Definitive Proof of the Near-Square Prime Conjecture, Landau's Fourth Problem

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1 Abstract

The Near-Square Prime conjecture, states that there are an infinite number of prime numbers of the form x^2+1. In this paper, a function was derived that determines the number of prime numbers of the form x^2+1 that are less than n^2+1 for large values of n. Then by mathematical induction, it is proven that as the value of n goes to infinity, the function goes to infinity, thus proving the Near-Square Prime conjecture.

2 Functions and Sets

Let the function $l(x)$ be the largest prime number of the form $4i+1$ that is less than x. For example, $l(10.5) = 5, l(20) = 17, l(17) = 13$.
Let the function $\pi^*(n)$ represent the number of primes of the form x^2+1 that are less than or equal to n^2+1.
Let the set $\mathbb{K}_n$ equal the set of odd integers of the form x^2+1 less than or equal to n^2+1 where n is an even integer.
Let the set $\mathbb{P}$ equal the set of prime numbers of the form $4i+1$.
Let the function $z_p(n) =$ the number of elements in $\mathbb{K}_n$ that are evenly divisible by prime number p excluding p, that are not divisible by another prime number less than p. For example, if n is 12, then $\mathbb{K}_{12} = \{5, 17, 37, 65, 101, 145\}$ and $z_5(12) = 2$ since 65 and 145 are evenly divisible by 5.

3 Methodology

We will look only at cases where n is an even number because if n is odd, then $n^2 + 1$ will be an even number and thus not prime.

Let $\mathbb{K}_n$ be the set of odd integers of the form $x^2 + 1$ that are less than or equal to $n^2 + 1$ is as follows:

$\mathbb{K}_n = \{5, 17, 37, 65, 101, 145, 197, 257, 325, 401, 485, \ldots, n^2 + 1\}$

These numbers are in the form $y = 4x^2 + 8x + 5$, where x is an integer greater than or equal to 0.

There are exactly $n/2$ elements in $\mathbb{K}_n$. Notice that not all of these numbers are prime.

To identify the elements in $\mathbb{K}_n$ that are prime, we will eliminate the values divisible by primes of the form $4i + 1$ since primes not of this form do not evenly divide numbers of the form $x^2 + 1$. This is a known theorem of quadradic residues.

Let $\mathbb{P}$ be the set of primes of the form $4i + 1$ as follows:

$\mathbb{P} = \{5,13,17,29,37,41,53,61,73,89,97,101,109,113,137,...\}$

According to Dirichlet's Theorem, there are an infinite number of prime numbers of the form $4i + 1$. Note that the minimum gap between primes of the form $4i + 1$ is 4, and there are no consecutive gaps of 4. This is because for the sequence 5,9,13,17,21,25,29,33..., every third number is divisible by 3.

We start by identifying all the elements in set $\mathbb{K}_n$ that are divisible by the prime number 5, the first prime number of the form $4x + 1$.

$\mathbb{K}_n = \{$5,17,37, 65 ,101, 145 ,197,257, 325 ,401, 485 , 577,677, 785 ,901, 1025 , 1157,1297, 1445 ,1601, 1765 ,1937$\ldots$,$n^2 + 1\}$

Notice that after 5, for every 5 elements in $\mathbb{K}_n$ there are two elements (highlighted in yellow) that are divisible by 5. This is a property of quadradic equations.

The equation $y = 4x^2 + 8x + 5$ can be written as $y = x(4x + 8) + 5$. Values of $x = 5k$ or $x = 5k + 3$ where k is an integer, will result in a value of y that is evenly divisible by 5. Plugging $5k$ for x gives $y = 5k(4x + 8)$ which is divisible by 5, plugging $5k + 3$ for x gives $y = x(4(5k + 3) + 8) = x(20k + 20)$ which is also divisible by 5.

Thus, as $n \to \infty$, the number of the elements in $\mathbb{K}_n$ that are evenly divisible by 5 approaches $2/5$.

$$z_5(n) \lim_{n\to\infty} = (n/2)(2/5)$$

Next, we identify all the elements in set $\mathbb{K}_n$ that are divisible by 13, the next higher prime of the form $4i+1$.
$\mathbb{K}_n = \{$5,17,37, 65 ,101,145,197,257, 325 ,401,485, 577,677,785,901,1025, 1157 , 1297,1445,1601, 1765, 1937 ,...,$n^2+1\}$

Notice that every 13 elements in $\mathbb{K}_n$, there are two elements (yellow) that are divisible by 13. If we subtract 65 from both sides of $y = 4x^2 + 8x + 5$, we get $y-65 = 4x^2+8x-60$ which can be written as $y-65 = (4x-12)(4x+20)$. Values of $x = 13k+3$ or $x = 13k+8$ will result in an integer value of $y/13$. If we plug $x = 13k+3$ in the left set of parentheses, we get $y-65 = 52k(4x+20)$ which is divisible by 13 since 52 is a multiple of 13. If we plug $13k+8$ in the right set of parentheses we get $y - 65 = (4x - 12)(52k + 52)$ which is also divisible by 13.

