$N-$dimensional AdS related spacetime and its transformation

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ABSTRACT: Recently, anti-de Sitter spaces are used in promising theories of quantum gravity like the anti-de Sitter/conformal field theory correspondence. The latter provides an approach to string theorie, which includes more than four dimensions. Unfortunately, the anti-de Sitter model contains no mass and is not able to describe our universe adequately. Nevertheless, the rising interest in higherdimensional theories motivates to take a deeper look at the $n-$dimensional AdS Spacetime. In this paper, a solution of Einstein's field equations is constructed from a modified anti-de Sitter metric in n dimensions. The idea is based on the connection between Schwarzschild- and McVittie metric: McVittie's model, which interpolates between a Schwarzschild Black Hole and an expanding global Friedmann–Lemaître–Robertson–Walker spacetime, can be constructed by a simple coordinate replacement in Schwarzschild's isotropic intervall, where radial coordinate and it's differential is multiplied by a time dependent scale factor $a(t)$. In a previous work I showed, that an exact solution of Einstein's equations can analogously be generated from a static transformation of de Sitter's metric. The present article is concerned with the application of this method on an AdS (Anti de Sitter) related spacetime in n dimensions. It is shown that the resulting isotropic intervall is a solution of the $n-$dimensional Einstein equations. Further, it is transformed into a spherical symmetric but anisotropic form, analogously to the transformtion found by Kaloper, Kleban and Martin for McVittie's metric.

1. Introduction

As already mentioned in the abstract, higher dimensional anti-de Sitter spacetime is currently receiving more and more attention: "*Recently, it has been proposed by Maldacena that large N limits of certain conformal field theories in d dimensions can be described in terms of supergravity (and string theory) on the product of $d+1$-dimensional AdS space with a compact manifold.*" cf. [12]. In this article, the coordinate replacement (Multiplication of radial coordinate and it's differential by a time dependent scale factor) which turns Schwarzschild's metric into McVittie's intervall is applied to a modified AdS_n model. In previous considerations, see [6, 7], I used this replacement to receive a de-Sitter based solution in four dimensions. These results can be transfered to construct a metric based on the modified AdS_n spacetime. Anti de Sitter spacetime has negative scalar curvature and solves Einstein's empty space equations $R^i_k - \frac{1}{2}R\delta^i_k + \Lambda\delta^i_k = 0$ for negative cosmological constant $\Lambda < 0$. The following conventions are used in this article: The signature of the metric is choosen to be $(-,+,...,+)$, Ricci tensor and curvature scalar are given by $R_{nk} = \sum_a \left(\partial_a\Gamma^a_{kn} - \partial_k\Gamma^a_{an} + \sum_b \left(\Gamma^a_{ab}\Gamma^b_{kn} - \Gamma^a_{kb}\Gamma^b_{an}\right)\right)$ and $R = \sum_n\sum_k g^{nk}R_{nk}$. The notation $d\Omega^2_{n-2}$ is used for the line element on the unit $(n-2)-$sphere:

$$d\Omega^2_{n-2} = d\theta_1^2 + \sin^2(\theta_1)\, d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)\, d\theta_3^2 + \cdots + \prod_{k=1}^{n-3}\sin^2(\theta_k)\, d\theta^2_{n-2}$$

Anti-de Sitter space AdS_n in n dimensions with respect to the coordinates $\{t,\, r,\, \theta_1, \ldots, \theta_{n-2}\}$ is given by

$$ds^2 = -\left(1 - \frac{2\Lambda}{(n-2)(n-1)}r^2\right)c^2dt^2 + \left(1 - \frac{2\Lambda}{(n-2)(n-1)}r^2\right)^{-1}dr^2 + r^2 d\Omega^2_{n-2} \tag{1}$$

cf. for example [5]. This line element can be transformed to a system of isotropic coordinates $\{t,\, q,\, \theta_1, \ldots, \theta_{n-2}\}$ by

$$r = q\left(1 + \frac{\Lambda}{2(n-2)(n-1)}q^2\right)^{-1}. \tag{2}$$

After some calculation, metric (1) turns into the static and isotropic form:

$$ds^2 = -\left(\frac{1 - \frac{\Lambda}{2(n-2)(n-1)}q^2}{1 + \frac{\Lambda}{2(n-2)(n-1)}q^2}\right)^2 c^2dt^2 + \frac{1}{\left(1 + \frac{\Lambda}{2(n-2)(n-1)}q^2\right)^2}\left(dq^2 + q^2 d\Omega^2_{n-2}\right) \tag{3}$$

