Special relativity in complex space-time Part 5. Sketch of the mathematical structure of complex space-time

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Abstract

The article compiles the existing information on complex space-time, from which a sketch of its mathematical structure can be outlined. We divided space-time into geometric and physical layers. The elements of the geometric layer are any four-vectors because, although it is an orthogonal space, it does not have a metrics. The physical layer is created by objects having energy, whose state coordinates are proper or singular paravectors, therefore it is posible to define a metrics for these objects. However, this metrics has no classic properties and it is not a pseudometric of the Minkowski space-time either, so we called it para-metrics. In the physical complex space-time the triangle inequality and the Cauchy inequality have been proved but, however, they have the opposite directions as their equivalents known from Euclidean geometry. At the end, we give a few tips on how to simplify the idea of a complex space-time.

Keywords: *Complex space-time, alterntive special relativity, paravectors, metrics*

Introduction

Algebra of paravectors is presented in the article [1]. As could be seen in the next works, some paravectors are additive, and others are not. These first ones are called four-vectors and we write them in a large double font ($\mathbb{X}, \mathbb{F}$) or in the form of a matrix in brackets. We got to know the orthogonal transformation (equivalent to the Lorentz transformation) [4], which is invariant for the phase interval [4], wave equation [2], electric field potential, charge density function [5], Klein-Gordon equation and the function of action [7]. The phase interval describes the changes in the location of the object in the observer's system and associates them with the proper time of this object (which is the same in each system, so it is a universal time). The time has come to gather our knowledge and to first attempt to organize it in a concise mathematical form.

At the beginning, we will present assumptions and symbols that apply in this and all previous works.

Assumption 1. Physical formulas are stored in the natural units system[1], it means that

- the value of the speed of light is equal to 1
- the linear velocity is a dimensionless quantity and its value is a fraction of the speed of light,

Assumption 2. Time is a quantum size. This means that if we are talking about time, we mean its interval.

In our previous papers, for greater readability, when we were deal with complex sizes, and when with real ones, the complex sizes were marked with Greek letters and the real ones with Latin letters. The coordinates of objects have always been the same: time Δt and the route $\Delta\mathbf{x}$, regardless of whether they were complex or real. Now, because we will use once real coordinates and once complex ones, we will underline the complex coordinates for greater clarity.

Below we present the important information gethered from our previous works that will be needed for further reasoning.

1. A physical object is an object that is in constant motion and has energy (eg a material point, photon or wave front). By movement, we mean both movement in space and/or in time. Each object in the observer's space-time is described by two parameters: the position change relative to the observer $\Delta\mathbf{x} \in R^3$ and the observation time $\Delta t > 0$. The proper time of an observer and spatial coordinates in his frame are real. We assume that the time structure is granular with an infinitesimal positive step.

2. The state of the physical object in the observer's space-time is specified by its phase, i.e. the product of its state paravectors.

$$\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$$

   Due to the quantum of time the above formula has no physical meaning, but the movement of the object is specified by the phase interval

$$\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} \Delta t - \mathbf{v}\Delta\mathbf{x} \\ \Delta\mathbf{x} - \mathbf{v}\Delta t - i\mathbf{v} \times \Delta\mathbf{x} \end{pmatrix} = k \in R_+ \quad . \tag{1}$$

   The above formula shows the relationship between changes in the position of the object and its speed as well as the limitations that must be imposed on these sizes, which is shown in the next point.

3. Physical phenomena are those in which physical objects participate. Physical objects have two basic features:

[1] In cases when we want to mark the speed of light explicitly as c, the formulas are written in Heaveside-Lorentz units system, which we refer to by (H-L)

- they can not back in time
- they have energy

The first feature means that the proper time interval[2] of the object is a non-negative real number. As for the second feature, it was not found that the energy moved at a speed greater than the speed of light. On the mathematical side, this means that for a physical object, the value of speed meets the condition $v \in \langle 0,1\rangle$. When $v = 1$, the object moves at the speed of light. Phase interval of objects slower than light, corresponding to the proper time interval of this object, we write in the form

$$V^{-}\mathbb{X} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix}\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} \Delta t - \mathbf{v}\Delta\mathbf{x} \\ \Delta\mathbf{x} - \mathbf{v}\Delta t - i\mathbf{v}\times\Delta\mathbf{x} \end{pmatrix} = \Delta t^0 \in R_+ \setminus \{0\} \quad (2)$$

4. The dynamics of a physical object is characterized by the product of its velocity paravector by itself VV, which is proportional to the kinetic energy of this object.

5. When an observer describes two objects (or more), their mutual relations cause that one of them has to move in a compound motion, so the coordinates of its state paravectors must be complex ($\Lambda, \underline{\mathbb{X}} \in C^4 \setminus \{0\}$)

$$\Lambda^{-}\underline{\mathbb{X}} = \begin{bmatrix} \alpha \\ -\boldsymbol{\beta} \end{bmatrix}\begin{pmatrix} \underline{\Delta t} \\ \underline{\Delta\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \alpha\underline{\Delta t} - \boldsymbol{\beta}\underline{\Delta\mathbf{x}} \\ \alpha\underline{\Delta\mathbf{x}} - \boldsymbol{\beta}\underline{\Delta t} - i\boldsymbol{\beta}\times\underline{\Delta\mathbf{x}} \end{pmatrix} = \Delta t^0 \in R_+ \setminus \{0\} \quad \text{and} \quad \det\Lambda = 1 \quad (3)$$

In the case of compound motion (complex velocity Λ), the dynamics of the object is determined by the vigor $\Lambda\Lambda^*$.