Thus, as $n \to \infty$, the number of elements in $\mathbb{K}_n$ that are divisible by 13 approaches $2/13$. However, notice that 65 and 325 are also divisible by 5. About 2/5ths of the numbers divisible by 13 are also divisible by 5. So to avoid double counting, we must multiply the number of elements divisible by 13 by $3/5$. The number of elements in $\mathbb{K}_n$ that are evenly divisible by 13 and not divisible by 5 limit $n \to \infty$ are:

$$z_{13}(n)\lim_{n\to\infty} = (n/2)(3/5)(2/13)$$

Next, we identify all the elements in set $\mathbb{K}_n$ that are divisible by 17, the next higher prime of the form $4i+1$.
$\mathbb{K}_n = \{$5,17,37,65,101,145,197,257,325,401,485, 577,677,785, 901 ,1025,1157, 1297, 1445 ,1601, 1765,1937...,$n^2+1\}$
Notice that every 17 elements in $\mathbb{K}_n$ after 17, there are two elements that are divisible by 17. If we subtract 17 from both sides of $y = 4x^2 + 8x + 5$, we get $y-17 = 4x^2+8x+5-17$ which can be written as $y-17 = (4x-4)(4x+12)$. Values of $x = 17k + 1$ or $x = 17k + 14$ will result in an integer value of $y/17$. Thus, there will always be at least 2 values of x every 17 numbers that will result in a value of y that is evenly divisible by 17.

Thus, as $n \to \infty$, about 2/17ths of the elements in $\mathbb{K}_n$ are divisible by 17. However, about 2/5ths of the elements divisible by 17 are also divisible by 5 and 2/13ths of them are also divisible by 13. So to avoid double counting, we must multiply the number of elements divisible by 17 by $3/5$ and $11/13$. The number of elements in $\mathbb{K}_n$ that are evenly divisible by 17 excluding 17, and not divisible by 5 or 13 limit $n \to \infty$ are:

$$z_{17}(n)\lim_{n\to\infty} = (n/2)(3/5)(11/13)(2/17)$$

The fact that $y = 4x^2 + 8x + 5$ is quadratic, for every p numbers, there will always be 2 values of x that will result in a y that is evenly divisible by p.

The general formula for number of values in the set $\mathbb{K}_n$ that are evenly divisible by prime number p of the form $4i + 1$ excluding p, and not evenly divisible by a prime less than p is:

$$z_p(n) \lim_{n\to\infty} = (n/2)(3/5)(11/13)(15/17)...(2/p)$$

This can be written as:

$$z_p(n) \lim_{n\to\infty} = \left(\frac{n}{2}\right)\left(\frac{2}{p}\right)\prod_{\substack{q=5 \\ q\in\mathbb{P}}}^{p} \frac{(q-2)}{q}$$

where the product is over prime numbers of the form $4i + 1$.

We only need to go up to $l(n)$ since prime numbers greater than $l(n)$ will not evenly divide any odd number less than $n^2 + 1$ that is not already divisible by a lower prime. Let $k(n)$ equal the total number of composite numbers in set $\mathbb{K}_n$ limit $n \to \infty$ that are less than or equal to $n^2 + 1$.

$$k(n) = z_5(n) + z_{13}(n) + z_{17}(n) + ... + z_{l(n)}(n)$$

Substituting the values for $z_p(n)$ gives:

$$k(n) = \left(\frac{n}{2}\right)\sum_{\substack{p=5 \\ p\in\mathbb{P}}}^{l(n)} \left(\left(\frac{2}{p}\right)\prod_{\substack{q=5 \\ q\in\mathbb{P}}}^{p} \frac{(q-2)}{q}\right)$$

If we define the function $W(x)$, which represents the fraction of elements in $\mathbb{K}_n$ that are composite numbers, as follows:

$$W(x) = \sum_{\substack{p=5 \\ p\in\mathbb{P}}}^{x} \left(\left(\frac{2}{p}\right)\prod_{\substack{q=5 \\ q\in\mathbb{P}}}^{p} \frac{(q-2)}{q}\right)$$

where x is a prime number of the form $4i + 1$ and the sum and products are over prime numbers of the form $4i + 1$.

