The Richardson Nobelian experiment in magnetic field

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Abstract

The Richardson thermal effect is considered for the situation where the thermal electrons are inserted into the homogenous magnetic field. The electron flow in magnetic field produces the synchrotron radiation. We calculate the spectral distribution of the synchrotron photons.

Key words: Thermodynamics, work function, band model, synchrotron radiation.

1 Introduction

Thermionic emission is the thermally induced flow of charged particles from a surface, or, over a potential-energy barrier. This occurs because the thermal energy given to the particles overcomes the work function of the material. The charge carriers can be electrons or ions, referred earlier as thermions. If the emitter is connected to a battery, the charge left behind is neutralized by charge supplied by the battery as the emitted charge particles move away from the emitter, and finally the emitter will be in the same state as it was before emission.

The classical example of thermionic emission is the emission of of electrons from a hot cathode into a vacuum known as the Edison effect in a vacuum tube. The hot cathode can be a metal filament, a coated metal filament, or a separate structure of metal or carbides or borides of transition metals. Vacuum emission from metals tends to become significant only for temperatures over $T = 1000^0$ K.

Richardson writes (Richardson, 1929): "*In 1901 I was able to show that each unit area of a platinum surface emitted a limited number of electrons. This number increased very rapidly with the temperature, so that the maximum current i at any absolute temperature T was governed by the law*

$$i = A\sqrt{T}e^{-\frac{w}{kT}}. \quad (1)$$

In this equation k is Boltzmann constant, and A and w are specific constants of the material. This equation was completely accounted for by the simple hypothesis that the freely moving electrons in the interior of the hot conductor escaped when they reached the surface provided that the part of their energy which depended on the component of velocity normal to the surface was greater than the work function w."

Later, Richardson writes (Richardson, 1929): "*In 1911 as a result of pursuing some difficulties in connection with the thermodynamic theory of electron emission I came to the conclusion that*

$$i = AT^2e^{-\frac{w}{kT}} \quad (2)$$

was a theoretically preferable form of the temperature emission equation to Eq. (1) with, of course, different values of the constants A and w from those used with (1). It is impossible to distinguish between these two equations by experimenting."

From band theory, there are one or two electrons per atom in a solid that are free to move from atom to atom. This is sometimes collectively referred to as a sea of electrons. Their velocities follow a statistical distribution, and occasionally an electron will have enough velocity to exit the metal without being pulled back in. The minimum amount of energy needed for an electron to leave a surface is called the work function. The work function is characteristic of the material and for most metals is on the order of several electronvolts (Lide, 2008). Thermionic currents can be increased by decreasing the work function. This often-desired goal can be achieved by applying various oxide coatings to the wire. Now, it is proposed that the emission law should have the mathematical form (Crowell, 1965)

$$J = A_G T^2 e^{-\frac{W}{kT}}, \quad (3)$$

where J is the emission current density, T is the temperature of the metal, W is the work function of the metal, k is the Boltzmann constant, and A_G is a parameter and there is still no clear what is the exact expression of A_G , but there is agreement that A_G must be written in the form $A_G = \lambda_R A_0$ where λ_R is a material-specific correction factor that is typically of order 0.5, and A_0 is a universal constant given by Crowell (1965). Or,

$$A_0 = \frac{4\pi mk^2e}{h^3}. \quad (4)$$

2 The Richardson formula from the band model

Electron which is in the free zone of conductivity has zero kinetic energy. His total energy is the potential one.

The working function W is the work necessary for the escaping of an electron in solid state to vacuum without giving it some kinetical energy.

Let be the metal placed in the coordinate system in area with $x \leq 0$. Electron with energy

$$\frac{mv_x^2}{2} \geq W \tag{5}$$

can escape from the surface of this metal and its energy is

$$\frac{mv_x^2}{2} + E_C = E \geq 0, \tag{6}$$

where E_C is the chemical potential.

