Zero and infinity; their interrelation by means of division by zero

Saburou Saitoh
Institute of Reproducing Kernels
Kawauchi-cho, 5-1648-16, Kiryu 376-0041, JAPAN
saburou.saitoh@gmail.com

February 14, 2019

Abstract: In this paper, we first fix the definitions of zero and infinity in very general senses and we will give their simple and definite relation by means of division by zero. On this problem and relation we have considered over the long history beyond mathematics. As our mathematics, we will be able to obtain some definite result for the relation clearly with new concept and model since Aristotle and Euclid.

Key Words: Division by zero, division by zero calculus, singular point, $0/0 = 1/0 = z/0 = 0$, infinity, discontinuous, point at infinity, stereographic projection, Riemann sphere, horn torus, Laurent expansion, conformal mapping.

2010 Mathematics Subject Classification: 00A05, 00A09, 30C25, 30F10

1 Division by zero calculus

For the long history of division by zero, see [2, 19]. Further quite recently, the authors S.K.S. Sen and R. P. Agarwal [25] published the book on ZERO and they discuss also on the division by zero in connection with the paper [5], deeply. However, as a conclusion, they stated that “Thou shalt not divide by zero” remains valid eternally.

Therefore, we see that division by zero is still in confusions, see also [16] for the recent situations.

The division by zero with the mysterious and long history was indeed trivial and clear as in the followings:

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $a/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [5] for example.

Division by zero is trivial and clear from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [5, 26].

The simple field structure containing division by zero was established by M. Yamada ([8]). For a simple introduction, see Okumura [17].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [11, 12, 13, 14, 15, 16] for example.

The division by zero opens a new world since Aristotele-Euclid. See the references for recent related results.

As the number system containing the division by zero, the Yamada field structure is completed. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z - i}{z + i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z - (-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z - i) \cdot \frac{1}{z + i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.5}$$

we **define** the identity, by the division by zero

$$f(a) = C_0. \tag{1.6}$$

Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.

For the correspondence (1.6) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (1.5), we can **define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

For the fundamental function $W = 1/z$ we did not consider any value at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [27, 28, 29]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([9, 10]) for example.

In order to show clearly this strong discontinuity of the function $y = 1/x$ at the origin, we gave the simple and clear physical examples by Ctesíbio (BC. 286-222) and E. Torricelli (1608 -1646) in [23].

Therefore, the division by zero will give great impacts to calculus, Euclidian geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic ideas for the space and universe.

For the extended complex plane, we consider its stereographic projection mapping as the Riemann sphere and the point at infinity is realized as the north pole in the Alexsandroff's one point compactification. The Riemann sphere model gives a beautiful and complete realization of the extended complex plane through the stereographic projection mapping and the mapping has beautiful properties like isogonal (equiangular) and circle to circle correspondence (circle transformation). Therefore, the Riemann sphere is a very classical concept [1].

Now, with the division by zero we have to admit the strong discontinuity at the point at infinity, because the point at infinity is represented by zero. In [10], a formal contradiction for the classical result $1/0 = \infty$ was given and the strong discontinuity was shown in many and many examples. See the papers in the references.

On this situation, V. V. Puha discovered the mapping of the extended complex plane to a beautiful horn torus at (2018.6.4.7:22) and its inverse at (2018.6.18.22:18).

Incidentally, independently of the division by zero, Wolfgang W. Däumler has various special great ideas on horn torus as we see from his site:

Horn Torus & Physics (https://www.horntorus.com/) 'Geometry Of Everything', intellectual game to reveal engrams of dimensional thinking and proposal for a different approach to physical questions ...

Now, we will be able to realize simply and surprisingly the basic relation between zero and infinity. Here, for infinity, we have many senses and feelings, however, in this paper it will mean that infinity is the point at infinity.

In this paper we would like to establish clearly the basic relation of zero and infinity for the long interest by means of the division by zero. See for example [19, 25] for the related problems.

The horn torus model realizes this relation since Aristotele-Euclid. We introduce simply a horn torus model for the classical Riemann sphere from the viewpoint of the division by zero based on [4]. This model seems to be important for us.

2 Conclusion

Our conclusion of this short paper is that:

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea.

For the identity $1/0 = 0$, we have two interpretations. First of all, $1/0$ is not the usual sense, because if it is a usual sense, when we set $1/0 = a$, then we have the contradiction that $a \times 0 = 0$, immediately.

The point at infinity may be considered as the limiting

$$\lim_{z \to 0} \frac{1}{z} = \infty;$$

these meanings in the both sides are given by strictly sense of the $\epsilon - \delta$ logic. In this sense, we write it as follows:

$$\frac{1}{0} = \infty.$$

Meanwhile, in the sense of the division by zero, for the function

$$f(z) = \frac{1}{z}$$

we have the identity

$$f(0) = 0.$$

Here, note that there is no any problem, because it was given by the value of the function $f(z)$ at $z = 0$. We also can write it as follows:

$$\frac{1}{0} = 0,$$

from the form. **For these two interpretations, we have ∞ and** 0.

This statement is the conclusion of this paper. Indeed, we think we were able to state the basic relation between ZERO and INFINITY by means of the division by zero against the classical idea.

3 Horn torus model

In order to see the good model that shows geometrically the relation of zero and the point at infinity, we see simply the horn torus model.

