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Anomaly in sign function - probability function integration III

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Abstract In the paper it is demonstrated that integration of products of sign functions and probability density functions such as in Bell's formula for ± 1 measurement functions, leads to inconsistencies.

Keywords Inconsistency, Bell's theorem.

1 Introduction

In 1964, John Bell wrote a paper [1] on the possibility of hidden variables [2] causing the entanglement correlation $E(a,b)$ between two particles. In his famous paper, Einstein [2] argued that the quantum description must be supplemented with extra variables to explain the entanglement phenomenon. von Neuman [4] presented a mathematical proof that any hidden variables theory is in conflict with quantum mechanics. However, one can doubt if von Neuman's view on the matter was completely related to the physics. In the present paper, an inconsistency in the starting formula of Bell [1] will be demonstrated.

Bell, based his hidden variable description on particle pairs with entangled spin, originally formulated by Bohm [3]. Modern research from e.g. Nordén arrives at the conclusion that we don't need the Bell experiments and theorem to make a difference between classical and quantum explanations [7]. In an earlier article Norden [8], already draws attention is to the strange linear dependence on angle between polarizers that the Bell model assumes but for which it gives no explanation.

In his correlation expression, Bell [1] uses hidden variables λ that are elements of a universal set Λ and are distributed with a density $\rho(\lambda) \geq 0$. Suppose, $E(a,b)$ is the correlation between measurements with distant A and B that have unit-length, i.e. $||a|| = ||b|| = 1$, real 3 dim parameter vectors a and b. Then with the use of hypotherical λ we can write down the classical probability correlation between the two simultaneously measured spins of the particles. This is what we will call Bell's formula.

$$E(a,b) = \int_{\lambda\in\Lambda} \rho(\lambda)A(a,\lambda)B(b,\lambda)d\lambda \tag{1.1}$$

The spin measurement functions are, $A(a,\lambda) \in \{-1,1\}$ and $B(b,\lambda) \in \{-1,1\}$. The probability density is normalized, $\int \rho(\lambda)d\lambda = 1$.

In our present paper we look at a particular subset of models and question the methodology of the derivation of inequalities from Bell's formula.

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2 Example computation

Bell's formula (1.1) is very general. That means that it has to be valid for all kinds sub-cases where $\pm$ functions are used.

In the present example we will concentrate the attention on the following expression for e.g. the measurement function $A = A(a,\lambda) \in \{-1,1\}$. Let us also, for the sake of the example, make use of one single Gaussian density function and change the notation slightly. We have

$$\mathbb{P}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \tag{2.1}$$

Therefore, $\rho_{Gauss}(x) = \frac{d}{dx}\mathbb{P}(x)$ in this example. To be even more precise, let us concentrate on a sub-case of Bell's formula where

$$E = \int_{-\infty}^{+\infty} A(a,x)\rho_{Gauss}(x)dx \tag{2.2}$$

is a part of the computation of a more complete correlation. Let us define, for $a \in \mathbb{R} \cap [-1,1]$,

$$A(a,x) = \lim \left(\frac{H_n(x-a)+1}{3H_n(x-a)-1} \right) = \begin{cases} +1, & x \geq a \\ -1, & x < a \end{cases} \tag{2.3}$$

The lim notation is introduced to abbreviate $\lim_{n\to\infty}$. In order to remain close to the physics of the problem, we select $a \in [-1,1]$. Here we have $A(a,x) = \text{sign}(x-a)$. If a critic believes that (2.3) cannot be used then he has to explain if Bell's theorem and the associated experiments are still sufficiently general to allow all kinds of conclusions about the yes/no existence of Einstein hidden variables and inseparability [2] and [5]. It is based on, $H(x) = 1 \Leftrightarrow x \geq 0$ and $H(x) = 0 \Leftrightarrow x < 0$. The closed limit Heaviside form is here equal to:

$$H(x) = \lim H_n(x) = \lim \exp\left(-\frac{e^{-nx}}{n}\right) \geq 0 \tag{2.4}$$

It must be noted that (2.3) is perfectly in order when in a Bell formula we are looking for a function $A \in \{-1,1\}$.

