pi}{2} - q) = \cos(q)$,
(d) $\sin(2q) = 2\sin(q)\cos(q)$,
(e) $\cos(2q) = \cos^2(q) - \sin^2(q)$,

are always valid in terms of the above stated domains of definition.

* * *

PROOF:–

(a) From Theorem TI–11, it is evident, upon noting the real variable trigonometric formulae,

$\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$,

that

$$\begin{aligned}\sin(-q) &= \sin(-x - iy - j\hat{x} - k\hat{y}) \\ &= \sin(-x + i(-y) + j(-\hat{x}) + k(-\hat{y}))\end{aligned}$$

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$$= \sin(-x)\cosh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right) + i\left[\frac{\cos(-x)(-y)\sinh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right] +$$

$$j\left[\frac{\cos(-x)(-\hat{x})\sinh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right] + k\left[\frac{\cos(-x)(-\hat{y})\sinh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right]$$

$$= -\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\cos(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\cos(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] - k\left[\frac{\cos(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= -\left[\begin{array}{l}\sin(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\dfrac{\cos(x)\,y\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + \\ j\left[\dfrac{\cos(x)\,\hat{x}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\dfrac{\cos(x)\,\hat{y}\,\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\end{array}\right]$$

$= -\sin(q)$, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sin) \subseteq \mathbb{H}$. <u>Q.E.D.</u>

(b) Similarly, in terms of the same theorem, we likewise perceive that

$$\begin{aligned}\cos(-q) &= \cos(-x - iy - j\hat{x} - k\hat{y}) \\ &= \cos(-x + i(-y) + j(-\hat{x}) + k(-\hat{y}))\end{aligned}$$

$$= \cos(-x)\cosh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-y)^2}\right) - i\left[\frac{\sin(-x)(-y)\sinh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right] -$$

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$$j\left[\frac{\sin(-x)(-\hat{x})\sinh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right]-k\left[\frac{\sin(-x)(-\hat{y})\sinh\left(\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}\right)}{\sqrt{(-y)^2+(-\hat{x})^2+(-\hat{y})^2}}\right]$$

$$= \cos(x)\cosh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - i\left[\frac{\sin(x)y\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] -$$

$$j\left[\frac{\sin(x)\hat{x}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]-k\left[\frac{\sin(x)\hat{y}\sinh\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]$$

$$= \cos(q),\ \forall q = x+iy+j\hat{x}+k\hat{y} \in dom(\cos) \subseteq \mathbb{H}.$$ <u>Q.E.D.</u>

(c) By initially setting

$$\begin{aligned} q_1 &= -q \\ &= -x-iy-j\hat{x}-k\hat{y} \\ &= -x+i(-y)+j(-\hat{x})+k(-\hat{y}) \quad \text{and} \end{aligned}$$

$$q_2 = \tfrac{\pi}{2}$$

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$$\begin{aligned} &= \tfrac{\pi}{2} - i0\cdot y - j0\cdot\hat{x} - k0\cdot\hat{y} \\ &= \tfrac{\pi}{2} + i0\cdot(-y) + j0\cdot(-\hat{x}) + k0\cdot(-\hat{y}), \end{aligned}$$

we perceive, in accordance with the criteria laid down in Theorem TI−14, that

$$\begin{aligned} \sin\left(\tfrac{\pi}{2}-q\right) &= \sin\left(-q+\tfrac{\pi}{2}\right) \\ &= \sin(-q)\cos\left(\tfrac{\pi}{2}\right) + \cos(-q)\sin\left(\tfrac{\pi}{2}\right) \\ &= -\sin(q)\cdot 0 + \cos(q)\cdot 1 \\ &= \cos(q), \end{aligned}$$

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$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subseteq \mathbb{H}$, as required. Q.E.D.

(d) By initially setting

$q_1 = q_2 = q$,

we perceive by virtue of Theorem TI–14 that

$$\begin{aligned} \sin(2q) &= \sin(q+q) \\ &= \sin(q)\cos(q) + \cos(q)\sin(q) \\ &= \sin(q)\cos(q) + \sin(q)\cos(q) \\ &= 2\sin(q)\cos(q), \end{aligned}$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subseteq \mathbb{H}$, as required. Q.E.D.

(e) By initially setting

$q_1 = q_2 = q$,

we perceive in the light of Theorem TI–14 that

$$\begin{aligned} \cos(2q) &= \cos(q+q) \\ &= \cos(q)\cos(q) - \sin(q)\sin(q) \\ &= \cos^2(q) - \sin^2(q), \end{aligned}$$

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$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sin), dom(\cos) \subseteq \mathbb{H}$, as required. Q.E.D.

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We conclude our discussion of the trigonometric quaternion hypercomplex functions with the following remarks :-

(a) The results of the preceding Theorems TI-12, TI-13 and TI-15 are completely analogous with Eqs.(1-54) $\longrightarrow$ (1-60) and Eqs.(1-63) $\longrightarrow$ (1-67).