The formula of the current density j_x of an electron escaping to vacuum is obviously given as follows (Kireev, 1969):

$$j_x = e\int_{v_{x-min}}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x 2\frac{m}{h^3}e^{-\frac{E-F}{kT}} dv_x dv_y dv_z =$$

$$= em\int_{v_{x-min}}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{E-F}{kT}} v_x \frac{d\tau}{h^3}, \tag{7}$$

which formula is derived on the assumption that there are $2d\tau/h^3$ states ocupied by electrons with probability

$$f_0(E,T) = \frac{1}{e^{-\frac{E-F}{kT}}} \approx e^{-\frac{E-F}{kT}}, \tag{8}$$

where the approximation is valid for $E - F \gg kT$.

Using

$$E = \frac{mv^2}{2} + E_C = E_C + \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) \geq 0, \tag{9}$$

we get

$$j_x = 2e\frac{e^{-\frac{E-F}{kT}}}{h^3}\int_{-\infty}^{\infty} e^{-\frac{mv_y^2}{2kT}} dv_y \int_{-\infty}^{\infty} e^{-\frac{mv_z^2}{2kT}} dv_z \int_{v_{x-min}}^{\infty} v_x e^{-\frac{mv_x^2}{2kT}} dv_x. \tag{10}$$

With regard to identity

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}, \tag{11}$$

we get with transformation

$$\frac{mv_y^2}{2kT} = \xi^2; \quad dv_y = \sqrt{\frac{2kT}{m}}d\xi \tag{12}$$

the following formulas:

$$\int_{-\infty}^{\infty} e^{-\frac{mv_y^2}{2kT}} dv_y = \int_{-\infty}^{\infty} e^{-\frac{mv_z^2}{2kT}} dv_z = \sqrt{\frac{2k\pi T}{m}}. \tag{13}$$

The integration over v_x gives (Kireev, 1969)

$$\int_{v_{x-min}}^{\infty} v_x e^{-\frac{mv_x^2}{2kT}} dv_x = \frac{kT}{m} \int_{v_{x-min}}^{\infty} e^{-\frac{mv_x^2}{2kT}} d\left(\frac{mv_x^2}{2kT}\right) = \frac{kT}{m}.e^{-\frac{W}{kT}} = \frac{kT}{m} e^{-\frac{E_C}{kT}}. \tag{14}$$

So,

$$j_x = \frac{4\pi e m^2 k^2 T^2}{h^3} e^{-\frac{-F-E_C+E_C}{kT}} = AT^2 e^{\frac{F}{kT}} \tag{15}$$

with

$$A = \frac{4\pi e m^2 k^2}{h^3}. \tag{16}$$

If we identify F with the thermodynamical work Φ, or, $F = -\Phi$, then we get

$$j_x = AT^2 e^{\frac{-\Phi}{kT}}. \tag{17}$$

Numerically, the thermodynamical work Φ is equal to the work necessary for the escaping of an electron from the Fermi level to vacuum. Formula (17) is so called Richardson-Dushman law of the thermoelectrical emission.

Let us remark that there is the equilibrium between escaping electron and the electrons returning to the metal. So, in order to measure the current of escaping electron it is necessary introduce the external field to suck the thermoelectric electrons and at the same to compensate the charge loss by metal.

The thermoelectrical emission depends as it is seen on then temperature T. For $\Phi = 2,5$ eV and $T = 300^0$ K, we have $j_T = 10^{-36}$ Amper/cm^2 and for $T = 1500^0$ K, we have $j_T = 0,8$ Amper/cm^2.

3 Thermal electrons in magnetic field

The quantity Φ is called the thermodynamic work of the escaped electron from the metal. Numerically it is equal to the work necessary for escaping of electrons being on the Fermi level. In real conditions emission current cannot have the value J_T. It is possible under special conditions. The surface over the metal with escaping electron is charged. The field can be considered as the homogenous electric field.

If we now insert the metal in the magnetic field then electrons move in the electromagnetic field and it is easy to see that the situation can be described by the synchrotron equation. In such a way the electrons radiates the synchrotron radiation and the known formulas can be applied to the Richardson problem. The experimental goal is to determine the spectrum of the synchrotron radiation and compare it with the theoretical values.