2. Basic line element and Einstein's equations

Let l be a constant with physical unit m^{-1}. Based on the latter metric (3) we get a suitable ansatz for the modified line element. Instead of the negative factor $\frac{\Lambda}{2(n-2)(n-1)}$ an arbitrary constant $-l^2 < 0$ is used. Now, according to the method used in [7], the radial coordinate q is replaced by $a(t)\, q$ and the differential dq by $a(t)\, dq$. It should be mentioned that this replacement is different from a coordinate transformation, e.g. $q = a\bar{q}$, since the $\dot{a}\bar{q}dt$ term in $dq = \dot{a}\bar{q}dt + ad\bar{q}$ is ignored. Thereby one receives from (3) the isotropic line element:

$$ds^2 = -\left(\frac{1 + l^2\, a^2\,(t)\, q^2}{1 - l^2\, a^2\,(t)\, q^2}\right)^2 c^2 dt^2 + \frac{a^2\,(t)}{\left(1 - l^2\, a^2\,(t)\, q^2\right)^2}\left(dq^2 + q^2 d\Omega_{n-2}^2\right) \tag{4}$$

In order to work out Einstein's equations for this metric, the abbreviation

$$\chi = \chi\,(t, q) := l^2\, a^2\,(t)\, q^2 \tag{5}$$

is used in the following. Metric (4) then takes the simple form

$$ds^2 = -\left(\frac{1+\chi}{1-\chi}\right)^2 c^2 dt^2 + \frac{a^2\,(t)}{(1-\chi)^2}\left(dq^2 + q^2 d\Omega_{n-2}^2\right). \tag{6}$$

Obviously, the metric is a special case of a general isotropic line element, which is given by:

$$ds^2 = -e^{\nu(q,t)} c^2 dt^2 + e^{\mu(q,t)}\left(dq^2 + q^2 d\Omega_{n-2}^2\right) \tag{7}$$

Hereinafter, let overdots and primes stand for partial differentiation with respect to the time coordinate t and the spatial coordinate q, respectively. Patel, Tikekar and Dadhich published equations that result from Einstein's equations for a perfect fluid in such an isotropic spacetime with $n+2$ dimensions, cf appendix A. Their paper [2] contains an obvious typing error, located in equation (7), which does not reproduce the corresponding equation in the well known four-dimensional case (see for example the article of Israelit and Rosen [3]). Up to this, their set of equations fits to the following Einstein-tensor for the intervall (7), which reads:

$$\begin{aligned}
G^t_{\ t} &= \frac{n-2}{2}\left\{-\frac{n-1}{4}\dot{\mu}^2\frac{e^{-\nu}}{c^2} + \left(\mu'' + \frac{n-3}{4}\mu'^2 + \frac{n-2}{q}\mu'\right)e^{-\mu}\right\} + \Lambda \\
G^t_{\ q} &= \frac{n-2}{2}\left\{\dot{\mu}' - \frac{1}{2}\dot{\mu}\nu'\right\}\frac{e^{-\nu}}{c^2}, \qquad G^q_{\ t} = \frac{n-2}{2}\left\{\frac{1}{2}\dot{\mu}\nu' - \dot{\mu}'\right\}e^{-\mu} \\
G^q_{\ q} &= \frac{n-2}{2}\left\{-\left(\ddot{\mu} + \frac{n-1}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right)\frac{e^{-\nu}}{c^2} + \left(\frac{\nu'\mu'}{2} + \frac{\nu'}{q} + \frac{n-3}{q}\mu' + \frac{n-3}{4}\mu'^2\right)e^{-\mu}\right\} + \Lambda \\
G^{\theta_k}_{\ \theta_k} &= \frac{n-2}{2}\left\{-\left(\ddot{\mu} + \frac{n-1}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right)\frac{e^{-\nu}}{c^2} + \frac{1}{2}\left(\nu'' + (n-3)\,\mu'' + \frac{\nu'^2}{2} + \frac{\nu'}{q} + \frac{n-3}{q}\mu'\right)e^{-\mu}\right\} + \Lambda
\end{aligned} \tag{8}$$

