Vortex Gradient Formula

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We expound the gradient of Vorticity tensor formula in general coordinates as treated in relativistic mechanics.

The formula of the gradient of the Vorticity tensor (eq.A.26) is derived in general coordinates as treated in classical and relativistic continuum mechanics and as groundwork of Tailherer's theory, a sort of extension of general relativity as shall be clearer afterwards [2].
The basic equations are those of vortex kinematics encountered in lagrangian description of continua [1] relating the angular velocity tensor to the deformation velocity $K_{\alpha\beta} = ½\partial_\tau g_{\alpha\beta}$ (as remarked in [2] identified with the second fundamental tensor relative to V_4): let us start by considering all the points-event of the space spanned by the particles of a continuum as parameterized with their co-ordinates representing the position vector OP with respect to an arbitrary origin O , and a local frame referred to a local basis of vectors $\boldsymbol{e}_\alpha = \partial OP/\partial x^\alpha$ whose the metric tensor $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ and the countervariant frame $g^{\alpha\beta}$ associated with, such that $g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha{}^\beta$. We consider the lagrangian metric $g_{\alpha\beta}(x^\alpha/\tau)$ as function of the trajectory line's variables x^α and time τ , and so $\boldsymbol{e}_\alpha$. Since our reasoning might be done in the 4-dimensional cronotope too (so τ is referred to as the proper time), it follows that if the relations hold in each tern subspace, as we shall see they do, they will keep holding in the whole 4-dimensional space for the same equation that we shall get. So, let us choose without loss of generality the tern referring to the space indexes h=1,2,3. Let us consider now the gradient of the space components of the velocity which will be of the type:

$$\partial_h \boldsymbol{v} = q_{hk} \boldsymbol{e}^k \qquad (\partial_h = \partial / \partial x^h \quad h,k = 1,2,3) \tag{A.1}$$

The matrix q_{hk} can always be split up in a symmetrical part and a skew-symmetric one

$$q_{hk} = \partial_h \boldsymbol{v} \cdot \boldsymbol{e}_k = K_{hk} + \omega_{hk} \tag{A.2}$$

with symmetrical part

$$K_{hk} = 1/2(\partial_h \boldsymbol{v} \cdot \boldsymbol{e}_k + \partial_k \boldsymbol{v} \cdot \boldsymbol{e}_h) = K_{kh} \tag{A.3}$$

and skew-symmetric

$$\omega_{hk} = 1/2(\partial_h \boldsymbol{v} \cdot \boldsymbol{e}_k - \partial_k \boldsymbol{v} \cdot \boldsymbol{e}_h) = -\omega_{kh} \tag{A.4}$$

Since

$$\partial_\tau \boldsymbol{e}_h = \frac{\partial^2 OP}{\partial\tau\,\partial x^h} = \frac{\partial^2 OP}{\partial x^h\,\partial\tau} = \partial_h \boldsymbol{v} \tag{A.5}$$

equ.(A.3) will be written as:

$$K_{hk} = 1/2(\partial_\tau \boldsymbol{e}_h \cdot \boldsymbol{e}_k + \partial_\tau \boldsymbol{e}_k \cdot \boldsymbol{e}_h) = 1/2\partial_\tau (\boldsymbol{e}_h \cdot \boldsymbol{e}_k) = 1/2\partial_\tau g_{hk} \tag{A.6}$$

which can be referred to the second fundamental tensor as already outlined in [2] where it was denoted as deformation velocity of the metric. For what concerns ω_{hk} by taking (A.5) into account let us introduce the vector

$$\boldsymbol{\omega} = ½\, \boldsymbol{e}^h \times \partial_\tau \boldsymbol{e}_h = ½\, \boldsymbol{e}^h \times \partial_h \boldsymbol{v} \qquad \text{with} \ \times \ \text{exterior product} \tag{A.7}$$

Then from the (A.2) and (A.3) we have successively:

$$\boldsymbol{\omega} = ½\, \boldsymbol{e}^{h} \times (K_{hk} + \omega_{hk})\, \boldsymbol{e}^{k} = ½\, (K_{hk}\, \boldsymbol{e}^{h} \times \boldsymbol{e}^{k}) + ½\, (\omega_{hk}\, \boldsymbol{e}^{h} \times \boldsymbol{e}^{k}) \tag{A.8}$$

which in account of the symmetry of K_{hk} and the skew-symmetry of $\boldsymbol{e}^{h} \times \boldsymbol{e}^{k}$ becomes:

$$\boldsymbol{\omega} = ½\, (\omega_{hk}\, \boldsymbol{e}^{h} \times \boldsymbol{e}^{k}) \tag{A.9}$$