The power spectral formula for the synergic synchrotron-Čerenkov radiation was derived by the Schwinger source methods in the form (Schwinger et al., 1976):

$$P(\omega,t) = -\frac{\omega}{4\pi^2}\frac{\mu}{n^2}\int\int\int d\mathbf{x}d\mathbf{x}'dt'\frac{\sin\left[\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|\right]}{|\mathbf{x}-\mathbf{x}'|}\cos[\omega(t-t')]\times$$

$$\times\left\{\varrho(\mathbf{x},t)\varrho(\mathbf{x}',t') - \frac{n^2}{c^2}\mathbf{J}(\mathbf{x},t)\cdot\mathbf{J}(\mathbf{x}',t')\right\}. \tag{18}$$

Now, we apply the last formula to the case of an electron moving in a magnetic field and then we generalize the derived results in case of thermal electrons forming the Richardson constelation.

We write for the charge density ϱ and for the current density, $\mathbf{J}$, the equations concerning only the circular motion of the electron and then we will show how to apply the derived formulas for the circular motion of an electron. We write for the circular motion (Schwinger et al., 1976):

$$\varrho(\mathbf{x},t) = e\delta(\mathbf{x}-\mathbf{R}(t)), \qquad \mathbf{J}(\mathbf{x},t) = e\mathbf{v}(t)\delta(\mathbf{x}-\mathbf{R}(t)) \tag{19}$$

with

$$\mathbf{R}(t) = R(\mathbf{i}\cos(\omega_0 t) + \mathbf{j}\sin(\omega_0 t)). \tag{20}$$

In this specific case, we have:

$$\mathbf{v}(t) = d\mathbf{R}/dt, \quad \omega_0 = v/R, \quad \beta = v/c, \quad v = |\mathbf{v}|. \tag{21}$$

After insertion of eq. (19) into eq. (18), we get

$$P(\omega,t) = \sum_{l=1}^{\infty}\delta(\omega - l\omega_0)P_l(\omega,t) \tag{22}$$

with

$$P_l(\omega,t) = \frac{e^2}{4\pi^2 n^2}\frac{\omega\mu\omega_0}{v}\left(2n^2\beta^2 J'_{2l}(2ln\beta) - (1-n^2\beta^2)\int_0^{2ln\beta}dx J_{2l}(x)\right) \tag{23}$$

where during the derivation of eq. (23), we have used the relations:

$$t' - t = \tau, \quad dt' = d\tau \tag{24}$$

$$|\mathbf{R}(t+\tau) - \mathbf{R}(t)| = 2R\left|\sin\frac{1}{2}\omega_0\tau\right| \tag{25}$$

$$\mathbf{v}(t)\cdot\mathbf{v}(t+\tau) = v^2\cos\omega_0\tau \tag{26}$$

$$\omega_0\tau = \varphi + 2\pi l, \qquad \varphi \in (-\pi, \pi),\ l = 0,\ \pm 1,\ \pm 2,\ .... \tag{27}$$

Using the formulas

$$J'_{2l}(2ln\beta) \sim \frac{1}{\sqrt{3}}\frac{1}{\pi}\left(\frac{3}{2l_n}\right)^{2/3} K_{2/3}(l/l_n), \quad l \gg 1 \tag{28}$$

$$\int_0^{2ln\beta} J_{2l}(y)dy \sim \frac{1}{\sqrt{3}}\frac{1}{\pi}\int_{l/l_n}^{\infty} K_{1/3}(y)dy, \quad l \gg 1 \tag{29}$$

$$l_n = \frac{3}{2}(1 - n^2\beta^2)^{-3/2}, \tag{30}$$

$$K'_{2/3} = -\frac{1}{2}(K_{1/3} + K_{5/3}), \tag{31}$$

$$\kappa(\xi) = \xi\int_{\xi}^{\infty} K_{5/3}(y)dy, \quad \xi = l/l_n \tag{32}$$

and

$$\kappa(\xi) \approx \sqrt{\frac{\pi}{2}}\xi^{1/2}e^{-\xi}, \quad \xi \gg 1, \tag{33}$$

we obtain, for the specific situation $2n^2\beta^2 \approx 1$, the following l-harmonics (Schwinger et al., 1976):

$$P_l(\omega, t) = \frac{\omega\mu e^2}{4\pi^2 n^2 R}\sqrt{\frac{\pi}{6}}\left(\frac{3}{2l}\right)^{2/3}\xi^{1/6}e^{-\xi}. \tag{34}$$