Einstein's tensor (8) for the general isotropic interval includes the corresponding Einstein-tensor for metric (6) as a special case. The general isotropic metric (7) takes the form (6) if

$$\nu = 2\left\{\ln(1+\chi) - \ln(1-\chi)\right\}, \qquad \mu = 2\left\{\ln(a) - \ln(1-\chi)\right\}.$$

With $H := \dot{a}/a$ it is $\dot{\chi} = 2H\chi$ and $\chi' = 2\chi/q$ and the required derivatives of the functions ν and μ are given by

$$\dot{\nu} = 8H\frac{\chi}{1-\chi^2}, \quad \nu' = \frac{8}{q}\frac{\chi}{1-\chi^2}, \quad \dot{\mu} = 2H\frac{1+\chi}{1-\chi}, \quad \mu' = \frac{4}{q}\frac{\chi}{1-\chi}, \quad \nu'' = \frac{8}{q^2}\frac{\chi\left(1+4\chi^2\right)}{\left(1-\chi^2\right)^2},$$

$$\ddot{\mu} = \frac{2}{(1-\chi)^2}\left\{\frac{\ddot{a}}{a}\left(1-\chi^2\right) + H^2\left(\chi^2 + 4\chi - 1\right)\right\}, \quad \dot{\mu}' = \frac{8H}{q}\frac{\chi}{(1-\chi)^2}, \quad \mu'' = \frac{4}{q^2}\frac{\chi\,(1+\chi)}{(1-\chi)^2}.$$

By using the above derivatives in (6), one receives the Einstein's tensor for the metric (4) after a relatively simple but cumbersome calculation. The remaining nonzero components are:

$$G^t_{\ t} = \frac{(n-2)(n-1)}{2}\left[4l^2 - \frac{H^2}{c^2}\right] + \Lambda \tag{9}$$

$$G^q_{\ q} = G^\phi_{\ \phi} = G^{\theta_k}_{\ \theta_k} = \frac{n-2}{2}\left\{\frac{2}{c^2}\left(H^2 - \frac{\ddot{a}}{a}\right)\frac{1-x}{1+x} + \left[4l^2 - \frac{H^2}{c^2}\right](n-1)\right\} + \Lambda \tag{10}$$

Correspondingly, Einstein's empty space equations lead to

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(4l^2 + \frac{2\Lambda}{(n-2)(n-1)}\right)c^2 \tag{11}$$

and

$$\frac{n-2}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2-\frac{\ddot{a}}{a}\right]\frac{1-x}{1+x}+\frac{(n-2)(n-1)}{2}\left[4l^2-\frac{H^2}{c^2}\right]+\Lambda=0 \tag{12}$$

3. Empty space solution

Equation (11) can be solved easily, it is

$$a(t)=a_0\exp\left(\pm\sqrt{4l^2+\frac{2\Lambda}{(n-2)(n-1)}}\,ct\right) \tag{13}$$

where a_0 contains all constants of integration. Now it has to be shown, that (13) also solves (12). Generally, if $(\dot{a}/a)^2$ is an arbitrary nonzero constant, the term is equal to $\ddot{a}/a$:

$$0=\frac{d}{dt}\left[\left(\frac{\dot{a}}{a}\right)^2\right]=2\frac{\dot{a}}{a}\cdot\frac{\ddot{a}a-\dot{a}^2}{a^2}=2\frac{\dot{a}}{a}\left[\frac{\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2\right]$$

Thereby, the first addend in (12) vanishes and the whole equation reduces to (11). Thus, we have a $n-$dimensional exact solution of Einstein's field equations given by

$$ds^2=-\left[\frac{1+l^2q^2a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{2\Lambda}{(n-2)(n-1)}}\,ct\right)}{1-l^2q^2a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{2\Lambda}{(n-2)(n-1)}}\,ct\right)}\right]^2c^2dt^2+\frac{a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{2\Lambda}{(n-2)(n-1)}}\,ct\right)\left(dq^2+q^2d\Omega_{n-2}^2\right)}{\left[1-l^2q^2a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{2\Lambda}{(n-2)(n-1)}}\,ct\right)\right]^2} \tag{14}$$