6. The transition from a rest frame to an inertial moving frame (boost) is shown by the transformation described by the complex orthogonal paravector Λ defined as a transformation of a four-vector $\underline{\mathbb{X}}$ onto $\underline{\mathbb{X}}'$ one such that the phase interval is the invariant of this transformation

$$\underline{\mathbb{X}} \xrightarrow{\Lambda} \underline{\mathbb{X}}' \qquad \text{and} \qquad \Gamma \xrightarrow{\Lambda} \Gamma' \qquad \text{such that} \qquad \Gamma^{-}\underline{\mathbb{X}} = \Gamma'^{-}\underline{\mathbb{X}}'$$

7. The state of a single object in the observer's real system is defined by real paravectors where the speed of this object is determined by

   - a paravector in the form of $V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$, for objects moving at a speed less than light,
   - a paravector in the form of $C = \begin{bmatrix} 1 \\ \mathbf{c} \end{bmatrix}$ and $c = 1$, for objects moving at a speed equal to light,

[2] In our opinion, the time is the universal time that passes in each system at the same pace.

and a real 4-vector $\mathbb{X} \in R_+ \times R^3$ describes the position change. When an observer describes an object moving in relation to another also moving object, complex paravectors describe the state of the subject object. The observer can always change complex paravectors of the state to the real form using the realisation.

8. Since the only information carrier is energy (which is always real), it is indifferent to the observer if the object's conditions are showed using real or complex paravectors. He can transform the state of each object to the real form (and vice versa) by means of the realization [6] which is defined on the basis of equivalence

$$\Lambda^- \underline{\mathbb{X}} \to V^- \mathbb{X}' = \underline{\Lambda^-\big|}\mathbb{X}' \qquad \Leftrightarrow \qquad \Lambda\Lambda^* = VV$$

where Λ is an orthogonal paravector, $\underline{\mathbb{X}}$ is a complex proper 4-vector, $\mathbb{X}' \in R_+ \times R^3$, and V is the real velocity paravector. Realisation distorts the mutual spatio-temporal relationship between these objects because it is not an orthogonal transformation. However, it preserves the scalar product of vigors that are proportional to the real kinetic energy paravectors of these objects.

Since realization does not preserve a scalar product, it can be used only to describe the motion of a single physical object (or space-time parallel objects, i.e., mutually immobile) and it serves to select the frame associated with this object(s) to be the most convenient for the observer. However, realization keeps the parallelism of the paravectors, so we interpret it as a kind of projection on the real space-time of an immobile observer.

The observer is free to choose the system in which he describes the states of the observed objects, but he describes these objects in his real coordinate system using real paravectors. These real coordinates are realized coordinates from the complex space-time and do not reflect the mutual relations between the observed objects. When describing two objects, choosing real coordinates for one of them and wanting to maintain their mutual relations, the other one can choose only the system in which time will be real. An example of how to go from real to complex coordinates was demonstrated in the paper [6] chapter 5.

1 Metrics in the physical space.

Let us imagine the following experiment:

From the center of the laboratory O a material point starts and moves away at velocity $\mathbf{v}_1$, then bounces perpendicularly from the wall and returns to the point O at velocity $\mathbf{v}_2$ (Left figure). The observer located in the laboratory describes the coordinates of the point motion in his real coordinate system.

In the frame of the moving point, the whole experiment lasted $\Delta t^0 = \Delta t_1^0 + \Delta t_2^0$. In the observer's frame, the point motion can be described by vector equations from the following equations:

$$\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix} 1 \\ -\mathbf{v}_1 \end{bmatrix}\begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} = \begin{pmatrix} \Delta t_1^0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix} 1 \\ -\mathbf{v}_2 \end{bmatrix}\begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \Delta t_2^0 \\ 0 \end{pmatrix} \tag{4}$$