We observe that in (2.2) there is the case of "$\int$ before lim". In Appendix 1 it is demonstrated that "$\int$ before lim" and "lim before $\int$" give the same result. It is noted that there are many limits to only *one* sign function. We are therefore allowed to use the "lim before $\int$" form equipped with (2.3) and write in Riemanian integration form:

$$E = \lim \int_{-\infty}^{+\infty} \frac{d}{dx}\mathbb{P}(x) \left(\frac{H_n(x-a)+1}{3H_n(x-a)-1} \right) dx \tag{2.5}$$

Because all normal concrete mathematical operations can be performed on (2.5), we may use partial integration. If this is not the case then the critic has to explain why he believes the Bell theorem is still general. Pending that verdict,

$$E = 1 - \lim \int_{-\infty}^{\infty} \mathbb{P}(x) \frac{d}{dx} \left(\frac{H_n(x-a)+1}{3H_n(x-a)-1} \right) dx = \tag{2.6}$$

$$1 - \lim \int_{-\infty}^{\infty} \mathbb{P}(x) \frac{\frac{dH_n(x-a)}{dx}}{(3H_n(x-a)-1)^2} \left\{ (3H_n(x-a)-1) - 3\left(H_n(x-a)+1\right) \right\} dx$$

Therefore

$$E = 1 + 4 \lim \int_{-\infty}^{\infty} \mathbb{P}(x) \frac{\frac{dH_n(x-a)}{dx}}{(3H_n(x-a)-1)^2} dx \tag{2.7}$$

From the definition (2.4) it can be concluded that $\frac{dH_n(x-a)}{dx} \geq 0$ for all $x \in \mathbb{R}$ and arbitrary $n > 0$. Note that x and n are independent. Moreover, note that $\forall_{x \in \mathbb{R}} 0 \leq \mathbb{P}(x) \leq 1$. Hence, with a similar inequality producing procedure such as in [1] we have

$$\mathbb{P}(x) \frac{\frac{dH_n(x-a)}{dx}}{(3H_n(x-a)-1)^2} \leq \frac{\frac{dH_n(x-a)}{dx}}{(3H_n(x-a)-1)^2} \tag{2.8}$$

This implies looking at (2.7),

$$\begin{gathered} E \leq 1 + 4 \lim \int_{-\infty}^{\infty} \frac{\frac{dH_n(x-a)}{dx}}{(3H_n(x-a)-1)^2} dx = \\ 1 - \frac{4}{3} \lim \int_{-\infty}^{\infty} \frac{d}{dx} \left(3H_n(x-a)-1\right)^{-1} dx = \\ 1 - \frac{4}{3} \lim \left\{ \frac{1}{3H_n(\infty)-1} - \frac{1}{3H_n(-\infty)-1} \right\} = \\ 1 - \frac{4}{3} \left\{ \frac{1}{(3*1)-1} - \frac{1}{(3*0)-1} \right\} = \\ 1 - \frac{4}{3} \left\{ \frac{1}{2} - \frac{1}{-1} \right\} = 1 - \frac{4}{3} \left\{ \frac{1}{2} + 1 \right\} = 1 - \frac{4}{3}\frac{3}{2} = 1 - 2 = -1 \end{gathered} \tag{2.9}$$

Therefore, this excercise gives $E \leq -1$. However, this holds true for all values of $a \in [-1, 1]$. Hence, the outcome $E \leq -1$ is unacceptable.

3 Conclusion

In the paper it is demonstrated that the operations leading to Bell inequalities are in some cases leading to unacceptable outcomes. The expected $E = 1 - 2\mathbb{P}(a)$ that can be obtained from the sign-and-density integration stands in contrast to the derived $E \leq -1$. The latter is based on the same operations such as employed in derivations of Bell inequialities. If one wants to contest the conclusion that Bell's formula is ill defined and want to use persistent poles in the (2.5) integral then

- one destroys the ± 1 meaning of the Bell formula because H then projects in e.g. $\{0, \frac{1}{3}, 1\}$. Sign functions like $2H - 1$ then no longer project in $\{-1, 1\}$.
- one has to explain why the expected outcome $E = 1 - 2\mathbb{P}(a)$ can be obtained as well from (2.7).

We contest the validity of the objection "there are poles in the integrand of (2.5)" and believe this is untrue and contradictory with a meaningful Bell formula. The confusion around this theme is related to Bell's formula and can *not* be attributed to the methods such as presented here and in [6]. Key question is: can the present sign function limit representation be excluded from consideration for spin measurement function.

The author is aware of the fact that the claim is controversial. At its least we can argue that a Bell experiment [9] most likely excludes certain Einstein type models [2] and [5] from consideration.

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Appendix 1

Here we look at interchanging between limit and integral and note that limit parameter, n, is *independent* of the integration variable $x \in \mathbb{R}$. We will show in this appendix that, with a specific example "$\int$ before lim" gives the same as "lim before $\int$". The example is specific but generalizes to the case under attentionm in the main text because there is only one sign function but a multitude of limits that may approach it.