We will consider the three circles represented by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2, \tag{3.1}$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The stereographic projection mapping from (x, y) plane to the Riemann sphere is given by

$$\xi = \frac{x}{x^2 + y^2 + 1},$$

$$\eta = \frac{y}{x^2 + y^2 + 1},$$

and

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Of course,

$$\xi^2 + \eta^2 = \zeta(1 - \zeta).$$

and

$$x = \frac{\xi}{1 - \zeta}, y = \frac{\eta}{1 - \zeta}.$$

In these formulas, we can see the division by zero

$$0 = \frac{0}{0}$$

that shows the mapping from $(0, 0, 1)$ to $(0, 0)$. The point at infinity is mapped to the origin, by the division by zero. However, of course, for the limiting $z \to \infty$, z tends to ∞: the point at infinity.

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

$$\eta = \frac{2y\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

and

$$\zeta = \frac{(x^2 + y^2 - 1)\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center (0,0,1/2) and the horn torus. This is the desired point on the horn torus for the plane point.

The inversion is given by

$$x = \xi \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} \tag{3.2}$$

and

$$y = \eta \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} . \tag{3.3}$$

In these formulas, we can see the division by zero

$$0 = \frac{0}{0}$$

that shows the mapping from $(0, 0, 1/2)$ to $(0, 0)$.

For the properties of horn torus with physical applications, see [3].

4 Conformal mapping from the plane to the horn torus with a modified mapping

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model (2018.8.18.09):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

Our situation is invariant by rotation around ζ axis, and so we shall consider the problem on the ξ, ζ plane.

Let $N(0,0,1)$ be the north pole. Let $P'(\xi,\eta,\zeta)$ denote a point on the Riemann sphere and let $z = x+iy$ be the common point with the line NP' and $\zeta = 0$ plane (: $z = x+iy$); that is P' is the stereographic projection map of the point $z = x+iy$ onto the unit sphere.

Let $M(1/4,0,1/2)$ be the center of the circle (2.1). Let P'' be the common point of the line $SP'(S = S(0,0,1/2))$ and the circle (2.1).

Let Q' be $(0,0,\zeta)$ that is the line $Q'P'$ is parallel to the x axis. Let Q'' and M'' be the common points with the ζ axis and $\xi = 1/4$ with the parallel line to the x axis through the point P'', respectively.

Further, we set $\alpha = \angle OSP' = \angle P''IS = (1/2)\angle P''MS$ (I := I(1/2,0,1/2)). We set P for the point on the horn torus such that $\phi = \angle SMP$ and Q be the point on the ζ axis such that the line QP is parallel to the x axis.

Then, we have:

$$\overline{P'Q'} = \frac{1}{2}\sin\alpha,$$

$$\overline{P''M''} = \frac{1}{4}|\cos(2\alpha)|,$$

$$\overline{P''Q''} = \frac{1}{4}(1-\cos(2\alpha)),$$

the length of latitude through P' is

$$2\pi\overline{P'Q'} = \pi\sin\alpha,$$

and the length of latitude through P''

$$2\pi\overline{P''Q''} = \frac{\pi}{2}(1-\cos(2\alpha)) = \pi\sin^2\alpha.$$

Similarly, we have

$$2\pi\overline{QP} = \frac{\pi}{2}(1 - \cos\phi).$$

In order to become the conformal mapping from the point P' to the point P, we have the identity

$$d\alpha : d\phi = \sin\alpha : 1 - \cos\phi;$$

that is we have the differential equation

$$\frac{d\alpha}{\sin\alpha} = \frac{d\phi}{1 - \cos\phi}.$$

Note here that the radius of the circle (2.1) is half of the stereographic projection mapping circle (the Riemann sphere). We solve this differential equation as, with an integral constant C

$$\log \mid \tan\frac{\alpha}{2} \mid = -\cot\frac{\phi}{2} + C.$$

For this derivation of the differential equation, see the detail comments in the site : **conformal mapping sphere ↔ horn torus** with beautiful figures and many informations, by W. W. Däumler. In order to check his idea, we gave a complete proof analytically, in [4].

Using the correspondence

$$\alpha = 0 \leftrightarrow \phi = 0,$$

or

$$\alpha = \pi/2 \leftrightarrow \phi = \pi$$

or

$$\alpha = \pi \leftrightarrow \phi = 2\pi,$$

we have $C = 0$. Note that $\tan(\pi/2) = 0$, $\cot(\pi/2) = 0$ and $\log 0 = 0$ ([7]). Note also that the function $y = e^x$ takes two values 1 and 0 at $x = 0$. By solving for ϕ we have the result

$$\phi = 2\cot^{-1}(-\log \mid \tan(\alpha/2) \mid) \tag{4.1}$$

or

$$\alpha = 2\tan^{-1}(e^{(-\cot(\phi/2))}). \tag{4.2}$$

Next, note that

$$\tan\frac{\alpha}{2} = |z|$$

and

$$\alpha = 2\tan^{-1}|z|. \tag{4.3}$$

We thus have

$$\phi = 2\cot^{-1}(-\log|z|) \tag{4.4}$$

and the inverse is

$$|z| = e^{-\cot(\phi/2)}. \tag{4.5}$$

We thus obtain the complicated conformal mapping for the extended z plane to the horn torus by (4.4) and (4.2). The inverse conformal mapping for the horn torus to the extended complex z plane is given by (4.1) and (4.5).

We can represent the direct Däumler mapping from the extended z plane onto the horn torus as follows (V. V. Puha: 2018.8.28.22:31): With (4.4)

$$\xi = \frac{x\cdot(1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}}, \tag{4.6}$$

$$\eta = \frac{y\cdot(1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}}, \tag{4.7}$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}. \tag{4.8}$$

In order to check the conformality of the Däumler mapping, we gave the analytical proof in the paper [4].

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