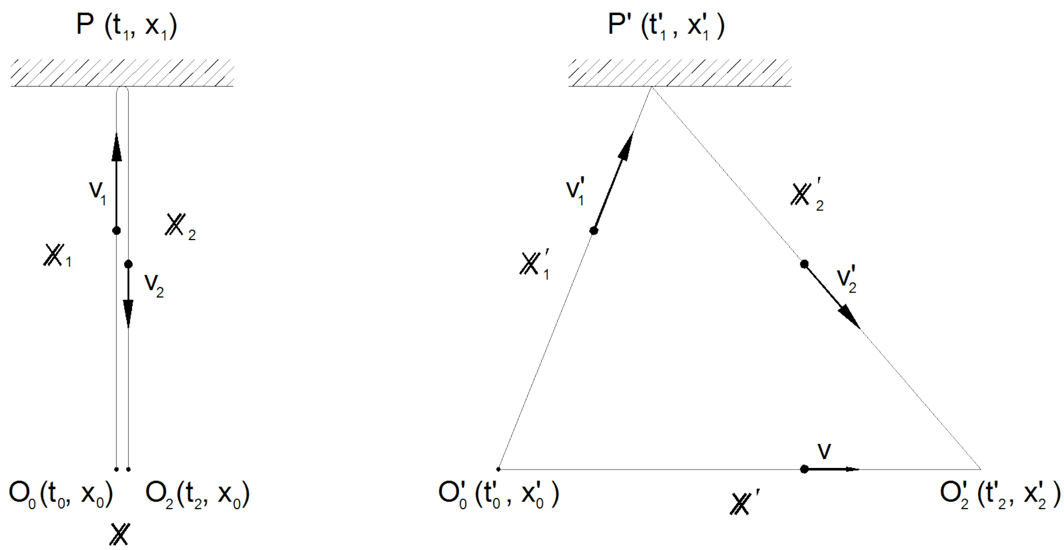

Figure 1: A space-temporal locked triangle.

Since the whole experiment lasts $\Delta t_1 + \Delta t_2$, and the observed point at the end will be at the start point, then the coordinates have to meet the dependencies:

$$\begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} + \begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \Delta t \\ 0 \end{pmatrix}, \quad \text{i.e.} \quad \mathbb{X}_1 + \mathbb{X}_2 = \mathbb{X}, \tag{5}$$

where $\begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} = \frac{\Delta t_1^0}{\sqrt{1-v_1^2}} \begin{bmatrix} 1 \\ \mathbf{v}_1 \end{bmatrix}$ and $\begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} = \frac{\Delta t_2^0}{\sqrt{1-v_2^2}} \begin{bmatrix} 1 \\ \mathbf{v}_2 \end{bmatrix}$.

We are talking about the coordinates of a physical object, that is all the four-vectors above are proper.

Another observer is in a vehicle moving at velocity $-\mathbf{v}$ and describes the movement of this point (Figure 1 right) with the following formulas

$$- \text{ first line segment} \qquad V^{-}\underline{\mathbb{X}}_1' = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_1' \\ \underline{\Delta \mathbf{x}}_1' \end{pmatrix} = \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} \tag{6}$$

$$- \text{ secound line segment} \qquad V^{-}\underline{\mathbb{X}}_2' = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_2' \\ \underline{\Delta \mathbf{x}}_2' \end{pmatrix} = \begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} \tag{7}$$

$$- \text{ centrum of laboratory} \qquad V^{-}\mathbb{X}' = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \tag{8}$$

In his frame equation (5) has a form $\underline{\mathbb{X}}_1' + \underline{\mathbb{X}}_2' = \mathbb{X}'$ or $V\mathbb{X}_1 + V\mathbb{X}_2 = V\mathbb{X}$. The result is obvious: In both frames, the corresponding four vectors are additive.

Since the velocity paravector (uniform motion[3]) represents the quotient

$$V = \frac{\mathbb{X}}{|\mathbb{X}|} = \frac{1}{\sqrt{\Delta^2 t - \Delta^2 x}} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}, \tag{9}$$

still another way is that by pulling out the times we can express the sum of four-vectors

$$\Delta t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \Delta t_1 \begin{bmatrix} 1 \\ \mathbf{v}_1 \end{bmatrix} + \Delta t_2 \begin{bmatrix} 1 \\ \mathbf{v}_2 \end{bmatrix} \tag{10}$$

In the system moving at the speed $-\mathbf{v}$ we have

$$\frac{\Delta t}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} = \frac{\Delta t_1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v}_1 \end{bmatrix} + \frac{\Delta t_2}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v}_2 \end{bmatrix} \tag{11}$$

or
$$\Delta t' \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} = \Delta t'_1 \begin{bmatrix} 1 \\ \underline{\mathbf{v}}'_1 \end{bmatrix} + \Delta t'_2 \begin{bmatrix} 1 \\ \underline{\mathbf{v}}'_2 \end{bmatrix}, \tag{12}$$

where
$$\Delta t'_i = \frac{1 + \mathbf{v}\mathbf{v}_i}{\sqrt{1-v^2}} \Delta t_i \qquad \text{and} \qquad \underline{\mathbf{v}}'_i = \frac{\mathbf{v} + \mathbf{v}_i + i\mathbf{v} \times \mathbf{v}_i}{1 + \mathbf{v}\mathbf{v}_i} \tag{13}$$

or otherwise
$$\begin{pmatrix} \Delta t' \\ \Delta t' \mathbf{v} \end{pmatrix} = \begin{pmatrix} \Delta t'_1 \\ \Delta t'_1 \underline{\mathbf{v}}'_1 \end{pmatrix} + \begin{pmatrix} \Delta t'_2 \\ \Delta t'_2 \underline{\mathbf{v}}'_2 \end{pmatrix}, \tag{14}$$

it means
$$\begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \Delta t'_1 \\ \underline{\Delta \mathbf{x}}'_1 \end{pmatrix} + \begin{pmatrix} \Delta t'_2 \\ \underline{\Delta \mathbf{x}}'_2 \end{pmatrix} \tag{15}$$

At this point, we draw attention to one more important conclusion that will allow us to understand the metrics of physical space. Since the orthogonal transformation maintains the scalar product so that the triangle closes (ie. that the particles meet in one place and time), the 4-vectors describing the position of the physical particles in the triangle must satisfy the theorem:

Theorem 1.1. If $\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2, \underline{\mathbb{X}}_3$ are the proper four-vectors and $\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2 = \underline{\mathbb{X}}_3$ then

1. $\left\langle \underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2 \right\rangle \in R_+$
2. $\langle \underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2 \rangle \geqslant \left|\underline{\mathbb{X}}_1\right| \left|\underline{\mathbb{X}}_2\right|$

Proof.