The equation for the total number of composite values in set $\mathbb{K}_n$ is:

$$k(n) = \left(\frac{n}{2}\right)(W(l(n))$$

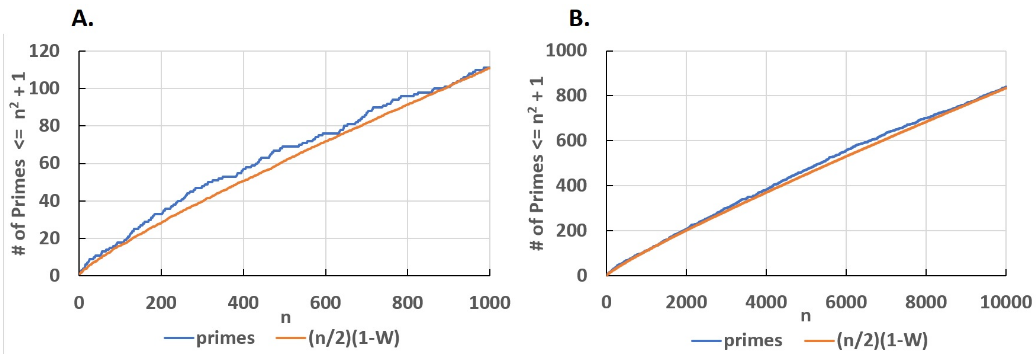

Figure 1: The actual number of primes of the form x^2+1 that are less than or equal to n^2+1 is very closely approximated by $\pi^*(n) = (n/2)(1-W(l(n)))$.

The number of primes of the form x^2+1 in $\mathbb{K}_n$ that are less than $n^2+1 \lim n \to \infty$ equals the total number of values in $\mathbb{K}_n$, which is $(n/2)$, minus the total number of composite elements in $\mathbb{K}_n$. Let $\pi^*(n)$ represent the predicted number of primes in the set $\mathbb{K}_n$ that are prime.

$$\begin{aligned} \pi^*(n) &= \left(\frac{n}{2}\right) - k(n) \\ \pi^*(n) &= \left(\frac{n}{2}\right) - \left(\frac{n}{2}\right)(W(l(n)) \\ \textbf{Equation 1: } \pi^*(n) &= \left(\frac{n}{2}\right)(1 - W(l(n))) \end{aligned}$$

To verify that I derived equation 1 properly, I plotted the number of primes of the form x^2+1 that are less than or equal to n^2+1 (blue line) and $\pi^*(n)$ (orange line) for values of n up to 1000 (Figure 1A) and as can be seen, the lines correspond very closely. As n increases to 10,000 (Figure 1B) the lines are virtually on top of each other.

Since I will be using mathematical induction to prove the Near-Square Prime conjecture, I need to define $1 - W(p_{i+1})$ in terms of $W(p_i)$. Below are the values of $1 - W(p_i)$.

$1 - W(5) = 1 - \left(\frac{2}{5}\right) = \frac{3}{5}$

$1 - W(13) = 1 - \left(\frac{2}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{2}{13}\right) = \left(\frac{3}{5}\right)\left(\frac{11}{13}\right)$

$1 - W(17) = 1 - \left(\frac{2}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{2}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{11}{13}\right)\left(\frac{2}{17}\right) = \left(\frac{3}{5}\right)\left(\frac{11}{13}\right)\left(\frac{15}{17}\right)$

$1 - W(29) = 1 - \left(\frac{2}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{2}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{11}{13}\right)\left(\frac{2}{17}\right) - \left(\frac{3}{5}\right)\left(\frac{11}{13}\right)\left(\frac{15}{17}\right)\left(\frac{2}{29}\right) =$

$\left(\frac{3}{5}\right)\left(\frac{11}{13}\right)\left(\frac{15}{17}\right)\left(\frac{27}{29}\right)$

Notice the value of $1 - W(p_{i+1})$ is equal to $((p_{i+1} - 2)/p_{i+1})$ times the previous value of $1 - W(p_i)$. This gives us the following recursive definition for $1 - W(p_{i+1})$:

$$\textbf{Equation 2: } 1 - W(p_{i+1}) = \frac{(p_{i+1} - 2)}{p_{i+1}}(1 - W(p_i))$$