Every circular trajectory of electron with velocity in the intervals dv_x, dv_y in the magneto-thermal field with $\mathbf{H}||z$ is realized with the thermal probability

$$2e\frac{e^{-\frac{E-F}{kT}}}{h^3}e^{-\frac{mv_x^2}{2kT}}e^{-\frac{mv_y^2}{2kT}}dv_x dv_y \tag{35}$$

So, the intensity of the synchrotron radiation of all trajectories with l-harmonics denoted by Π_l is given by the formula $v_{x-min} = v_0$

$$\Pi_l(\omega, t) = 2e\frac{e^{-\frac{E-F}{kT}}}{h^3}\int_{v_0}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{mv_x^2}{2kT}}e^{-\frac{mv_y^2}{2kT}}dv_x dv_y \times$$

$$\frac{\omega\mu e^2}{4\pi^2 n^2 R}\sqrt{\frac{\pi}{6}}\left(\frac{3}{2l}\right)^{2/3}\xi^{1/6}e^{-\xi}, \tag{36}$$

where $(\frac{mv_0^2}{2} + E_C = E \geq 0)$ and

$$\xi = l/l_n = \frac{2l}{3}(1 - n^2\beta^2)^{3/2} \quad \beta^2 = (v_x^2 + v_x^2)/c^2 \tag{37}$$

and

$$\xi^{1/6} = \left[\frac{2l}{3}(1-n^2\beta^2)^{3/2}\right]^{1/6} \tag{38}$$

$$e^{-\xi} = \exp\left[-\frac{2l}{3}(1-n^2\beta^2)^{3/2}\right]. \tag{39}$$

If we go to cylindric coordinates ϱ, φ, then

$$\int_{v_0}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y)dv_x dv_y \to \int_{v_0}^{\infty}\int_{-\pi/2}^{\pi/2} f(\varrho, \varphi)\varrho d\varrho d\varphi. \tag{40}$$

After the φ-integration $\to \pi$, we get final spectral power formula for Richardson emission of electron in magnetic field: $\Pi_l(\omega, t)$:

$$\Pi_l(\omega, t) = 2e\frac{e^{-\frac{\Phi}{kT}}}{h^3}\pi\int_{v_0}^{\infty} e^{-\frac{m\varrho^2}{2kT}}\varrho d\varrho\frac{\omega\mu e^2}{4\pi^2 n^2 R}\sqrt{\frac{\pi}{6}}\left(\frac{3}{2l}\right)^{2/3}\xi^{1/6}e^{-\xi};$$

$$\xi^{1/6} = \left[\frac{2l}{3}(1-n^2\varrho^2)^{3/2}\right]^{1/6}; \quad e^{-\xi} = \exp\left[-\frac{2l}{3}(1-n^2\varrho^2)^{3/2}\right]. \tag{41}$$

4 Discussion

We can see that the thermoelectric effect substantially differs from the blackbody photon emission derived by Planck from the modification of the thermodynamical entropy, and later, by Einstein from the Bohr model of atom with two postulates: 1. every atom can exist in the discrete series of states in which electrons do not radiate even if they are moving at acceleration (the postulate of the stationary states), 2. transiting electron from the stationary state to other, emits the energy according to the law $\hbar\omega = E_m - E_n$, called the Bohr formula, where E_m is the energy of an electron in the initial state, and E_n is the energy of the final state of an electron to which the transition is made and $E_m > E_n$.

Einstein introduced coefficients of spontaneous and stimulated emission and in case of spontaneous emission, the excited atomic state decays without external stimulus as an analogue of the natural radioactivity decay. In the process of the stimulated emission, the atom is induced by the external stimulus to make the same transition. The external stimulus is a blackbody photon that has an energy given by the Bohr formula. So, we see the substantial difference between black body emission and thermal electron emission.

There exists the thermionic emission from a single-layer graphene which has been verified with an experiment (Liang et al., 2015). So our theory can be immediately applied to the graphhene physics. In 2003 author suggested an invention called the magnetronic laser (Pardy, 2003). In case that we in this laser use the thermal electrons, we get so called thermal magnetronic laser and it is not excluded that such new laser will play the substational role in the laser physics.

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