A new divergence measure of belief function in D–S evidence theory

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Abstract

Dempster–Shafer (D–S) evidence theory is useful to handle the uncertainty problems. In D–S evidence theory, however, how to handle the high conflict evidences is still an open issue. In this paper, a new reinforced belief divergence measure, called as $\mathcal{RB}$ is developed to measure the discrepancy between basic belief assignments (BBAs) in D–S evidence theory. The proposed $\mathcal{RB}$ divergence is the first work to consider both of the correlations between the belief functions and the subset of set of belief functions. Additionally, the $\mathcal{RB}$ divergence has the merits for measurement. It can provide a more convincing and effective solution to measure the discrepancy between BBAs in D–S evidence theory.

Keywords: Dempster–Shafer evidence theory, Belief divergence measure, Evidential conflict, Belief function

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1. Introduction

In this paper, we mainly focus on the research of handling uncertainty based on Dempster–Shafer (D–S) evidence theory [1, 2]. Through the study of the advantages of D–S evidence theory, we know that it has the capability of handling the uncertainty in a flexibly and effectively way without prior information [3]. Whereas, an open issue in D–S evidence theory is about how to handle the high conflict evidences, since the counter-intuitive results may be generated by using Dempster's combination rule [4, 5].

So far, a substantial amount of works have been done from two different kinds of perspectives, namely, the modification of Dempster's combination rule and the pre-procession of the body of evidence [6, 7]. In this paper, we consider the second perspective, i.e., the pre-procession of the body of evidence to solve the problem of conflict evidences. A recent improved work is Xiao's method [8], which weights the body of evidence in the view of divergence. After carefully studying the existing methods, we found that Xiao's method [8] has the best performance to handle conflict evidences. However, it takes into account the conflict evidence only in the level of belief functions, leaving out of consideration in the relationship of the subset of set of belief functions.

In this paper, therefore, a reinforced belief divergence measure, called as $\mathcal{RB}$ is proposed to measure the discrepancy between BBAs in the D–S evidence theory. The proposed $\mathcal{RB}$ divergence is the first work to consider both of the correlations between the belief functions and the subset of set of belief functions, so that it can provide a more effective solution to measure the discrepancy between BBAs.

The rest of this paper is organized as follows. Section 2 briefly introduces the preliminaries of this paper, including the D–S evidence theory and belief divergence measure. In Section 3, a reinforced belief divergence measure is derived, in which the performance measure and comparative analysis are discussed subsequently. Section 4 concludes this study.

2. Preliminaries

2.1. Dempster-Shafer evidence theory

Dempster–Shafer (D–S) evidence theory [1, 2], as a generalization of Bayes probability theory is effective to deal with uncertain information.

Definition 2.1 *(Frame of discernment).*

Let $\mathbb{H}$ *be a set of mutually exclusive and collectively exhaustive events,*

$$\mathbb{H} = \{e_1, e_2, \ldots, e_i, \ldots, e_h\}, \tag{1}$$

which is defined as a frame of discernment.

The power set of $\mathbb{H}$*, denoted as* $2^{\mathbb{H}}$*, is defined by*

$$2^{\mathbb{H}} = \{\emptyset, \{e_1\}, \ldots, \{e_h\}, \{e_1, e_2\}, \ldots, \{e_1, e_2, \ldots, e_i\}, \ldots, \mathbb{H}\}, \tag{2}$$

where $\emptyset$ *is an empty set.*

If $A \in 2^{\mathbb{H}}$*,* A *is called a hypothesis.*

Definition 2.2 *(Mass function).*

In the frame of discernment $\mathbb{H}$*, a mass function, denoted as* m *is a mapping from* $2^{\mathbb{H}}$ *to [0, 1], which is defined by*

$$m : 2^{\mathbb{H}} \to [0, 1], \tag{3}$$

satisfying the following conditions,

$$m(\emptyset) = 0 \quad \textit{and} \quad \sum_{A \in 2^{\mathbb{H}}} m(A) = 1. \tag{4}$$

In the D–S theory, the mass function m can be also called a basic belief assignment (BBA) [9, 10].