(b) The algebraic properties enunciated in Theorem TI-14, however, are comparable to but not wholly analogous with Eqs.(1-61) and (1-62) in view of the restrictions placed on the quaternions, q_1 and q_2, via the formulae,

$$\sin(q_1+q_2) = \sin(q_1)\cos(q_2) + \cos(q_1)\sin(q_2) \qquad (1\text{-}68);$$

$$\cos(q_1+q_2) = \cos(q_1)\cos(q_2) - \sin(q_1)\sin(q_2) \qquad (1\text{-}69).$$

5. The Hyperbolic Functions.

From complex variable analysis, we recall that the hyperbolic functions, namely - $\sinh(z)$, $\cosh(z)$, $\tanh(z)$, $\mathrm{sech}(z)$, $\mathrm{cosech}(z)$ and $\coth(z)$, are ultimately derived from the exponential functions, e^z and e^{-z}. Indeed, Churchill et al. [1] provide us with the following definitive formulae to that effect :-

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad (1\text{-}70);$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \qquad (1\text{-}71);$$

$$\tanh(z) = \sinh(z)/\cosh(z) \quad (\cosh(z) \neq 0) \quad (1\text{-}72);$$

$$\mathrm{sech}(z) = 1/\cosh(z) \quad (\cosh(z) \neq 0) \quad (1\text{-}73);$$

$$\mathrm{cosech}(z) = 1/\sinh(z) \quad (\sinh(z) \neq 0) \quad (1\text{-}74);$$

$$\coth(z) = \cosh(z)/\sinh(z) \quad (\sinh(z) \neq 0) \quad (1-75).$$

Subsequently, we shall enunciate the quaternion analogues of $\sinh(z)$, $\cosh(z)$, etc., by means of the next two definitions :-

Definition DI-12.

Let there exist two hyperbolic quaternion hypercomplex functions, which are respectively denoted as $\sinh(q)$ and $\cosh(q)$ such that their domains,

$$\mathrm{dom}(\sinh),\ \mathrm{dom}(\cosh) \subseteq \mathbb{H}.$$

Henceforth, we define

(a) the hyperbolic sine function,

$$\sinh(q) = \frac{\exp(q) - \exp(-q)}{2},$$

(b) the hyperbolic cosine function,

$$\cosh(q) = \frac{\exp(q) + \exp(-q)}{2},$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sinh),\ \mathrm{dom}(\cosh) \subseteq \mathbb{H}.$$

Definition DI-13.

Let there exist four *hyperbolic* quaternion hypercomplex functions, which shall be respectively denoted as $\tanh(q)$, $\operatorname{sech}(q)$, $\operatorname{cosech}(q)$ and $\coth(q)$. Henceforth, we define

(a) the *hyperbolic tangent* function,

$$\tanh(q) = \frac{\sinh(q)}{\cosh(q)} \qquad (\cosh(q) \neq 0) \,,$$

(b) the *hyperbolic secant* function,

$$\operatorname{sech}(q) = \frac{1}{\cosh(q)} \qquad (\cosh(q) \neq 0) \,,$$

(c) the *hyperbolic cosecant* function,

$$\operatorname{cosech}(q) = \frac{1}{\sinh(q)} \qquad (\sinh(q) \neq 0) \,,$$

(d) the *hyperbolic cotangent* function,

$$\coth(q) = \frac{\cosh(q)}{\sinh(q)} \qquad (\sinh(q) \neq 0),$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sinh),\ \mathrm{dom}(\cosh) \subset \mathbb{H}.$$

As a further consequence of Eqs. (1–70) ⟶ (1–75), it may be proven that the following trigonometric identities from complex variable analysis likewise exist:–

$$\cosh^2(z) - \sinh^2(z) = 1 \qquad (1-76);$$

$$1 - \tanh^2(z) = \operatorname{sech}^2(z) \qquad (1-77);$$

$$\coth^2(z) - 1 = \operatorname{cosech}^2(z) \qquad (1-78);$$

$$\sinh(z) = \sinh(x)\cos(y) + i\cosh(x)\sin(y) \qquad (1-79);$$

$$\cosh(z) = \cosh(x)\cos(y) + i\sinh(x)\sin(y) \qquad (1-80);$$

$$\sinh(z_1+z_2) = \sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2) \qquad (1-81);$$

$$\cosh(z_1+z_2) = \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2) \qquad (1-82);$$

$$\sinh(-z) = -\sinh(z) \qquad (1-83);$$

$$\cosh(-z) = \cosh(z) \qquad (1-84).$$

Hence, the purpose of our next four theorems is to derive suitable quaternion analogues of Eqs. (1–76) ⟶ (1–84).

Theorem TI–16.

Let there exist two *hyperbolic* quaternion hypercomplex functions, $\sinh(q)$ and $\cosh(q)$, each having as their respective domains,

$$\mathrm{dom}(\sinh),\ \mathrm{dom}(\cosh) \subseteq \mathbb{H}.$$

Subsequently, we may prove that the following hyperbolic identities, namely –

(a) $\cosh^2(q) - \sinh^2(q) = 1$,
(b) $1 - \tanh^2(q) = \mathrm{sech}^2(q)$,
(c) $\coth^2(q) - 1 = \mathrm{cosech}^2(q)$,

are always valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sinh), \mathrm{dom}(\cosh) \subseteq \mathbb{H}$.

