Two-Colouring of a Map

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We provide a characterization of maps which are colourable with two colours.

1 Introduction

Since the paper [1] of two US Americans Kenneth Appel and Wolfgang Haken it is well known that the 'Four Colour Theorem' is true. It was proven 1976 by the aid of computers, which have to consider nearly 2000 subcases. The four mathematicians Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas in the year 1996 presented a new proof [2], where they reduced the number of subcases to ca. 600, but still it lacks an elementary proof without the aid of a computer. The following paper arose by the futile attempts to prove the Four Colour Theorem.

Let Map be a map of N elements, $\mathsf{Map} = \{S_1, S_2, S_3, \dots, S_{N-1}, S_N\}$, $N \in \mathbb{N}$. First we have to make something clear. An element $S_k \in \mathsf{Map}$ is called a *country*. Each country is a subset of $\mathbb{R}^2$ with finite but positive surface. Every country is confined by its border or boundary, a subset of $\mathbb{R}^2$ with no surface. We assume that each country is homeomorphic to a square. This means that a country is connected, and that it has a trivial fundamental group, i.e. it has no holes. This means further that each country S_k is homeomorphic to the whole closed unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, and that the *border* of a country is homeomorphic to $\{x^2 + y^2 = 1\}$ for $x, y \in \mathbb{R}$. Every circle with a center in the border of a country contains a point from that country. Two countries are *neighboring* if and only if they have some common border homeomorphic to $\{x \in \mathbb{R} \mid 0 < x < 1\}$. If A and B are neighboring countries, then A is called a *neighbor* of B and B is a neighbor of A, or $\{A, B\}$ is a *neighboring pair*. Two neighboring countries are also topological neighboring. Note that two countries which meet in a finite sets of points are not neighboring. A *colouring* of the map means that each country has a colour, and neighboring countries get different colours. In this case we call the map *stainable*.

We mention the following trivial propositions although we do not need them, since we read them nowhere, but we think they are important.

Proposition 1. *Let* Map *be any map. This map is stainable with four colours if and only if* Map *is the union of* Map_1 *and* Map_2*, i.e.* $\mathsf{Map} = \mathsf{Map}_1 \cup \mathsf{Map}_2$*, and both* Map_1 *and* Map_2 *are stainable with just two colours.*

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Proposition 2. *Let* Map *be any map. This map is stainable with three colours if and only if* Map *is the union of* Map_1 *and* Map_2*, i.e.* $\mathsf{Map} = \mathsf{Map}_1 \cup \mathsf{Map}_2$*, and* Map_2 *is stainable with two colours, while* Map_1 *is stainable with one colour.*

Remark 1. *In both propositions we can choose disjoint sets* Map_1 *and* Map_2*.*

The entire calculations occurs on the usual Euclidian plane $= \mathbb{R}^2$. A connected set of countries is called a *continent*. We assume that all contries have neighbors, and there is a single continent.

We need additional definitions.
We say that a *triple* are three countries such that each country have some common border to the two others. We call a *way* an ordered set $(L_1, L_2, L_3, \ldots, L_{K-1}, L_K)$ of countries such that L_{i-1} and L_i are neighboring for $2 \leq i \leq K$. The country L_1 is the *beginning*, and L_K is called the *end*. We call a *circle* an ordered set $(L_1, L_2, L_3, \ldots, L_{K-1}, L_K)$ of countries such that L_{i-1} and L_i are neighboring for $2 \leq i \leq K$, and L_1 and L_K are neighbors, too. Every circle is also a way. We call the set $\{L_1, L_2, L_3, \ldots, L_{K-1}, L_K\}$ the *used countries* of the way or the circle, respectively. Note that the number of used countries in a way may be less than K. We declare that the number of used countries in each way or circle is at least three. This avoids ways and circles like $(A, B, A, B, \ldots)$ with a neighboring pair $\{A, B\}$. We say the circle is *odd* or it is an *odd circle* if the number of the used countries of the circle is odd.