The assumption that the 4-vectors $\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2, \underline{\mathbb{X}}_3$ are proper and they are the sides of the space-temporary triangle is equivalent to the statement that there exists a system in which

[3]Please note that in the formula (9) after extracting Δt and dividing the denominator the bracket brace has been changed. One can see here why the velocity paravector is not a 4-vector.

the triangle is spatially degenerated. In other words, there is an orthogonal paravector Λ such that

$$\underline{\mathbb{X}}_1 = \Lambda \mathbb{X}_1 \quad , \quad \underline{\mathbb{X}}_2 = \Lambda \mathbb{X}_2 \quad \text{and} \quad \underline{\mathbb{X}}_3 = \Lambda \mathbb{X}_3 \quad , \quad \text{where} \tag{16}$$

$$\mathbb{X}_1 = \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x} \end{pmatrix} \quad , \quad \mathbb{X}_2 = \begin{pmatrix} \Delta t_2 \\ -\Delta \mathbf{x} \end{pmatrix} \quad , \quad \mathbb{X}_3 = \begin{pmatrix} \Delta t_1 + \Delta t_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbb{X}_1, \mathbb{X}_2 \quad \text{are proper.} \tag{17}$$

The frame in which the triangle is spatially degenerated is called *laboratory* (as in the Fig. 1 left).

1. From the sum of $\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2 = \underline{\mathbb{X}}_3$ it follows that $\det(\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2) = \det \underline{\mathbb{X}}_3$

   Based on the polarization identity ([1] Th.3.4)

   $$\det \underline{\mathbb{X}}_1 + \det \underline{\mathbb{X}}_2 + 2\langle \underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2 \rangle = \det \underline{\mathbb{X}}_3 \tag{18}$$

   Since the four-vectors $\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2$ and $\underline{\mathbb{X}}_3$ are proper, so from the above it is clear that the scalar product of the four-vector $\underline{\mathbb{X}}_1$ and $\underline{\mathbb{X}}_2$ is a real number.

   From the example above, we know that $\underline{\mathbb{X}}_1 = \Lambda \mathbb{X}_1$, $\underline{\mathbb{X}}_2 = \Lambda \mathbb{X}_2$ and $\underline{\mathbb{X}}_3 = \Lambda \mathbb{X}_3$, where Λ is a complex orthogonal paravector, and real coordinates are the coordinates in the laboratory system. So in the laboratory system we have the coordinates of 4-vectors (17), which gives the polarization identity in the form

   $$(\Delta t_1)^2 - (\Delta \mathbf{x})^2 + (\Delta t_2)^2 - (\Delta \mathbf{x})^2 + 2 \langle \mathbb{X}_1, \mathbb{X}_2 \rangle = (\Delta t_1 + \Delta t_2)^2 \quad , \tag{19}$$

   hence

   $$\langle \mathbb{X}_1, \mathbb{X}_2 \rangle = \Delta t_1 \Delta t_2 + (\Delta \mathbf{x})^2 \geqslant 0, \tag{20}$$

   and since the orthogonal transformation preserves the scalar product, then

   $$\left\langle \underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2 \right\rangle \in R_+ \quad . \tag{21}$$

2. Because we know that the orthogonal transformation does not change the determinant or the scalar product, then on the basis of the above formula we conclude that square of the vector being the result of vector product of the paravectors does not change, either.

   Below we will show that the square of the vector product of the real proper paravectors is a non-negative number.

   $$(\mathbb{X}_1 \mathbb{X}_2\} = \Delta t_2 \mathbf{x}_1 - \Delta t_1 \mathbf{x}_2 - i \mathbf{x}_1 \times \mathbf{x}_2 \tag{22}$$

   $$(\mathbb{X}_1 \mathbb{X}_2\}^2 = (\Delta t_2 \mathbf{x}_1 - \Delta t_1 \mathbf{x}_2)^2 - (\mathbf{x}_1 \times \mathbf{x}_2)^2 = t_1^2 t_2^2 \left[(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2 \right] \tag{23}$$