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PROOF:–

From Theorem TI–4, we recall that the exponential function,

$$\exp(q) = e^x \cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] e^x \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) ,$$

and hence, in an analogous manner to the above equation, we likewise obtain the function,

$$\exp(-q) = e^{-x} \cos(\sqrt{y^2+\hat{x}^2+\hat{y}^2}) - \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] e^{-x} \sin(\sqrt{y^2+\hat{x}^2+\hat{y}^2}).$$

(a) In view of these statements as well as the criteria specified in Theorem TI-10 and other theorems pertaining to the properties of the exponential function, we immediately deduce from Definition DI-12 that the expression,

$$\cosh^2(q) - \sinh^2(q) = \left[\cosh(q)\right]^2 - \left[\sinh(q)\right]^2$$

$$= \left[\frac{\exp(q) + \exp(-q)}{2}\right]^2 - \left[\frac{\exp(q) - \exp(-q)}{2}\right]^2$$

$$= \tfrac{1}{4}\left[(\exp(q))^2 + 2\exp(q)\exp(-q) + (\exp(-q))^2\right] - \tfrac{1}{4}\left[(\exp(q))^2 - 2\exp(q)\exp(-q) + (\exp(-q))^2\right]$$

$$= \tfrac{1}{4}\left[(\exp(q))^2 + 2\exp(q-q) + (\exp(-q))^2\right] - \tfrac{1}{4}\left[(\exp(q))^2 - 2\exp(q-q) + (\exp(-q))^2\right]$$

$$= \tfrac{1}{4}\left[(\exp(q))^2 + 2\exp(0) + (\exp(-q))^2\right] - \tfrac{1}{4}\left[(\exp(q))^2 - 2\exp(0) + (\exp(-q))^2\right]$$

$$= \tfrac{1}{4}\left[(\exp(q))^2 + 2 + (\exp(-q))^2\right] - \tfrac{1}{4}\left[(\exp(q))^2 - 2 + (\exp(-q))^2\right]$$

$$= \tfrac{1}{4}(\exp(q))^2 + \tfrac{1}{2} + \tfrac{1}{4}(\exp(-q))^2 - \tfrac{1}{4}(\exp(q))^2 + \tfrac{1}{2} - \tfrac{1}{4}(\exp(-q))^2$$

$$= 1\ , \ \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sinh), \mathrm{dom}(\cosh) \subseteq \mathbb{H},$$

as required. <u>Q.E.D.</u>

(b) Granted the above identity, we similarly deduce from both Theorem TI-10 and Definition DI-13 that the expression,

$$\left[\cosh^2(q) - \sinh^2(q)\right]\left(\frac{1}{\cosh^2(q)}\right) = \left[(\cosh(q))^2 - (\sinh(q))^2\right]\left(\frac{1}{(\cosh(q))^2}\right)$$

$$= \frac{(\cosh(q))^2}{(\cosh(q))^2} - \frac{(\sinh(q))^2}{(\cosh(q))^2} = \frac{1}{(\cosh(q))^2}$$

$$\therefore\ 1 - \left(\frac{\sinh(q)}{\cosh(q)}\right)^2 = \left(\frac{1}{\cosh(q)}\right)^2$$

$$\therefore\ 1 - (\tanh(q))^2 = (\operatorname{sech}(q))^2$$

$$\therefore\ 1 - \tanh^2(q) = \operatorname{sech}^2(q)\ ,$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sinh), \mathrm{dom}(\cosh) \subset \mathbb{H}$, as required. <u>Q.E.D.</u>

<u>(c)</u> After multiplying both sides of identity <u>(a)</u> by the term, $\frac{1}{\sinh^2(q)}$, we accordingly obtain the expression,

$$\left[\cosh^2(q) - \sinh^2(q)\right]\left(\frac{1}{\sinh^2(q)}\right) = \left[(\cosh(q))^2 - (\sinh(q))^2\right]\left(\frac{1}{(\sinh(q))^2}\right)$$

$$= \frac{(\cosh(q))^2}{(\sinh(q))^2} - \frac{(\sinh(q))^2}{(\sinh(q))^2} = \frac{1}{(\sinh(q))^2}$$

$$\therefore\ \left(\frac{\cosh(q)}{\sinh(q)}\right)^2 - 1 = \left(\frac{1}{\sinh(q)}\right)^2$$

$$\therefore\ (\coth(q))^2 - 1 = (\operatorname{cosech}(q))^2$$

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$$\therefore \coth^2(q) - 1 = \operatorname{cosech}^2(q),$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sinh), dom(\cosh) \subset \mathbb{H}, \text{ as required. } \underline{\underline{Q.E.D.}}$$

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Theorem TI-17.

Let there exist two hyperbolic quaternion hypercomplex functions, sin(q) and cosh(q), each having as their respective domains,

$$dom(\sinh), dom(\cosh) \subseteq \mathbb{H}.$$

Subsequently, it may be proven that these functions may be algebraically expressed as

$$\text{(a) } \sinh(q) = \sinh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cosh(x)\,y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cosh(x)\,\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cosh(x)\,\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

$$\text{(b) } \cosh(q) = \cosh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\sinh(x)\,y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^3}}\right] +$$

$$j\left[\frac{\sinh(x)\,\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\sinh(x)\,\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

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$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(sinh), dom(cosh) \subseteq \mathbb{H}$.