Definition 2.3 *(Belief function and plausibility function).*

Let A be a hypothesis in the frame of discernment $\mathbb{H}$.

A belief function $Bel : 2^{\mathbb{H}} \to [0, 1]$ is defined as

$$Bel(A) = \sum_{B \subseteq A} m(B). \tag{5}$$

A plausibility function $Pl : 2^{\mathbb{H}} \to [0, 1]$ is defined as

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B). \tag{6}$$

It is obvious that $Pl(A) \geq Bel(A)$, where Bel is the lower limit function of hypothesis A and Pl is the upper limit function of hypothesis A [11].

Definition 2.4 *(Dempster's rule of combination).*

Let m_1 and m_2 be two independent BBAs in the frame of discernment $\mathbb{H}$. The Dempster's rule of combination, which is called the orthogonal sum, denoted as $m = m_1 \oplus m_2$, is defined by

$$m(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{B \cap C = A} m_1(B) m_2(C), & A \neq \emptyset, \\ 0, & A = \emptyset, \end{cases} \tag{7}$$

with

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C), \tag{8}$$

where $B, C \in 2^{\mathbb{H}}$, and K is the coefficient of conflict between BBAs m_1 and m_2.

It is noteworthy that the Dempster's combination rule is practicable under the condition that $K < 1$ [12, 13].

2.2. Belief divergence measure

Recently, a Belief Jensen–Shannon divergence measure, called as BJS divergence was presented by Xiao [8] to measure the discrepancy and conflict degree between the evidences.

Definition 2.5 *(The BJS divergence of belief functions).*

Let A_i be a hypothesis of the belief function m, and let m_1 and m_2 be two belief functions on the same frame of discernment Ω, containing h mutually exclusive and exhaustive hypotheses. The BJS divergence between the belief functions m_1 and m_2 is denoted as:

$$BJS(m_1, m_2) = \frac{1}{2}\left[S\left(m_1, \frac{m_1+m_2}{2}\right) + S\left(m_2, \frac{m_1+m_2}{2}\right)\right], \tag{9}$$

where $S(m_1, m_2) = \sum_i m_1(A_i) \log \frac{m_1(A_i)}{m_2(A_i)}$ *and* $\sum_i m_j(A_i) = 1$ $(i = 1, \ldots, 2^h; j = 1, 2)$.

$BJS(m_1, m_2)$ can be also expressed in the following form

$$BJS(m_1, m_2) = H\left(\frac{m_1+m_2}{2}\right) - \frac{1}{2}H(m_1) - \frac{1}{2}H(m_2),$$
$$= \frac{1}{2}\left[\sum_i m_1(A_i)\log\left(\frac{2m_1(A_i)}{m_1(A_i)+m_2(A_i)}\right) + \sum_i m_2(A_i)\log\left(\frac{2m_2(A_i)}{m_1(A_i)+m_2(A_i)}\right)\right], \tag{10}$$

where $H(m_j) = -\sum_i m_j(A_i) \log\ m_j(A_i)$ $(i = 1, \ldots, 2^h; j = 1, 2)$ *is the Shannon entropy.*

The property of BJS divergence are:

(1) BJS(m_1, m_2) is symmetric and always well defined;

(2) BJS(m_1, m_2) is bounded, $0 \leq$ BJS$(m_1, m_2) \leq 1$;

(3) its square root, $\sqrt{\mathrm{BJS}(m_1, m_2)}$ verifies the triangle inequality.

3. A reinforced belief divergence measure

3.1. Correlation between belief functions

Definition 3.1 *(Correlation coefficient between belief functions).*

Let m_1 and m_2 be two belief functions in the frame of discernment $\mathbb{H}$, which includes h mutually exclusive and collectively exhaustive events, where A_i is a hypothesis of m_1 and A_j is a hypothesis of m_2 $(i, j = 1, \ldots, 2^h)$. A correlation coefficient between the sets of belief functions m_1 and m_2 is defined as

$$\Gamma(A_i, A_j) = \frac{|A_i \cap A_j|}{|A_j|}, \tag{11}$$

where $A_i \cap A_j$ denotes the intersection between A_i and A_j; $|A_j|$ represents the cardinality of A_j.