4. Coordinate transformation

In the previous paper [7], I used the subsequent method to transform a four-dimensional de Sitter-like metric. Analogously, our line element (4) can also be transformed to the coordinates $\{t,r,\theta_1,..,\theta_{n-2}\}$, whereas it takes the anisotropic form

$$ds^2=-g_{tt}dt^2+2g_{tr}dtdr+g_{rr}dr^2+r^2d\Omega_{n-2}^2. \tag{15}$$

Comparing the $g_{\theta\theta}$ components of (4) and (15) leads to $r=aq\left(1+l^2a^2q^2\right)^{-1}$ which can be rearranged to get the transformation for the $q-$coordinate, it is:

$$q=\frac{\pm1\pm\sqrt{1+4l^2a^2r^2}}{2l^2ar}$$

Therewith metric (4) can be transformed into:

$$ds^2=\left[H^2r^2-c^2\left(1+4l^2a^2r^2\right)\right]dt^2+\frac{2Hr\,dt\,dr}{\sqrt{1+4l^2a^2r^2}}+\frac{dr^2}{1+4l^2a^2r^2}+r^2d\Omega_{n-2}^2 \tag{16}$$

A brief sketch of the necessary calculations are given in the appendix B. Collectively, both transformations are comparable in structure with Kaloper, Kleban and Martin's method to transform Mc Vittie's metric, cf. [9]. It is remarkably that this method also works in arbitrary dimensions for the anti-de Sitter metric.

Appendix A: Equations given by Patel, Tikekar and Dadhich

One receives the field equations given in the Patel, Tikekar and Dadhich paper [2] by adopting the dimension and using the stress-energy tensor of a perfect fluid, further they have no cosmological constant. In our notation their equations for the $n-$dimensional case should read:

$$\begin{aligned}
\frac{8\pi\gamma\rho}{c^2} &= \frac{n-2}{2}\left\{-\frac{n-1}{4c^2}\dot{\mu}^2e^{-\nu}+\left(\mu''+\frac{n-3}{4}\mu'^2+\frac{n-2}{q}\mu'\right)e^{-\mu}\right\}\\
\frac{8\pi\gamma p}{c^4} &= \frac{n-2}{2}\left\{\frac{e^{-\nu}}{c^2}\left(\ddot{\mu}-\frac{\dot{\mu}\dot{\nu}}{2}+\frac{n-1}{4}\dot{\mu}^2\right)-e^{-\mu}\left(\nu'\left(\frac{\mu'}{2}+\frac{1}{q}\right)+\frac{n-3}{q}\mu'+\frac{n-3}{4}\mu'^2\right)\right\}\\
0 &= \dot{\mu}'-\frac{\dot{\mu}\nu'}{2}\quad\text{and}\quad 0=\nu''+(n-3)\mu''+\frac{\nu'^2}{2}-\frac{n-3}{2}\mu'^2-\mu'\nu'-\frac{\nu'}{q}-(n-3)\frac{\mu'}{q}
\end{aligned}$$

Appendix B: The Transformation

This section contains some intermediate data of the coodinate transformation given in the section 4, in order to make it easier to understand. At first it is comfortable to use the abbreviation

$$\zeta := \sqrt{1 + 4l^2a^2r^2}.$$

so that the transformation reads $q = \left(2l^2ar\right)^{-1}(\pm 1 \pm \zeta)$ and its differential is given by

$$dq = \left(-\frac{q}{r} \pm \frac{2}{a\zeta}\right) dr - qH\, dt \tag{17}$$

The identity $1 \mp \frac{2r}{aq\zeta} = \frac{1}{\zeta}$ is helpful to compute the part

$$g_{qq}dq^2 = \frac{a^2}{(1 - l^2a^2q^2)}dq^2 = \frac{r^2}{q^2}dq^2 = \left(Hr\, dt + \frac{dr}{\zeta}\right)^2 \tag{18}$$

of (4). Moreover one receives $q^2 = \frac{1+\zeta}{2l^4r^4a^2}$ and with some calculation

$$\frac{1 + l^2a^2q^2}{1 - l^2a^2q^2} = -\zeta \tag{19}$$

it follows $g_{tt} = \zeta^2$. Since the comparison of both metrics claims $g_{qq}q^2d\Omega_{n-2}^2 = r^2d\Omega_{n-2}^2$, we finally receive metric (16).

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