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PROOF:-

Once again, we recall from Theorem TI-4 that the exponential function,

$$exp(q) = e^{x} \cos(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] e^{x} \sin(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) ,$$

and hence, in an analogous manner to the above equation, we likewise obtain

$$exp(-q) = e^{-x} \cos(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) - \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] e^{-x} \sin(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) .$$

(a) As a natural consequence of these two statements, it logically follows from Definition DI-12 that

$$sinh(q) = \frac{exp(q) - exp(-q)}{2}$$

$$= \frac{exp(q)}{2} - \frac{exp(-q)}{2}$$

$$= \frac{1}{2}\left[e^{x} \cos(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] e^{x} \sin(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2})\right] -$$

$$\frac{1}{2}\left[e^{-x} \cos(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}) - \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right] e^{-x} \sin(\sqrt{y^2 + \hat{x}^2 + \hat{y}^2})\right]$$

$$= \tfrac{1}{2}e^{x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\tfrac{1}{2}e^{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) -$$

$$\tfrac{1}{2}e^{-x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\tfrac{1}{2}e^{-x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \left(\frac{e^{x}-e^{-x}}{2}\right)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{e^{x}+e^{-x}}{2}\right)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \sinh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\cosh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \sinh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\cosh(x)\,y\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\cosh(x)\,\hat{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\cosh(x)\,\hat{y}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sinh) \subseteq \mathbb{H}$, as required. Q.E.D.

(b) Similarly, by virtue of Definition DI-12, we also deduce that

$$\cosh(q) = \frac{\exp(q) + \exp(-q)}{2}$$

$$= \frac{\exp(q)}{2} + \frac{\exp(-q)}{2}$$

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$$= \frac{1}{2}\left[e^{x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]e^{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right] +$$

$$\frac{1}{2}\left[e^{-x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]e^{-x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)\right]$$

$$= \frac{1}{2}e^{x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{1}{2}e^{x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\frac{1}{2}e^{-x}\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) - \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\frac{1}{2}e^{-x}\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \left(\frac{e^{x}+e^{-x}}{2}\right)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\left(\frac{e^{x}-e^{-x}}{2}\right)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

$$= \cosh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sinh(x)\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)$$

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$$= \cosh(x)\cos\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right) + i\left[\frac{\sinh(x)\,y\,\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] +$$

$$j\left[\frac{\sinh(x)\,\hat{x}\,\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right] + k\left[\frac{\sinh(x)\,\hat{y}\,\sin\left(\sqrt{y^2+\hat{x}^2+\hat{y}^2}\right)}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right],$$

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$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\cosh) \subseteq \mathbb{H}$, as required. Q.E.D.

Theorem TI-18.

Let there exist two quaternions,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$
$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2.$$

Hence, we may prove that the following hyperbolic identities, namely –

(a) $\sinh(q_1 + q_2) = \sinh(q_1)\cosh(q_2) + \cosh(q_1)\sinh(q_2)$,
(b) $\cosh(q_1 + q_2) = \cosh(q_1)\cosh(q_2) + \sinh(q_1)\sinh(q_2)$,

are always valid, whenever

$$y_2 = \lambda y_1 ; \hat{x}_2 = \lambda \hat{x}_1 ; \hat{y}_2 = \lambda \hat{y}_1, \quad \forall x_1, x_2, y_1, \hat{x}_1, \hat{y}_1, \lambda \in \mathbb{R}.$$

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PROOF:–

By initially writing

$$y_2 = \lambda y_1 ; \hat{x}_2 = \lambda \hat{x}_1 ; \hat{y}_2 = \lambda \hat{y}_1, \quad \forall \lambda \in \mathbb{R},$$

it follows that the quaternion,

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$$\begin{aligned} q_2 &= x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2 \\ &= x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1 \\ &= x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)\lambda\,, \end{aligned}$$

and hence the quaternion sum,

$$\begin{aligned} q_1 + q_2 &= x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 + x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1 \\ &= x_1 + x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)(\lambda + 1). \end{aligned}$$

Thus, in accordance with Theorem TI-17, we further deduce that

$$\sinh(q_1) = \sinh(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= \sinh(x_1)\cos\left(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) + \left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cosh(x_1)\sin\left(\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right);$$

$$\sinh(q_2) = \sinh(x_2 + (iy_1 + j\hat{x}_1 + k\hat{y}_1)\lambda)$$

$$= \sinh(x_2)\cos\left(|\lambda|\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) +$$

$$\left(\frac{\lambda}{|\lambda|}\right)\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cosh(x_2)\sin\left(|\lambda|\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)$$

$$= \sinh(x_2)\cos\left(\lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cosh(x_2)\sin\left(\lambda\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right);$$

$$\cosh(q_1) = \cosh(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= \cosh(x_1)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_1)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) ;$$