$\boldsymbol{\omega}$ will be named angular velocity and characterised by the coefficients ω_{hk}.
Moreover, if we multiply (A.7) by $\boldsymbol{e}_{h}$ through the exterior product, taking into account that $\boldsymbol{e}^{h} \cdot \boldsymbol{e}_{k} = \delta^{h}_{k}$, we get:
$\boldsymbol{\omega} \times \boldsymbol{e}_{h} = ½\, (\omega_{lk}\, \boldsymbol{e}^{l} \times \boldsymbol{e}^{k}) \times \boldsymbol{e}_{h} = ½\, \omega_{lk} (\boldsymbol{e}^{l} \cdot \boldsymbol{e}_{h}\, \boldsymbol{e}^{k} - \boldsymbol{e}^{k} \cdot \boldsymbol{e}_{h}\, \boldsymbol{e}^{l}) = ½\, \omega_{lk} (\delta^{l}_{h}\, \boldsymbol{e}^{k} - \delta^{k}_{h}\, \boldsymbol{e}^{l})$ and therefore the relation:

$$\boldsymbol{\omega} \times \boldsymbol{e}_{h} = \omega_{hk}\, \boldsymbol{e}^{k} \qquad (h,k=1,2,3) \tag{A.10}$$

Let us now make some recalls. By differentiating the vectors $\boldsymbol{e}_h\, (x^{\mu}/\tau)$ of the local base with respect to the proper time we get the gradient of the space components of 4-velocity as from (A.5). In deriving them with respect to x^{μ} we get for definition the Christoffel symbols as well-known in differential geometry:

$$\partial_j \boldsymbol{e}_{h} = \Gamma^{k}_{\ jh} \boldsymbol{e}_{k} \qquad\qquad \partial_j \boldsymbol{e}^{h} = -\Gamma^{h}_{\ jk} \boldsymbol{e}^{k} \tag{A.11}$$

Let us recall the links between the Christoffel symbols of the first and second kind:

$$\Gamma^{k}_{\ jh} = g^{kr}\, \Gamma_{jh,r} \qquad\qquad \Gamma_{jh,r} = g_{rk}\, \Gamma^{k}_{\ jh} \tag{A.12}$$

From (A.11) it turns out that

$$\partial_j \boldsymbol{v} = \partial_j (v_h\, \boldsymbol{e}^{h}) = (\partial_j v_h - \Gamma^{k}_{\ jh}\, v_k)\, \boldsymbol{e}^{h} = (\nabla_j v_h)\, \boldsymbol{e}^{h} \tag{A.13}$$

leading via (A.4) to the expression:

$$\omega_{hk} = 1/2\, (\nabla_h v_k - \nabla_k v_h) = 1/2\, (\partial_h v_k - \partial_k v_h) \tag{A.14}$$

and by taking advantage of the symmetry of Christoffel symbols with respect to inferior indexes.
Analogously we get for (A.3):

$$K_{hk} = 1/2\, (\nabla_h v_k + \nabla_k v_h) \tag{A.15}$$

as usual for the deformation tensor.
The gradient of the velocity expressed in terms of deformation and angular velocity follows as from (A.1), (A.2) and (A.10):

$$\partial_h \boldsymbol{v} = \underline{K}_h + \boldsymbol{\omega} \times \boldsymbol{e}_{h} \tag{A.16}$$

with $\boldsymbol{K}_h$ following as from (A.6):

$$\boldsymbol{K}_h = K_{hk}\, \boldsymbol{e}^{k} = ½\, \partial_\tau K_{hk}\, \boldsymbol{e}^{k} \tag{A.17}$$

From (A.16) we can infer $\boldsymbol{\omega}$ to depend on $\boldsymbol{K}$, that is to say, on the deformation velocity as will be seen better next. To see that let us derive both the members of (A.9) with respect to x^{j}. We get:

$$\partial_j \boldsymbol{\omega} = ½\, \partial_j \boldsymbol{e}^{h} \times \partial_\tau \boldsymbol{e}_{h} + ½\, \boldsymbol{e}^{h} \times \partial_\tau (\partial_j \boldsymbol{e}_{h}) \tag{A.18}$$

as well as on using (A.11):

$$\partial_j \boldsymbol{\omega} = -\,½\,\Gamma^{h}{}_{jk}\,\boldsymbol{e}^{k}\times\partial_\tau\boldsymbol{e}_{h} + ½\,\boldsymbol{e}^{h}\times\Gamma^{k}{}_{jh}\,\partial_\tau\boldsymbol{e}_{k} + ½\,\partial_\tau(\Gamma^{k}{}_{jh})\,\boldsymbol{e}^{h}\times\boldsymbol{e}_{k} \qquad \text{(A.19)}$$