3.2. A new divergence measure of belief function

In this section, a new divergence measure of belief function is exploited on the basis of the correlation coefficient between belief functions.

Definition 3.2 *(Divergence measure of belief functions).*

Let $\mathbb{H}$ be the frame of discernment which has h mutually exclusive and collectively exhaustive events. Let m_1 and m_2 be two belief functions on $\mathbb{H}$, where A_i is a hypothesis of m_1 and A_j is a hypothesis of m_2 $(i, j = 1, \ldots, 2^h)$. The belief divergence measure, denoted as $\mathfrak{B}$ between the belief functions m_1 and m_2

is defined by

$$\mathfrak{B}(m_1, m_2) = \sum_{i=1}^{2^h} \sum_{j=1}^{2^h} m_1(A_i) \log \frac{m_1(A_i)}{\frac{1}{2}m_1(A_i) + \frac{1}{2}m_2(A_j)} \frac{|A_i \cap A_j|}{|A_j|} + \sum_{i=1}^{2^h} \sum_{j=1}^{2^h} m_2(A_j) \log \frac{m_2(A_j)}{\frac{1}{2}m_1(A_i) + \frac{1}{2}m_2(A_j)} \frac{|A_i \cap A_j|}{|A_j|}. \tag{12}$$

When the hypotheses of belief functions are composed of singleton sets, the $\mathfrak{B}$ divergence measure degrades into the BJS divergence [8]:

$$\mathfrak{B}(m_1, m_2) = \sum_{i=1}^{h} m_1(A_i) \log \frac{m_1(A_i)}{\frac{1}{2}m_1(A_i) + \frac{1}{2}m_2(A_i)} + \sum_{i=1}^{h} m_2(A_i) \log \frac{m_2(A_i)}{\frac{1}{2}m_1(A_i) + \frac{1}{2}m_2(A_i)}. \tag{13}$$

3.3. Reinforced divergence measure of belief function

In this section, a reinforced divergence measure of belief function is devised based on the newly defined $\mathfrak{B}$ divergence.

Definition 3.3 *(Reinforced belief divergence measure).*

Let $\mathbb{H}$ be the frame of discernment which has h mutually exclusive and collectively exhaustive events. Let m_1 and m_2 be two belief functions on $\mathbb{H}$. The reinforced belief divergence measure, denoted as $\mathcal{RB}$ between the belief functions m_1 and m_2 is defined by

$$\mathcal{RB}(m_1, m_2) = \sqrt{\frac{\mathfrak{B}(m_1, m_1) + \mathfrak{B}(m_2, m_2) - 2\mathfrak{B}(m_1, m_2)}{2}}, \tag{14}$$

where $\mathfrak{B}(\cdot)$ is the belief divergence measure function in Definition 3.2.

The properties for the $\mathcal{RB}$ divergence measure can be easily induced by

- Non-negativeness: $\mathcal{RB}(m_1, m_2) \geq 0$.

- Nondegeneracy: $\mathcal{RB}(m_1, m_2) = 0$ if and only if $m_1 = m_2$.
- Symmetry: $\mathcal{RB}(m_1, m_2) = \mathcal{RB}(m_2, m_1)$.
- Triangle inequality: $\mathcal{RB}(m_1, m_3) \leq \mathcal{RB}(m_1, m_2) + \mathcal{RB}(m_2, m_3)$.

4. Conclusion

In this paper, a new divergence measure of belief function in Dempster–Shafer evidence theory was proposed. The main contribution of this study was that this was the first work to consider the relationship between multiple sets of belief function for divergence measure in the evidence theory, rather than only the correlations between belief functions. In a word, the proposed method provided a promising solution to measure the discrepancy between the belief functions in the evidence theory.

Conflict of Interest

The author states that there are no conflicts of interest.

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