$$\cosh(q_2) = \cosh(x_2 + (iy_1+j\hat{x}_1+k\hat{y}_1)\lambda)$$

$$= \cosh(x_2)\cos\left(|\lambda|\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{\lambda}{|\lambda|}\right)\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_2)\sin\left(|\lambda|\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)$$

$$= \cosh(x_2)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_2)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) ;$$

$$\sinh(q_1+q_2) = \sinh(x_1+x_2+(iy_1+j\hat{x}_1+k\hat{y}_1)(\lambda+1))$$

$$= \sinh(x_1+x_2)\cos\left(|\lambda+1|\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{\lambda+1}{|\lambda+1|}\right)\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cosh(x_1+x_2)\sin\left(|\lambda+1|\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)$$

$$= \sinh(x_1+x_2)\cos\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cosh(x_1+x_2)\sin\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) ;$$

$$\cosh(q_1+q_2) = \cosh(x_1+x_2+(iy_1+j\hat{x}_1+k\hat{y}_1)(\lambda+1))$$

$$= \cosh(x_1+x_2)\cos(|\lambda+1|\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}) \quad +$$

$$\left(\frac{\lambda+1}{|\lambda+1|}\right)\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_1+x_2)\sin(|\lambda+1|\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2})$$

$$= \cosh(x_1+x_2)\cos((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}) \quad +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_1+x_2)\sin((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}).$$

(e) Upon utilising the previous results, it is evident, by virtue of Theorem TI-10, that the hyperbolic expression,

$$\sinh(q_1)\cosh(q_2) + \cosh(q_1)\sinh(q_2)$$

$$= \left[\sinh(x_1)\cos(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}) + \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cosh(x_1)\sin(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2})\right] \times$$

$$\left[\cosh(x_2)\cos(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}) + \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_2)\sin(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2})\right]$$

$$+$$

$$\left[\cosh(x_1)\cos(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}) + \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_1)\sin(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2})\right] \times$$

$$\left[\sinh(x_2)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cosh(x_2)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\right]$$

$$= \sinh(x_1)\cosh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_1)\sinh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cosh(x_1)\cosh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) -$$

$$\cosh(x_1)\sinh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\cosh(x_1)\sinh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\cosh(x_1)\cosh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\sinh(x_1)\sinh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) -$$

$$\sinh(x_1)\cosh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)$$

$$= \left[\begin{array}{l}\sinh(x_1)\cosh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) - \\ \cosh(x_1)\sinh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) + \\ \cosh(x_1)\sinh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) - \\ \sinh(x_1)\cosh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right] +$$

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$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\left[\begin{array}{l}\sinh(x_1)\sinh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \cosh(x_1)\cosh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \cosh(x_1)\cosh(x_2)\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \sinh(x_1)\sinh(x_2)\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right]$$

$$= \sinh(x_1)\cosh(x_2)\left[\begin{array}{l}\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ -\\ \sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right] +$$

$$\cosh(x_1)\sinh(x_2)\left[\begin{array}{l}\cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ -\\ \sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right] +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\left[\begin{array}{l}\sinh(x_1)\sinh(x_2)\left[\begin{array}{l}\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right] +\\ \cosh(x_1)\cosh(x_2)\left[\begin{array}{l}\sin\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\cos\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \cos\left(\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\sin\left(\lambda\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right]\end{array}\right]$$

$$= \left[\begin{array}{l}\sinh(x_1)\cosh(x_2)\cos\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \cosh(x_1)\sinh(x_2)\cos\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right] +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\left[\begin{array}{l}\sinh(x_1)\sinh(x_2)\sin\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\ +\\ \cosh(x_1)\cosh(x_2)\sin\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)\end{array}\right]$$

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$$= \left(\sinh(x_1)\cosh(x_2)+\cosh(x_1)\sinh(x_2)\right)\cos\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right) +$$

$$\left(\frac{iy_1+j\hat{x}_1+k\hat{y}_1}{\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}}\right)\left(\cosh(x_1)\cosh(x_2)+\sinh(x_1)\sinh(x_2)\right)\sin\left((\lambda+1)\sqrt{y_1^2+\hat{x}_1^2+\hat{y}_1^2}\right)$$

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$$= \sinh(x_1 + x_2)\cos\left((\lambda+1)\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right) \ +$$

$$\left(\frac{iy_1 + j\hat{x}_1 + k\hat{y}_1}{\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}}\right)\cosh(x_1 + x_2)\sin\left((\lambda+1)\sqrt{y_1^2 + \hat{x}_1^2 + \hat{y}_1^2}\right)$$

$= \sinh(q_1 + q_2)$, as required. $\underline{\underline{Q.E.D.}}$

(b) From Theorem TI-16, we initially recognise that

$$\cosh^2(q_1+q_2) = 1 + \sinh^2(q_1+q_2),$$

bearing in mind our previously established criteria, namely –

$$\left.\begin{aligned} q_1 &= x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 \\ q_2 &= x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1 \end{aligned}\right\}, \ \forall \lambda \in \mathbb{R}.$$