Let $l(n) = p_i$ and let's approximate $n = p_i$. Since n is an even integer, n is at least $p_i + 1$ so this approximation errs on the side of caution. Plugging p_i for $l(n)$ and n into equation 1 gives the following:

$$\begin{aligned}
\pi^*(p_i) &= \left(\frac{p_i}{2}\right)(1 - W(p_i)) \\
\pi^*(p_{i+1}) &= \left(\frac{p_{i+1}}{2}\right)(1 - W(p_{i+1})) \\
\pi^*(p_{i+1}) &= \left(\frac{p_{i+1}}{2}\right)\left(\frac{p_{i+1} - 2}{p_{i+1}}\right)(1 - W(p_i)) \qquad \text{Using equation 2} \\
\pi^*(p_{i+1}) &= \left(\frac{(p_{i+1} - 2)}{2}\right)(1 - W(p_i))
\end{aligned}$$

Taking the ratio of $\pi^*(p_{i+1})/\pi^*(p_i)$ gives:

$$\begin{aligned}
\pi^*(p_{i+1})/\pi^*(p_i) &= \frac{\left(\frac{(p_{i+1}-2)}{2}\right)(1 - W(p_i))}{\left(\frac{p_i}{2}\right)(1 - W(p_i))} \\
\pi^*(p_{i+1})/\pi^*(p_i) &= \frac{(p_{i+1} - 2)}{p_i} > 1
\end{aligned}$$

Since p_{i+1} is at least p_i+4, this proves that $\pi^*(p_{i+1})$ will always be bigger than $\pi^*(p_i)$. However, plugging in $p_i + 4$ for p_{i+1} gives $(p_i + 4 - 2)/p_i = (p_i + 2)/p_i$ which approaches 1 as p_i goes to infinity. This could mean that $\pi^*(p_i)$ may approach a constant.

To prove that $\pi^*(p_i)$ goes to infinity as p_i goes to infinity, I will prove that $\pi^*(p_i)^2$ goes to infinity. This is done because it is easier to prove that

$\pi^*(p_i)^2$ goes to infinity than $\pi^*(p_i)$.

$$\begin{aligned}
\pi^*(p_i)^2 &= \left(\frac{p_i^2}{4}\right)(1-W(p_i))^2 \\
\pi^*(p_{i+1})^2 &= \left(\frac{(p_{i+1}-2)^2}{4}\right)(1-W(p_i))^2
\end{aligned}$$

Let $\Delta\pi(p_i)$ represent the difference between $\pi^*(p_{i+1})^2$ and $\pi^*(p_i)^2$.

$$\begin{aligned}
\Delta\pi(p_i) &= \pi^*(p_{i+1})^2 - \pi^*(p_i)^2 \\
\Delta\pi(p_i) &= \left(\frac{(p_{i+1}-2)^2}{4}\right)(1-W(p_i))^2 - \left(\frac{p_i^2}{4}\right)(1-W(p_i))^2 \\
\Delta\pi(p_i) &= \left(\frac{(p_{i+1}-2)^2 - p_i^2}{4}\right)(1-W(p_i))^2
\end{aligned}$$

We know that p_{i+1} is at least $p_i + 4$, so to simplify things, let's substitute p_{i+1} with $p_i + 4$. We will call this new function $\Delta\pi^*(p_i)$ which will always be less than or equal to $\Delta\pi(p_i)$.

$$\begin{aligned}
\Delta\pi^*(p_i) &= ((p_i+4-2)^2 - p_i^2)(1-W(p_i))^2/4 \\
\Delta\pi^*(p_i) &= ((p_i+2)^2 - p_i^2)(1-W(p_i))^2/4 \\
\Delta\pi^*(p_i) &= ((p_i^2+4p_i+4) - p_i^2)(1-W(p_i))^2/4 \\
\Delta\pi^*(p_i) &= (4p_i+4)(1-W(p_i))^2/4 \\
\Delta\pi^*(p_i) &= (p_i+1)(1-W(p_i))^2
\end{aligned}$$

I will prove $\Delta\pi^*(p_i) > 0$ by mathematical induction. Base case: $p_0 = 5$.

$$\begin{aligned}
\Delta\pi^*(5) &= (5+1)(1-W(5))^2 \\
\Delta\pi^*(5) &= (6)(1-2/5)^2 \\
\Delta\pi^*(5) &= 6(3/5)^2 \\
\Delta\pi^*(5) &= 6(9/25) \\
\Delta\pi^*(5) &= 72/25 > 1
\end{aligned}$$