2 The Theorem

Since we have a single continent, all countries are on this continent.

Theorem 1. *Any map is stainable with two colours if and only if it contains neither a triple nor an odd circle.*

Proof. As abbreviations we define the assertions $\mathsf{H} :=$ A map is stainable with two colours, and $\mathsf{J} :=$ A map contains no triple and no odd circle.

We want to prove: $\mathsf{H} \iff \mathsf{J}$. Instead this we show $\overline{\mathsf{J}} \Rightarrow \overline{\mathsf{H}}$ and $\mathsf{J} \Rightarrow \mathsf{H}$, where $\overline{\mathsf{J}}$ means the negation of J. If a map contains a triple or an odd circle it is not possible to colour it with two colours, i.e. $\overline{\mathsf{J}} \Rightarrow \overline{\mathsf{H}}$. Let us assume J, i.e. we assume a map without a triple or an odd circle. We shall colour each connected component CC with just two colours, which we call r (red) and g (green). We start with an arbitrary country $X \in CC$. We colour it with r. Assume two neighbors of X. We call them Y and Z. The pair $\{Y, Z\}$ cannot be neighboring, otherwise $\{X, Y, Z\}$ would be a triple, which is forbidden by J. We colour Y and Z with g. All neighbors of Y and Z are coloured with r, and so on. We colour each country in CC alternating with r and g.

Assume a country W which should be coloured both with r and g. We show that this is not possible. We have two ways $Way_1 := (A_1, A_2, A_3, \ldots, A_{k-1}, A_k)$ and $Way_2 := (B_1, B_2, B_3, \ldots, B_{j-1}, B_j)$, where $X = A_1 = B_1$ and $W = A_k = B_j$ and both Way_1 and Way_2 are ways, the beginning is X, the end is W. Since the country W should be coloured both with r and g, one number of the used countries of the ways Way_1 and Way_2 is odd and the other is even. We glue both ways in W together and we get a circle

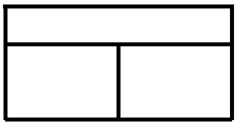
$$(X, A_2, A_3, \ldots, A_{k-2}, A_{k-1}, W, B_{j-1}, B_{j-2}, \ldots, B_3, B_2)$$

which is an odd circle, since we count the contries X and W of Way_1 and Way_2 twice. An odd circle is not possible due to J. Hence there can not occur a contradiction during the colouring of CC.

Assume a neighboring pair $\{S, T\}$ of countries, which are both coloured with the same colour. We show that this is not possible. There are two ways $(X, C_2, C_3, \ldots, C_{m-2}, C_{m-1}, S)$ and $(X, D_2, D_3, \ldots, D_{n-2}, D_{n-1}, T)$. Since the two countries S and T have the same colour, either the numbers of used countries of both ways are even or both numbers are odd. We put the two ways together into an odd circle

$$(X, C_2, C_3, \ldots, C_{m-2}, C_{m-1}, S, T, D_{n-1}, D_{n-2}, \ldots, D_3, D_2)$$

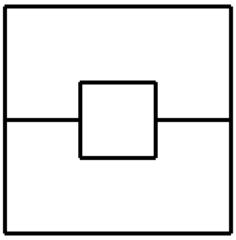
which is not possible due to J. $\square$



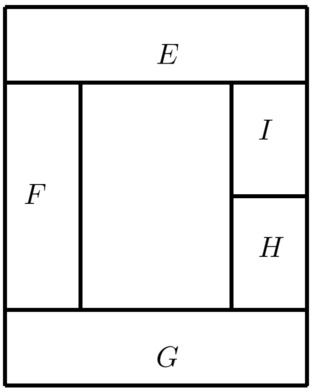
We show an example of a triple.

A triple is not stainable with two colours.

We need three colours.



We show another example of a triple.



On the left hand side is an odd circle

with countries $\{E, F, G, H, I\}$.

It is not stainable with two colours.

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References

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