   The proof is reduced to show that expression $(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2$, where $|\mathbf{v}_1|, |\mathbf{v}_2| \in \langle 0, 1)$ is a non-negative real number. We have so

   $$(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2 = v_1^2 + v_2^2 - 2\mathbf{v}_1 \mathbf{v}_2 - v_1^2 v_2^2 + (\mathbf{v}_1 \mathbf{v}_2)^2 = \tag{24}$$

$$=(1-\mathbf{v}_1\mathbf{v}_2)^2-(1-v_1^2)(1-v_2^2)\geqslant(1-v_1v_2)^2-(1-v_1^2)(1-v_2^2)= \tag{25}$$

$$=-2v_1v_2+v_1^2+v_2^2=(v_1-v_2)^2\geqslant 0 \tag{26}$$

In point 1, we proved that for proper paravectors a scalar product must have a non-negative real value. An orthogonal transformation preserves determinants and scalar product. From the above and based on the formula ([1] 16)

$$\langle\underline{\mathbb{X}}_1,\underline{\mathbb{X}}_2\rangle^2-(\underline{\mathbb{X}}_1,\underline{\mathbb{X}}_2\}^2=\det\underline{\mathbb{X}}_1\det\underline{\mathbb{X}}_2$$

it follows that $\langle\underline{\mathbb{X}}_1,\underline{\mathbb{X}}_2\rangle^2\geqslant\det\underline{\mathbb{X}}\det\underline{\mathbb{X}}_2$. Because the 4-vectors $\underline{\mathbb{X}}_1$, $\underline{\mathbb{X}}_2$ are proper and the scalar product $\langle\underline{\mathbb{X}}_1,\underline{\mathbb{X}}_2\rangle\geqslant 0$, so

$$\langle\underline{\mathbb{X}}_1,\underline{\mathbb{X}}_2\rangle\geqslant\left|\underline{\mathbb{X}}_1\right|\left|\underline{\mathbb{X}}_2\right| \quad . \tag{27}$$

□

From the above theorem, it can be seen that Cauchy-Buniakowski-Schwarz (CBS) inequality does not apply in the physical space-time constructed by us. More! For physical objects, i.e. moving at a speed not higher than light, **inequality is reversed**, it means that after entering time, the CBS inequality changes direction, but in fact for the constructed geometry it fulfills the same role.

Now is the best moment to reflect on the basic geometric property characterizing the space, which is the triangle inequality. Does it also apply in space-time? And if not, is there any other dependence between the four-vectors, which corresponds to this relationship?

We can guess that, in contrast to Euclidean geometry, in a complex space-time the triangle inequality does not have the same generality because not every 4-vector has a module. Only proper and singuliar paravectors have a module. Therefore, we guess that the relation of the triangle can be formulated only for physical objects of which we spoke above. The experience described above corresponds to space-time triangles. Two physical objects simultaneously set off from one place (point O_0') to meet again at the same time elsewhere (point O_2'). One object rushes directly to the target, and the other one takes a longer route, visiting the point P. Applying paravector formalism and referencing the trajectories of a particle moving along a longer path to the track of a particle flying directly to the target, we denote times and ways

$$\mathbb{X}' = \underline{\mathbb{X}}_1'+\underline{\mathbb{X}}_2' \tag{28}$$

$$\begin{pmatrix} t_2'-t_0' \\ \mathbf{x}_2'-\mathbf{x}_0' \end{pmatrix} = \begin{pmatrix} t_1'-t_0' \\ \mathbf{x}_1'-\mathbf{x}_0'+i\mathbf{y} \end{pmatrix}+\begin{pmatrix} t_2'-t_1' \\ \mathbf{x}_2'-\mathbf{x}_1'-i\mathbf{y} \end{pmatrix} \tag{29}$$

Since the above 4-vectors describe the location of physical objects, they must be proper or singular. This assumption implies that the values of $\Delta t'$ and $\Delta\mathbf{x}'$ is not arbitrary. It should also be remembered that they do not have to be real. To explore the relationship between the above 4-vectors, we will use the following reasoning:

Let's go back to Figure 1. An observer in the lab describes the movement of a physical object from point O to point P and back to point O. All coordinates are real. The other observer moves towards the laboratory at any speed less than c. As a result of the transformation of 4-vectors from the laboratory system to his own, he will receive 4-vectors that form the triangle that we want.

Four-vectors in a stationary frame can be denoted:

$$\begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} + \begin{pmatrix} \Delta t \\ -\Delta \mathbf{x} \end{pmatrix} = \begin{pmatrix} 2\Delta t \\ 0 \end{pmatrix}, \tag{30}$$

and the relation R between the modules of its sides

$$|\mathbb{X}| + |\mathbb{X}^{-}| \quad R \quad |\mathbb{X} + \mathbb{X}^{-}| \tag{31}$$

or

$$\sqrt{\Delta^2 t - \Delta^2 \mathbf{x}} + \sqrt{\Delta^2 t - \Delta^2 \mathbf{x}} \quad R \quad 2\Delta t \tag{32}$$

Since on both sides of the relationship there are non-negative real values, so we can square both sides, and the relationship will not change.

$$\Delta^2 t - \Delta^2 \mathbf{x} \qquad R \qquad \Delta^2 t \tag{33}$$

The left side of the relation is less than or equal to the right one. Any interesting for us triangle of 4 vectors we get by going to the system that moves with respect to the previous one. This means that we multiply the equation (30) by any orthogonal paravector. We have so

$$\underline{\mathbb{X}}'_1 = \Lambda \mathbb{X} \qquad , \qquad \underline{\mathbb{X}}'_2 = \Lambda \mathbb{X}^{-} \qquad , \qquad \underline{\mathbb{X}}'_1 + \underline{\mathbb{X}}'_2 = \Lambda(\mathbb{X} + \mathbb{X}^{-})$$