Furthermore, in accordance with both part (a) of this theorem and Theorem TI-10, we likewise deduce that

$$\cosh^2(q_1+q_2) = 1 + \left(\sinh(q_1)\cosh(q_2) + \cosh(q_1)\sinh(q_2)\right)^2$$

$$= 1 + \sinh^2(q_1)\cosh^2(q_2) + 2\sinh(q_1)\cosh(q_1)\sinh(q_2)\cosh(q_2) + \cosh^2(q_1)\sinh^2(q_2)$$

$$= 1 + (\cosh^2(q_1) - 1)\cosh^2(q_2) + 2\sinh(q_1)\cosh(q_1)\sinh(q_2)\cosh(q_2) + (1 + \sinh^2(q_1))\sinh^2(q_2)$$

$$= 1 + \cosh^2(q_1)\cosh^2(q_2) - \cosh^2(q_2) + 2\sinh(q_1)\cosh(q_1)\sinh(q_2)\cosh(q_2) + \sinh^2(q_2) + \sinh^2(q_1)\sinh^2(q_2)$$

$$= 1 + \sinh^2(q_2) - \cosh^2(q_2) + \cosh^2(q_1)\cosh^2(q_2) +$$
$$2\sinh(q_1)\cosh(q_1)\sinh(q_2)\cosh(q_2) + \sinh^2(q_1)\sinh^2(q_2)$$

$$= \cosh^2(q_2) - \cosh^2(q_2) + \cosh^2(q_1)\cosh^2(q_2) +$$
$$2\sinh(q_1)\cosh(q_1)\sinh(q_2)\cosh(q_2) + \sinh^2(q_1)\sinh^2(q_2)$$

$$= \cosh^2(q_1)\cosh^2(q_2) + 2\sinh(q_1)\cosh(q_1)\sinh(q_2)\cosh(q_2) + \sinh^2(q_1)\sinh^2(q_2)$$

$$= \left(\cosh(q_1)\cosh(q_2) + \sinh(q_1)\sinh(q_2)\right)^2$$

$$\therefore \cosh(q_1+q_2) = \pm\left(\cosh(q_1)\cosh(q_2) + \sinh(q_1)\sinh(q_2)\right).$$

Finally, by setting

$$q_1 = x_1 \text{ and } q_2 = x_2 \quad (y_1 = \hat{x}_1 = \hat{y}_1 = 0),$$

we deduce from the real variable hyperbolic cosine function,

$$\cosh(x_1+x_2) = \frac{e^{(x_1+x_2)} + e^{-(x_1+x_2)}}{2} \geq 1, \ \forall x_1, x_2 \in \mathbb{R},$$

that the above quaternion addition formula reduces to

$$\cosh(q_1+q_2) = \cosh(x_1+x_2) = \frac{e^{(x_1+x_2)} + e^{-(x_1+x_2)}}{2}$$

$$= \cosh(x_1)\cosh(x_2) + \sinh(x_1)\sinh(x_2) \geq 1, \ \forall x_1, x_2 \in \mathbb{R},$$

$$\Longrightarrow \cosh(x_1+x_2) \neq -\left[\cosh(x_1)\cosh(x_2) + \sinh(x_1)\sinh(x_2)\right],$$

and hence we obtain in the more general setting,

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$$\cosh(q_1+q_2) = \cosh(q_1)\cosh(q_2) + \sinh(q_1)\sinh(q_2),$$

whenever $q_2 = x_2 + i\lambda y_1 + j\lambda\hat{x}_1 + k\lambda\hat{y}_1$, $\forall \lambda \in \mathbb{R}$, as required. Q.E.D.

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Theorem TI−19.

Let there exist two *hyperbolic* quaternion hypercomplex functions, $\sinh(q)$ and $\cosh(q)$, whose respective domains,

$$\mathrm{dom}(\sinh),\ \mathrm{dom}(\cosh) \subseteq \mathbb{H}.$$

Subsequently, it may be proven that the formulae,

(a) $\sinh(-q) = -\sinh(q)$,
(b) $\cosh(-q) = \cosh(q)$,

are always valid, $\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathrm{dom}(\sinh), \mathrm{dom}(\cosh) \subseteq \mathbb{H}$.

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PROOF:−

From Definition DI−12, we recall that the hyperbolic sine and cosine functions are respectively denoted by the formulae:−

(i) $\sinh(q) = \dfrac{\exp(q) - \exp(-q)}{2}$;

(ii) $\cosh(q) = \dfrac{\exp(q) + \exp(-q)}{2}$.

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Hence, it immediately follows from these formulae that the hyperbolic functions,

(a) $\sinh(-q) = \dfrac{\exp(-q) - \exp(-(-q))}{2}$

$= \dfrac{\exp(-q) - \exp(q)}{2}$

$= -\left(\dfrac{\exp(q) - \exp(-q)}{2}\right)$

$= -\sinh(q),$

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and similarly,

(b) $\cosh(-q) = \dfrac{\exp(-q) + \exp(-(-q))}{2}$

$= \dfrac{\exp(-q) + \exp(q)}{2}$

$= \dfrac{\exp(q) + \exp(-q)}{2}$

$= \cosh(q),$

$\forall q = x + iy + j\hat{x} + k\hat{y} \in dom(\sinh), dom(\cosh) \subseteq \mathbb{H}$, as required. <u>Q.E.D.</u>

We conclude our discussion of the hyperbolic quaternion hypercomplex functions with the following remarks:-

(a) The results of the preceding Theorems TI-16, TI-17 and TI-19 are completely analogous with Eqs. (1-76) $\rightarrow$ (1-80) and Eqs. (1-83) & (1-84).