Since the first two terms vanish as it is understood by changing the indexes h and k, it turns out:

$$\partial_j \boldsymbol{\omega} = ½\,\partial_\tau(\Gamma^{k}{}_{jh})\,\boldsymbol{e}^{h}\times\boldsymbol{e}_{k} \qquad \text{(A.20)}$$

On the other hand, since $\Gamma_{jh,r} = ½\,(\partial_j g_{hr} + \partial_h g_{rj} - \partial_r g_{jh})$ and taking into account (A.6) we have:

$$\partial_\tau\Gamma_{jh,r} = \partial_j K_{hr} + \partial_h K_{rj} - \partial_r K_{jh} = \nabla_j K_{hr} + \nabla_h K_{rj} - \nabla_r K_{jh} + 2\,\Gamma^{k}{}_{jh} K_{kr}$$

where we used the definition of covariant derivative:

$$\nabla_j K_{hr} = \partial_j K_{hr} - \Gamma^{k}{}_{jh} K_{kr} - \Gamma^{k}{}_{jr} K_{hk} \qquad \text{(A.21)}$$

Making use of the triple tensor $q_{jh,r} = \nabla_j K_{hr} + \nabla_h K_{rj} - \nabla_r K_{jh}$ we obtain the following expression of the time derivative of Christoffel symbols of first kind:

$$\partial_\tau\Gamma_{jh,r} = q_{jh,r} + \Gamma^{k}{}_{jh}\,\partial_\tau g_{kr} \qquad \text{(A.22)}$$

Moreover, by differentiating (A.12)$_2$ with respect to proper time we get for the precedent relation:

$$\partial_\tau g_{rk}\,\Gamma^{k}{}_{jh} + g_{rk}\,\partial_\tau\Gamma^{k}{}_{jh} = q_{jh,r} + \Gamma^{k}{}_{jh}\,\partial_\tau g_{kr} \qquad \text{i.e.}$$

$$\partial_\tau\Gamma^{k}{}_{jh} = g^{kr}\,q_{jh,r} = q^{k}{}_{jh} \qquad \text{(A.23)}$$

which is plainly a tensor. Hence equ.(A.20) becomes:

$$\partial_j \boldsymbol{\omega} = ½\,q_{jh,k}\,\boldsymbol{e}^{h}\times\boldsymbol{e}^{k} \qquad \text{(A.24)}$$

or because of (A.21) and the skew-symmetry of the exterior product:

$$\partial_j \boldsymbol{\omega} = \nabla_h K_{kj}\;\boldsymbol{e}^{h}\times\boldsymbol{e}^{k} \qquad \text{(A.25)}$$

Then, by differentiating (A.9) we obtain:

$\partial_j\,\boldsymbol{\omega} = ½\;\partial_j\omega_{hk}\,(\boldsymbol{e}^{h}\times\boldsymbol{e}^{k}) + ½\;\omega_{hk}\,\partial_j\,(\boldsymbol{e}^{h}\times\boldsymbol{e}^{k})$ and taking (A.11)$_2$ into account and the definition of covariant derivative for ω_{hk} we finally arrive to the differential expressions:

$$\nabla_j\,\omega_{hk} = \nabla_h K_{kj} - \nabla_k K_{hj} \qquad (j,h,k = 1,2,3) \qquad \text{(A.26)}$$

Extending (A.26) to the 4-dimensional cronotope (also making $K_{\mu\nu}$ and $\omega_{\mu\nu}$ dimensionally as a [length]$^{-1}$ by re-defining them dividing by the light speed c) and entering the Tailherer's ansatz: $C_{\mu\nu} = S\omega_{\mu\nu}$, $C_{\mu\nu} = R_{\mu\nu\rho\sigma}\in^{\rho\sigma}$, with $\in^{\alpha\beta}$ any constant skew-symmetric tensor, we have a second gravitational equation:

$$\nabla_\sigma C_{\mu\nu} = S\,(\nabla_\mu K_{\nu\sigma} - \nabla_\nu K_{\mu\sigma}) \qquad (\mu,\nu,\sigma = 1,2,3,4)$$

with S = (2.5±1.2)E-19 m^{-1} [3] . By choosing $\in^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ Lorentz invariance is yet preserved, however general one is broken as discussed in [4], just regarding the gravitational wave phenomenon as symmetry breaking of general relativity .

References

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[2] Antonelli S 2014 Outstanding Outcomes from a Recent Theory of Gravity *International Journal of Physics,* Vol. 2, No.6, pp.267-276.

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[4] Antonelli, S. 2018 Appraisal of a new gravitational constant. *International Journal of Physics,* 3, 4, 139-149.