Because $|\Lambda| = 1$, so for any orthogonal Λ the relationship will be the same as in the equation (33). We have shown that for physical objects the space-time inequality of a triangle has the form

$$|\underline{\mathbb{X}}'_1| + |\underline{\mathbb{X}}'_2| \quad \leqslant \quad |\underline{\mathbb{X}}'_1 + \underline{\mathbb{X}}'_2| \tag{34}$$

that means, it is the reverse to that in Euclidean geometry. This inequality can also be proved by polarization identity and theorem 1.1.

In the above reasoning, we were talking about a physical triangle whose sides were paths of particles having energy, so their velocity was less than c and which had come out at the same moment from one point and they also met. The situation if the particles never meet, even if their tracks intersected, but they were at the point of intersection at different times, is no longer a closed triangle.

From the calculational side everything looks simple, but the above equations describe the movement of physical objects. Then, how to interpret the imaginary components? The answer is simple: **one should get used to them**. The imaginary distance has no physical meaning in the sense of the path to go. It should be treated as a quantity characterizing a certain deformation of real coordinates caused by discrepancy of the simultaneity. Or in the simplest way: treat it as a dependent variable needed to balance the calculations.

From the above considerations one can see that the metrics, in the sense similar to the concept known from Euclidean geometry, is the property not of space but of the physical objects located in this space. If we define the norm of the proper/singular four-vector as its module ([1] def 1.12), then we obtain a metrics with the following properties

1. $|\underline{\mathbb{X}}| \in R_+$
2. jeżeli $|\underline{\mathbb{X}}| = 0$ i $\underline{\mathbb{X}} \neq 0$, then an object moves at speed of light
3. $|\underline{\mathbb{X}}| = |\underline{\mathbb{X}}^-|$
4. If $\det(\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2) \in R_+$, then $|\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2| \geqslant |\underline{\mathbb{X}}_1| + |\underline{\mathbb{X}}_2|$ (reversed triangle inequality)

Since this metric applies to paravectors, we suggest a name *para-metrics* for it. It should be remembered that it applies only to physical objects, that is these, whose state paravectors are proper or singular.

The above considerations were carried out in the complex space-time, and let's see how the situation is perceived by the observer in his real space-time.

$$V_1^- \mathbb{X}_1 + V_2^- \mathbb{X}_2 = V^-(\mathbb{X}_1 + \mathbb{X}_2) \tag{35}$$

hence

$$V = \frac{(V_1^- \mathbb{X}_1 + V_2^- \mathbb{X}_2)(\mathbb{X}_1 + \mathbb{X}_2)^-}{\det(\mathbb{X}_1 + \mathbb{X}_2)} \tag{36}$$

Here all coordinates are real, the experiment is correctly described by the observer, but the image is deformed.

2 Mathematical structure of space-time

For better ordering of the obtained results, we will now present a sketch of the mathematical structure of the space in which we built our model. In the current state of knowledge about the complex space, the following construction should be treated only as a working mathematical model for imagining the complex space-time, and not as a strictly defined structure, because it does not include objects moving at the speed of light. As a structure symbol, we use the element's symbol in brackets, and as the symbol of the set - a symbol of its element.

1. We define geometric space-time $[\underline{\mathbb{X}}] = [C^{1+3}, +; \odot[\Gamma]]$ whose elements are four-vectors

   $$\underline{\mathbb{X}} = \begin{pmatrix} \underline{\Delta t} \\ \underline{\Delta \mathbf{x}} \end{pmatrix} \qquad \text{, where} \quad \underline{\Delta t} \in C, \quad \underline{\Delta \mathbf{x}} \in C^3,$$

   and $[\Gamma] = [C^{1+3}, \cdot]$ is a group of paravectors $\Gamma = \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}$ with the operation of multiplication. An operation $\odot$ is the non-commutative external multiplication of four-vectors whose result belongs to the group $[\Gamma]$.

   $$\underline{\mathbb{X}}_1 \underline{\mathbb{X}}_2 \in [\Gamma] \qquad \text{and} \qquad \Gamma \underline{\mathbb{X}} \in [\underline{\mathbb{X}}]$$

   a) In the geometric space-time $[\underline{\mathbb{X}}]$ we define relationships:

– integrated product:
  right $(\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2\rangle = \underline{\mathbb{X}}_1\underline{\mathbb{X}}_2^-$ and left $\langle\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2) = \underline{\mathbb{X}}_1^-\underline{\mathbb{X}}_2$
– scalar product $\langle\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2\rangle = (\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2\rangle_S = \langle\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2)_S \in C$
– determinant $\det\underline{\mathbb{X}} = \underline{\mathbb{X}\mathbb{X}^-} \in C$

b) The determinant introduces a skeleton into the space-time, on which the metrics of physical objects will be built. The skeleton of geometric space-time has following properties:
   – parallelogram identity $2\det\underline{\mathbb{X}}_1 + 2\det\underline{\mathbb{X}}_2 = \det\left(\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2\right) + \det\left(\underline{\mathbb{X}}_1 - \underline{\mathbb{X}}_2\right)$
   – polarization identity $\det(\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2) = \det\underline{\mathbb{X}}_1 + 2\left\langle\underline{\mathbb{X}}_1, \underline{\mathbb{X}}_2\right\rangle + \det\underline{\mathbb{X}}_2$

c) In the group $[\{\Gamma\}, \cdot]$ we distinguish a subgroup of boosts $[\{\Lambda\}, \cdot]$, where $\{\Lambda\}$ is a set of complex orthogonal paravectors, and operation is a multiplication.