Assuming that $\Delta\pi^*(p_i) > 1$, I will prove that $\Delta\pi^*(p_{i+1}) > 1$

$$\begin{aligned}
\Delta\pi^*(p_i) &= (p_i+1)(1-W(p_i))^2 \\
\Delta\pi^*(p_{i+1}) &= (p_{i+1}+1)(1-W(p_{i+1}))^2 \\
\Delta\pi^*(p_{i+1}) &= (p_{i+1}+1)\left(\left(\frac{(p_{i+1}-2)}{p_{i+1}}\right)(1-W(p_i))\right)^2 \\
\Delta\pi^*(p_{i+1}) &= (p_{i+1}+1)\left(\frac{(p_{i+1}-2)^2}{p_{i+1}^2}\right)(1-W(p_i))^2
\end{aligned}$$

Taking the ratio of $\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i)$ gives the following:

$$\begin{aligned}
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}+1)\left(\frac{(p_{i+1}-2)^2}{p_{i+1}^2}\right)(1-W(p_i))^2}{(p_i+1)(1-W(p_i))^2} \\
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}+1)(p_{i+1}-2)^2}{p_{i+1}^2(p_i+1)} \\
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}+1)(p_{i+1}^2-4p_{i+1}+4)}{(p_{i+1}^2 p_i+p_{i+1}^2)} \\
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}^3-4p_{i+1}^2+4p_{i+1}+p_{i+1}^2-4p_{i+1}+4)}{(p_{i+1}^2 p_i+p_{i+1}^2)} \\
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}^3-3p_{i+1}^2+4)}{(p_{i+1}^2 p_i+p_{i+1}^2)}
\end{aligned}$$

The minimum p_{i+1} can be is p_i+4. Substituting p_i with $p_{i+1}-4$ gives

$$\begin{aligned}
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}^3-3p_{i+1}^2+4)}{(p_{i+1}^2(p_{i+1}-4)+p_{i+1}^2)} \\
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}^3-3p_{i+1}^2+4)}{(p_{i+1}^3-4p_{i+1}^2+p_{i+1}^2)} \\
\Delta\pi^*(p_{i+1})/\Delta\pi^*(p_i) &= \frac{(p_{i+1}^3-3p_{i+1}^2+4)}{(p_{i+1}^3-3p_{i+1}^2)} > 1
\end{aligned}$$

Since the numerator is greater than the denominator by 4, the ratio will always be greater than 1, thus proving that $\Delta\pi^*(p_{i+1}) > \Delta\pi^*(p_i)$ for any p_i and p_{i+1}. Since $\Delta\pi^*(p_0) = 72/25$, then $\Delta\pi^*(p_i) > 72/25$ for all p_i where p_i

is a prime number of the form $4i+1$.

Since $\Delta\pi^*(p_i)$ is always less than or equal to $\Delta\pi(p_i)$, then $\Delta\pi(p_i) > 72/25$.

Since $\Delta\pi(p_i) > 72/25$, then $\pi^*(p_{i+1})^2 - \pi^*(p_i)^2 > 72/25$.

Since the gap between $\pi^*(p_i)^2$ and $\pi^*(p_{i+1})^2$ is always greater than $72/25$, then as p_i goes to infinity, $\pi^*(p_i)^2$ goes to infinity. Therefore, $\pi^*(p_i)$ also goes to infinity as p_i goes to infinity. This proves that there are an infinite number of primes of the form n^2+1 thus proving the near square primes conjecture.

4 Summary

It has been shown that as n goes to infinity, the number of prime numbers of the form x^2+1 that are less than or equal to n^2+1 approaches the following equation:

$$\pi^*(n) = \left(\frac{n}{2}\right)(1 - W(l(n))$$

where $W(x)$ is defined as follows:

$$W(x) = \sum_{\substack{p=5 \\ p\in\mathbb{P}}}^{x} \left(\left(\frac{2}{p}\right) \prod_{\substack{q=5 \\ q\in\mathbb{P}}}^{p} \frac{(q-2)}{q} \right)$$

where x is a prime number of the form $4i+1$ and the sum and products are over prime numbers of the form $4i+1$. By mathematical induction, it is proven that $\pi^*(p_i)^2$ goes to infinity as p_i goes to infinity thus proving that there are an infinite number of prime numbers of the form x^2+1.

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