Suppose, for $x \in \mathbb{R}$, we look at a density

$$\rho(x) = \begin{cases} x, & 0 \leq x \leq \sqrt{2}, \\ 0, & elsewhere \end{cases} \tag{3.1}$$

Furthermore, let us model the sign function, for $0 < a < \sqrt{2}$ with

$$\text{sign}(x-a) = \lim \left\{ \left(\frac{1}{2}\right)^{2n\sqrt{(x-a)^2}} + \frac{1}{\pi} \arctan\left[n(x-a)\right] \right\} \tag{3.2}$$

with $\lim \equiv \lim_{n\to\infty}$ such as in the main text. Moreover, $\sqrt{(x-a)^2}$ is positive or zero. If $x \leq a$, then $\sqrt{(x-a)^2} = -(x-a)$. If $x \geq a$ then so $\lim \int \sqrt{(x-a)^2} = (x-a)$. The expression in (3.2) is a realization of $\text{sign}(x-a) = 1 \Leftrightarrow x \geq a$ and $\text{sign}(x-a) = -1 \Leftrightarrow x < a$. Note that $\lim \left(\frac{1}{2}\right)^{2n\sqrt{(x-a)^2}} = 0, \Leftrightarrow x \neq a$ and $\lim \left(\frac{1}{2}\right)^{2n\sqrt{(x-a)^2}} = 1, \Leftrightarrow x = a$.

We compute the limit first "outside the integral". Let us first look at the arctan integral.

$$\begin{aligned} B_a = \int_0^{\sqrt{2}} x\text{sign}(x-a)dx = \frac{2}{\pi} \lim \int_0^{\sqrt{2}} x \arctan\left[n(x-a)\right] dx = \\ \frac{2}{\pi} \lim \int_0^{\sqrt{2}} \frac{1}{2} \frac{d}{dx} x^2 \arctan\left[n(x-a)\right] dx \end{aligned} \tag{3.3}$$

Partial integration then gives, $0 < a < \sqrt{2}$, with a of course independent of n,

$$\begin{aligned} B_a = \frac{\{\sqrt{2}\}^2}{2} \frac{2}{\pi} \lim \arctan\left[n(\sqrt{2} \ - a)\right] + \\ -\frac{1}{\pi} \lim \int_0^{\sqrt{2}} x^2 \frac{d}{dx} \arctan\left[n(x-a)\right] dx \end{aligned} \tag{3.4}$$

Therefore,

$$B_a = 1 - \frac{1}{\pi} \lim \int_0^{\sqrt{2}} x^2 \frac{n}{1+n^2(x-a)^2} dx \tag{3.5}$$

Suppose $y = x - a$ then this gives for $x^2 = (y+a)^2 = y^2 + 2ya + a^2$

$$B_a = 1 - \frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \left(y^2 + 2ya + a^2\right) \frac{ndy}{1+n^2y^2} \tag{3.6}$$

Or,

$$B_a = 1 - \frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{ny^2}{1+n^2y^2} dy + \tag{3.7}$$

$$-\frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{2nay}{1+n^2y^2} dy - \frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{na^2}{1+n^2y^2} dy$$

Now, we may note that $\frac{ny^2}{1+n^2y^2} = \frac{1}{n}\left(1 - \frac{1}{1+n^2y^2}\right)$. The first right hand integral in (3.7) then is

$$\frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{ny^2}{1+n^2y^2} dy = \lim \frac{1}{n\pi} \int_{-a}^{\sqrt{2}-a} \left(1 - \frac{1}{1+n^2y^2}\right) dy \tag{3.8}$$

Hence, the combined first and third integral of (3.7) is,looking at (3.8)

$$\frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{ny^2}{1+n^2y^2} dy + \frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{na^2}{1+n^2y^2} dy = \tag{3.9}$$

$$\frac{1}{\pi} \lim \left\{ \int_{-a}^{\sqrt{2}-a} \left(\frac{1}{n} + \frac{na^2 - \frac{1}{n}}{1+n^2y^2} \right) dy \right\} =$$

$$\frac{1}{\pi} \lim \left\{ \frac{\sqrt{2}}{n} + \frac{1}{n}\left(na^2 - \frac{1}{n}\right) \left[\arctan(n(\sqrt{2}-a)) - \arctan(-na)\right] \right\} =$$

$$\frac{1}{\pi} \lim \left\{ \left(a^2 - \frac{1}{n^2}\right) \left[\frac{\pi}{2} - (-\frac{\pi}{2})\right] \right\} = a^2$$

The second integral of (3.7) is

$$B'_a = \frac{1}{\pi} \lim \int_{-a}^{\sqrt{2}-a} \frac{2nay}{1+n^2y^2} dy \tag{3.10}$$