The space-time defined in this way responds to the vector space of the complex four-vectors over a group of complex paravectors. It is a geometric structure - an empty space in which there are no physical objects.

2. We create a space-time of physical objects $[\underline{\mathbb{X}}_p, +]$ by entering objects whose coordinates have the following limitations:

   a) Material objects:
      - the four-vectors of position $\underline{\mathbb{X}}_{pm}$ are proper paravectors ($\det\underline{\mathbb{X}}_{pm} > 0$),
      - phase limitation $\Lambda^-\underline{\mathbb{X}}_{pm} \in R_+ \setminus \{0\}$, where $\det\Lambda = 1$
   b) Immaterial objects:
      - the four-vectors of position $\underline{\mathbb{X}}_{pw}$ are singular paravectors ($\det\underline{\mathbb{X}}_{pw} = 0$),
      - phase limitation $\Omega^-\underline{\mathbb{X}}_{pw} = 0$, where $\det\Omega = 0$

   From the above conditions, it can be seen that scalar and vector components as well as real and imaginary components of the physical objects 4-vector positions are not independent of each other, because condition a) is equivalent to $Re(\Delta\underline{t})Im(\Delta\underline{t}) = Re(\Delta\underline{\mathbf{x}})Im(\Delta\underline{\mathbf{x}})$. The condition b) results from the fact that the phase interval is equal to the interval of its proper time. The space-time of physical objects is thus a substructure of geometric space-time such that its domain is a subset of proper and singular vectors of the $C \times C^3$ file and the set of complex numbers with the $+$ operation creates a semi-group, and the set of complex vectors C^3 with the $+$ operation is an abelian group.

3. We introduce the phase space of physical material objects $[\Theta_m] = [\{\Lambda^-\underline{\mathbb{X}}\}, +] = [\{\Delta t\} = R_+ \setminus \{0\}, +]$ which, like time, is a semi-group.

4. In the group of boosts $(\{\Lambda\}, \cdot)$, we introduce a one-argument relation $\text{vig}(\Lambda) = \Lambda\Lambda^* \in (R_+ \setminus \{0\}) \times R^3$, which is proportional to the kinetic energy of the physical object.

5. Each observer has his own real Cartesian space-time $[\mathbb{X}] = [R_+ \times R^3, +] \subset [\underline{\mathbb{X}}_{pm}]$ in which he is able to describe any physical object. The set R_+ together with summation is a semi-group, and $[R^3, +]$ is an abelian group.

6. The only carrier of information is energy, which is why the observer sees the movement of objects in the complex space-time by measuring their energy, which can be interpreted as projection from the complex space-time onto the real space-time of the observer. The projection retains a kinetic energy and momentum, which corresponds to equivalence $V = \underline{\Lambda|} \Leftrightarrow VV = \Lambda\Lambda^*$ and retains the phase interval of any object $\Lambda^- \underline{\mathbb{X}} = V^- \mathbb{X}$. The phase interval is equivalent to the proper time interval. In article [6] this projection we called *realization*.

7. For physical objects, we introduce a function $|\underline{\mathbb{X}}| = \sqrt{\det \underline{\mathbb{X}}}$ that fulfills the role of the metrics. This function is called para-metrics. The para-metrics has the following properties:

   If $\underline{\mathbb{X}}$ is a four-vector of the physical object position change, then

   - $|\underline{\mathbb{X}}| \in R_+$
   - If $\underline{\mathbb{X}} \neq 0$ and $|\underline{\mathbb{X}}| = 0$, that object moves at the speed of light (the same as in pseudomatics)
   - $|\underline{\mathbb{X}}| = |\underline{\mathbb{X}^-}|$
   - If $\det(\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2) \in R_+$, that $|\underline{\mathbb{X}}_1 + \underline{\mathbb{X}}_2| \geqslant |\underline{\mathbb{X}}_1| + |\underline{\mathbb{X}}_2|$ (the triangle condition in the space-time)

At the transition from the rest frame to the moving one, the position 4-vectors $\mathbb{X}$ of the physical objects in the resting reference system are replaced by new complex 4-vectors $\underline{\mathbb{X}}'$ in such a way that the phase interval is invariant $V^- \mathbb{X} = \Lambda'^- \underline{\mathbb{X}}'$. In the case of a single physical object, one can choose such a system that the state paravectors of this object are real, i.e. for the object to move through real coordinates, we do it with the help of realization $\Lambda'^- \underline{\mathbb{X}}' = \underline{\Lambda'|}\mathbb{X}" = V"\mathbb{X}"$.