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(b) The algebraic properties enunciated in Theorem TI-18, however, are comparable to but not wholly analogous with Eqs.(1-81) and (1-82) in view of the restrictions placed on the quaternions, q_1 and q_2, via the formulae,

$$\sinh(q_1+q_2) = \sinh(q_1)\cosh(q_2) + \cosh(q_1)\sinh(q_2) \qquad (1-85);$$

$$\cosh(q_1+q_2) = \cosh(q_1)\cosh(q_2) + \sinh(q_1)\sinh(q_2) \qquad (1-86).$$

6. The Logarithmic Function.

From complex variable analysis, we recall that the logarithmic function, $\log(z)$, is a multi-valued function comprising an infinite number of values, that is to say –

$$\log(z) = \mathrm{Log}(z) + i\,2n\pi \quad (n \in \mathbb{Z}) \qquad (1-87),$$

whereupon the single-valued function,

$$\mathrm{Log}(z) = \mathrm{Log}(r) + i\Theta \quad (r > 0 \ \&\ -\pi < \Theta \leq \pi) \qquad (1-88).$$

Churchill et al. [1] accordingly designate $\mathrm{Log}(z)$ to be the principal value of $\log(z)$ and, moreover, by rewriting Eq.(1-88) as

$$e^{\mathrm{Log}(z)} = e^{\mathrm{Log}(r) + i\Theta} \qquad (1-89),$$

it therefore follows that

$$e^{Log(r)+i\theta} = e^{Log(r)}e^{i\theta}$$
$$= re^{i\theta}$$
$$= z$$

$$\Longrightarrow e^{Log(z)} = z \qquad (1-90),$$

in other words, $Log(z)$ is the inverse of the exponential function, $\exp(z) = e^z$.
Hence, in view of Eqs. (1-88) & (1-90), we shall derive the quaternion analogue of $Log(z)$ by means of the next definition and theorem :-

Definition DI-14.

Let there exist a logarithmic quaternion hypercomplex function, $Log(q)$, whose domain,

$$dom(Log) \subseteq \mathbb{H} - \{0\}.$$

Furthermore, we postulate that the existence of this function is based upon the definitive formula —

$$\exp(Log(q)) = q, \quad \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\}.$$

Theorem TI-20.

Let there exist a logarithmic quaternion hypercomplex function, $Log(q)$, whose domain,

$\mathrm{dom}(\mathrm{Log}) \subseteq \mathbb{H} - \{0\}$.

Subsequently, it may be proven that this function can be algebraically expressed as

$$\mathrm{Log}(q) = \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right) + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\Theta ,$$

such that the real variable function,

$$\Theta = \cos^{-1}\left[\frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}}\right] \in [0, \pi] ,$$

$$\forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\} .$$

* * *

PROOF:-

Let us define the logarithmic function, $\mathrm{Log}(q)$, such that

$$\begin{aligned} \mathrm{Log}(q) &= u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}) \\ &= u_1(x,y,\hat{x},\hat{y}) + \left(\frac{i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})}{R(x,y,\hat{x},\hat{y})}\right) R(x,y,\hat{x},\hat{y}) , \end{aligned}$$

where the auxiliary function,

$$R(x,y,\hat{x},\hat{y}) = \sqrt{(v_1(x,y,\hat{x},\hat{y}))^2 + (u_2(x,y,\hat{x},\hat{y}))^2 + (v_2(x,y,\hat{x},\hat{y}))^2} .$$

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From Definition DI-14, we recall that

$$\exp(\mathrm{Log}(q)) = q, \quad \forall q = x + iy + j\hat{x} + k\hat{y} \in \mathbb{H} - \{0\},$$

and hence the provisions of Theorems TI-3 and TI-5 allow us to expand this equation such that we obtain

$$\exp(\mathrm{Log}(q)) =$$

$$\exp\left(u_1(x,y,\hat{x},\hat{y}) + \left[\frac{iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y})}{R(x,y,\hat{x},\hat{y})}\right]R(x,y,\hat{x},\hat{y})\right)$$

$$= \exp(u_1(x,y,\hat{x},\hat{y}))\exp\left(\left[\frac{iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y})}{R(x,y,\hat{x},\hat{y})}\right]R(x,y,\hat{x},\hat{y})\right)$$

$$= \exp(u_1(x,y,\hat{x},\hat{y}))\left[\cos(R(x,y,\hat{x},\hat{y})) + \left[\frac{iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y})}{R(x,y,\hat{x},\hat{y})}\right]\sin(R(x,y,\hat{x},\hat{y}))\right]$$

$$= \exp(u_1(x,y,\hat{x},\hat{y}))\cos(R(x,y,\hat{x},\hat{y})) +$$

$$\left[\frac{iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y})}{R(x,y,\hat{x},\hat{y})}\right]\exp(u_1(x,y,\hat{x},\hat{y}))\sin(R(x,y,\hat{x},\hat{y}))$$

$$= q = x + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}\right]\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}\ .$$