Suppose $z = ny$. Then $dz = ndy$ and

$$\frac{ny}{1+n^2y^2} = \frac{1}{2n} \frac{1}{1+z^2} \frac{dz^2}{dz} = \frac{1}{2n} \frac{d}{dz} \log\left(1+z^2\right) \tag{3.11}$$

Hence, looking at (3.10)

$$B' = \frac{1}{\pi} \lim \int_{-an}^{n(\sqrt{2}-a)} \frac{1}{2n} \left(\frac{d}{dz} \log\left(1+z^2\right) \right) \frac{dz}{n} = \tag{3.12}$$

$$\frac{1}{\pi} \lim \frac{1}{2n^2} \left\{ \log\left(1 + (\sqrt{2}-a)^2 n^2\right) - \log\left(1 + (-a)^2 n^2\right) \right\} = 0$$

Therefore, from (3.7)-(3.12) and substitution in (3.5) it follows that

$$B_a = 1 - a^2 \tag{3.13}$$

Then let us look at the influence of $\left(\frac{1}{2}\right)^{2n\sqrt{(x-a)^2}} = \exp\left[-n\sqrt{(x-a)^2}\log 2\right]$ term and demonstrate that this vanishes for $n \to \infty$. We have

$$C_a = \lim \int_0^{\sqrt{2}} x \exp\left[-n\sqrt{(x-a)^2}\log 2\right] dx = \tag{3.14}$$

$$\lim \int_0^a x \exp\left[n(x-a)\log 2\right] dx + \lim \int_a^{\sqrt{2}} x \exp\left[-n(x-a)\log 2\right] dx$$

Let us look at the first integral on the right hand of (3.14) and call it C_{1a}. Hence,

$$C_{1a} = \lim \int_0^a x \exp\left[n(x-a)\log 2\right] dx = \tag{3.15}$$

$$\lim \frac{\exp\left[-na\log 2\right]}{n\log 2} \int_0^a x \frac{d}{dx} \exp\left[nx\log 2\right] dx =$$

$$\lim \left\{ \frac{a}{n\log 2} - \frac{\exp\left[-na\log 2\right]}{n\log 2} \int_0^a \exp\left[nx\log 2\right] dx \right\}$$

Therefore we have

$$C_{1a} = -\lim \frac{\exp\left[-na\log 2\right]}{(n\log 2)^2} \int_0^a \frac{d}{dx} \exp\left[nx\log 2\right] dx = \tag{3.16}$$

$$-\lim \frac{\exp\left[-na\log 2\right]}{(n\log 2)^2} \left(1 - \exp\left[na\log 2\right]\right) =$$

$$\left(1 - \exp\left[-na\log 2\right]\right) = 0$$

Subsequently let us, with $y = x - a$ rewrite the second integral on the right hand side of (3.14) as

$$C_{2a} = \lim \int_a^{\sqrt{2}} x \exp\left[-n(x-a)\log 2\right] dx = \tag{3.17}$$

$$\lim \int_0^b (y+a) \exp\left[-ny\log 2\right] dy =$$

$$\lim \frac{-1}{n\log 2} \int_0^b (y+a) \frac{d}{dy} \exp\left[-ny\log 2\right] dy$$

with $b = \sqrt{2} - a > 0$. Therefore, after partial integration we may write

$$C_{2a} = \lim \frac{-1}{n\log 2} \left\{ (b+a)\exp\left[-nb\log 2\right] - a - \int_0^b \exp\left[-ny\log 2\right] dy \right\} = \tag{3.18}$$

$$\lim \frac{1}{(n\log 2)^2} \int_0^b \frac{d}{dy} \exp\left[-ny\log 2\right] dy =$$

$$\lim \frac{1}{(n\log 2)^2} \left(\exp\left[-nb\log 2\right] - 1\right) = 0$$

Hence, $B_a + C_a = 1 - a^2$ is the result of the $\lim \int$ form of integration. If we then look at the limit beyond the integration, i.e. look at $\int \lim$, then we see

$$D_a = \int_0^{\sqrt{2}} x \ \mathrm{sign}(x-a) dx = -\int_0^a x dx + \int_a^{\sqrt{2}} x dx = \tag{3.19}$$

$$-\frac{1}{2}a^2 + 1 - \frac{1}{2}a^2 = 1 - a^2$$

The latter result, based on $\int \lim$, agrees with the former derived from $\lim \int$ given in (3.13). This likely ends the discussion about the interchange between integral and limits. We have that $\lim \int$ gives the same result as $\int \lim$, i.e. $D_a[\int \lim] = (B_a + C_a)[\lim \int]$. The limit representation in (3.2) is a particular form for the general sign (sign(0)=1) function. The probability density was a convenient choice.