So defined space-time responds to the vector space, but it is not a Cartesian space, i.e. there is no coordinate system. The while (minimum time) is the infinitesimal interval ($\delta t > 0$), not a point on the time axis. Each observer introduces his coordinate system (his own Cartesian space) and individually describes the object or phenomenon in it. His coordinate system is always real, as well as the coordinates of 4-vectors defining slow phenomena in relation to the observer. In other words: **physical space-time is real locally**. The word *locally* means first of all a low speed towards the observer.

3 Summary

In the space-time, physical objects are in permanent motion. Even a stationary observer also moves because he gets older, so he moves in one direction along the time axis. The movement is an immanent feature of physical objects in space-time. However, this movement has some limitations that matter and energy must meet. A physical object can

1. move in any direction in space but not faster than the speed of light,

2. move in time with a strictly defined speed and only in one direction.

These limitations, in the mathematical form, are contained in the phase interval which for the object moving inertically at the speed of $v < c$ describes the dependence

$$V^{-}\mathbb{X} = \Delta t^{0} \quad , \tag{37}$$

where

1. orthogonal paravector $V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$ characterizes the speed of the object with respect to the observer in his frame,
2. four-vector $\mathbb{X} = \begin{pmatrix} t_2 - t_1 \\ x_2 - x_1 \end{pmatrix}$ determines the change of the location of the object in the real system of the observer
3. $\Delta t^0 \in R_+ \setminus \{0\}$ is an interval of universal time (= proper time of the object).

In other words: 4-vector describing the change of the position of a physical object can be only a proper one, and the paravector of its speed must be orthogonal [4] For the wave front moving at the speed of light, the phase interval has the form

$$\begin{bmatrix} 1 \\ -\mathbf{c} \end{bmatrix}\begin{pmatrix} t_2 - t_1 \\ x_2 - x_1 \end{pmatrix} = 0 \tag{38}$$

and both the position 4-vector and the velocity paravector are singular paravectors.

The above description is correct for a single object only. If we want to describe in the real coordinate system two objects moving in relation to each other and to the observer, then the image will be deformed. That is, if the observer describes the coordinates of both objects using real paravectors, he can't describe the relationship between these objects with real coordinates. This is caused by a mutual shift in the time of the observer, the first and second object. In order to make all the relations correct the observer must:
- relate the relation of one of the objects to each other (real coordinates of the phase interval components),
- calculate the relation of the second object to the first (same phase phase component),
- then he must calculate the complex components of the phase interval of the second object in relation to each other, but taking into account the first object (the movement composed of the two previous movements).
He will then receive correct and interpretable relations between all the objects. One can imagine such space in the following way:

The local real space-times of any observers are „immersed" in the complex vector space-time which is objective for everyone. The carrier of information about observed

[4] In the local real space-time, it can not be a special paravector(!) but in a complex space-time it can be an orthogonal special paravector.

objects is the energy of light, which is always and in every system real but the space-time in which everybody is located is complex. What the observer sees is a projection on his real space-time. That is why the scalar product does not work with him, and the image he is watching is deformed. Deformation results from the "collapsing" of simultaneity of the phenomena viewed due to mutual movement and distance. The observer must then convert the real coordinates of objects into complex coordinates. He can do it thanks to the knowledge of the projection property (called realization), which was developed in the article [6].

The shift of imagination from the Cartesian space to the complex space can be compared to changing the view about the shape of the Earth. It was not easy, especially since the laws of physics were not known yet, which explains why planets somehow stick together and people can walk "upside down". Similarly in our case. Although many things point to the complexity of phenomena (eg magnetic field or gyroscopic effect), but they are not joined with imaginary space-time coordinates. To imagine the imaginary coordinates is all the more difficult because the real and imaginary components of the coordinates are not independent of each other. In order to understand the complex space-time, one has to revise the methodology of thinking based on the concepts of Cartesian space and simultaneity, and look differently at complex numbers. The Cartesian space is proper only for the observer, so each observer has his real coordinate system. In the complex space-time, although four-vectors can be designated as space-time pairs (the beginning and the end of the phenomenon) and the relations between these 4-vectors can be strictly defined, it is not possible to build a complex coordinate system. This is related to the idea of a set of complex numbers learned at school as a plane on which real and imaginary components correspond to independent variables. In the case of physical objects it is not so. Imaginary components appear with mutual movement, and what they are like - depends on the choice of the observer. Being in the same real Cartesian system, he can choose different combinations of real and imaginary coordinates depending on how convenient it is to coordinate the described objects. For the same reason, there is a certain freedom of time, which was not in the Euclidean space.

As far as time is concerned, all this indicates that although time is the fourth coordinate, it has completely different properties from spatial coordinates and is closer to the time imagined in the 19th century. In our opinion, time in each system expires at the same pace. The fact that in the system in motion it looks like it has flowed differently has the same cause as the fact that the electric field is visible as a magnetic field: What the observer can see can be interpreted as a projection from the "turned" system located in the complex space-time on its real Cartesian system. The word "turn" is in quotes because it is not a rotation in the Euclidean sense.

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