Furthermore, by writing

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$$v_1(x,y,\hat{x},\hat{y}) = \frac{\mathcal{V}(x,y,\hat{x},\hat{y})\,y}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} \; ; \; u_2(x,y,\hat{x},\hat{y}) = \frac{\mathcal{V}(x,y,\hat{x},\hat{y})\,\hat{x}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} \; ;$$

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$$v_2(x,y,\hat{x},\hat{y}) = \frac{\mathcal{V}(x,y,\hat{x},\hat{y})\,\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$$

$$\Longrightarrow R(x,y,\hat{x},\hat{y}) = \sqrt{(\mathcal{V}(x,y,\hat{x},\hat{y}))^2} = |\mathcal{V}(x,y,\hat{x},\hat{y})| \,,$$

we accordingly perceive, after making the appropriate algebraic substitutions, that

$$\exp(\mathrm{Log}(q)) = \exp(u_1(x,y,\hat{x},\hat{y}))\cos(|\mathcal{V}(x,y,\hat{x},\hat{y})|) +$$

$$\left(\frac{\mathcal{V}(x,y,\hat{x},\hat{y})}{|\mathcal{V}(x,y,\hat{x},\hat{y})|}\right)\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\exp(u_1(x,y,\hat{x},\hat{y}))\sin(|\mathcal{V}(x,y,\hat{x},\hat{y})|)$$

$$= \exp(u_1(x,y,\hat{x},\hat{y}))\cos(\mathcal{V}(x,y,\hat{x},\hat{y})) \;\; +$$

$$\left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\exp(u_1(x,y,\hat{x},\hat{y}))\sin(\mathcal{V}(x,y,\hat{x},\hat{y}))$$

$$= q = x + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\sqrt{y^2+\hat{x}^2+\hat{y}^2}$$

and hence we obtain the pair of simultaneous equations –

$$\left.\begin{aligned} \exp(u_1(x,y,\hat{x},\hat{y}))\cos(\mathcal{V}(x,y,\hat{x},\hat{y})) &= x \\ \exp(u_1(x,y,\hat{x},\hat{y}))\sin(\mathcal{V}(x,y,\hat{x},\hat{y})) &= \sqrt{y^2+\hat{x}^2+\hat{y}^2} \end{aligned}\right\} \;\; \underline{(i)}\,.$$

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Now, in order to solve these equations in terms of the real variable functions, $u_1(x,y,\hat{x},\hat{y})$ and $V(x,y,\hat{x},\hat{y})$, we firstly note that the sum,

$$[\exp(u_1(x,y,\hat{x},\hat{y}))\cos(V(x,y,\hat{x},\hat{y}))]^2 + [\exp(u_1(x,y,\hat{x},\hat{y}))\sin(V(x,y,\hat{x},\hat{y}))]^2$$

$$= x^2 + y^2 + \hat{x}^2 + \hat{y}^2$$

$$\therefore [\exp(u_1(x,y,\hat{x},\hat{y}))]^2[\cos^2(V(x,y,\hat{x},\hat{y})) + \sin^2(V(x,y,\hat{x},\hat{y}))]$$

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$$= x^2 + y^2 + \hat{x}^2 + \hat{y}^2$$

$$\therefore [\exp(u_1(x,y,\hat{x},\hat{y}))]^2 = x^2 + y^2 + \hat{x}^2 + \hat{y}^2$$

$$\therefore \exp(u_1(x,y,\hat{x},\hat{y})) = \sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2} \qquad \text{(ii)}$$

$$\Longrightarrow u_1(x,y,\hat{x},\hat{y}) = \log\left(\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}\right).$$

Direct substitution of Eq. (ii) into Eqs. (i) thus yields

$$\cos(V(x,y,\hat{x},\hat{y})) = \frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}} \qquad \text{and}$$

$$\sin(V(x,y,\hat{x},\hat{y})) = \frac{\sqrt{y^2 + \hat{x}^2 + \hat{y}^2}}{\sqrt{x^2 + y^2 + \hat{x}^2 + \hat{y}^2}}.$$

From elementary trigonometry and real variable analysis (viz. Appendix A1), we deduce that the real variable function,

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$$v(x,y,\hat{x},\hat{y}) = \cos^{-1}\left[\frac{x}{\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}}\right] = \Theta \in [0,\pi].$$

In summary, we conclude that the logarithmic function, $Log(q)$, can be algebraically expressed as

$$Log(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y})$$

$$= \log\left(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\right) + \frac{iy\Theta}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} +$$

$$\frac{j\hat{x}\Theta}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}} + \frac{k\hat{y}\Theta}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}$$

$$= \log\left(\sqrt{x^2+y^2+\hat{x}^2+\hat{y}^2}\right) + \left[\frac{iy+j\hat{x}+k\hat{y}}{\sqrt{y^2+\hat{x}^2+\hat{y}^2}}\right]\Theta,$$

To be continued via the author's next submission, namely -

> *"**A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 4/10.**"*

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