Note on the Golden Mean, Nonlocality in Quantum Mechanics and Fractal Cantorian Spacetime

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Abstract

Given the inverse of the Golden Mean $\tau^{-1} = \phi = \frac{1}{2}(\sqrt{5} - 1)$, it is known that the continuous fraction expansion of $\phi^{-1} = 1 + \phi = \tau$ is $(1, 1, 1, \cdots)$. Integer solutions for the pairs of numbers $(d_i, n_i), i = 1, 2, 3, \cdots$ are found obeying the equation $(1+\phi)^n = d+\phi^n$. The latter equation was inspired from El Naschie's formulation of fractal Cantorian space time $\mathcal{E}_\infty$, and such that it furnishes the continuous fraction expansion of $(1 + \phi)^n = (d, d, d, d, \cdots)$, generalizing the original expression for the Golden mean. Hardy showed that is possible to demonstrate nonlocality without using Bell inequalities for two particles prepared in *nonmaximally* entangled states. The maximal probability of obtaining his nonlocality proof was found to be precisely ϕ^5. Zheng showed that three-particle nonmaximally entangled states revealed quantum nonlocality without using inequalities, and the maximal probability of obtaining the nonlocality proof was found to be $0.25 \sim \phi^3 = 0.236$. Given that the two-parameter p, q quantum-calculus deformations of the integers $[n]_{p,q} = F_n$ *coincide* precisely with the Fibonacci numbers, as a result of Binet's formula when $p = (1+\phi) = \tau, q = -\phi = -\tau^{-1}$, we explore further the implications of these results in the quantum entanglement of two-particle spin-s states. We finalized with some remarks on the generalized Binet's formula corresponding to generalized Fibonacci sequences.

Keywords : Cantorian Fractal spacetime; Quantum Calculus; Golden Mean, Noncommutative Geometry, Quantum Mechanics; Nonlocality.

Recently [1] we reviewed the two-parameter quantum calculus used in the construction of Fibonacci oscillators and presented the (p, q)-deformed Lorentz transformations which (still) leave invariant the (undeformed) Minkowski spacetime interval $t^2 - x^2 - y^2 - z^2$. Such transformations required the introduction of three different types of exponential functions leading to the (p,q)-analogs of hyperbolic and trigonometric functions. The key finding was that composition law of two successive Lorentz boosts (rotations) was *no* longer additive $\xi_3 \neq \xi_1 + \xi_2$ ($\theta_3 \neq \theta_1 + \theta_2$). We finalized with a discussion on quantum groups, noncommutative spacetimes, κ-deformed Poincare algebra and quasi-crystals.

In this short note we shall explore further these ideas in relation to the old results of Hardy [3], pertaining the maximal nonlocal effect in a two-particle Quantum Mechanical entanglement process, and El Naschie's [2] Cantorian Fractal Spacetime.

As a reminder, the Fibonacci numbers (known also as the Hemachandra-Virahanka-Gopala numbers)

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \quad \cdots \tag{1}$$

obtained recursively from the relation $F_{n+1} = F_{n-1} + F_n$, obey the 2×2 matrix conditions

$$\mathbf{M}^n \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \;=\; \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{2}$$

The eigenvalues of the 2×2 matrix $\mathbf{M}$ are given by the Golden Mean and its Galois conjugate, respectively,

$$\tau \;=\; \frac{1+\sqrt{5}}{2}, \quad \tilde{\tau} \;=\; -\;1/\tau \;=\; \frac{1-\sqrt{5}}{2} \tag{3}$$

Their corresponding eigenvectors are

$$\begin{pmatrix} \tilde{\tau} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\tilde{\tau} \end{pmatrix} \tag{4}$$

From eqs-(2-4) one can infer that the powers of τ^n and τ^{-n} can be expressed themselves in terms of τ and the Fibonacci numbers as follows

$$\tau^n \;=\; F_{n+1} \;+\; \frac{F_n}{\tau}, \quad \tau^{-n} \;=\; (-1)^n \; F_{n-1} \;+\; (-1)^{n+1} \; \frac{F_n}{\tau} \tag{5}$$

and from the latter relations one arrives finally at Binet's formula

$$\frac{\tau^n \;-\; (-1)^n \tau^{-n}}{\sqrt{5}} \;=\; F_n \tag{6}$$

From Binet's formula one can deduce directly that $lim_{n\to\infty}(F_{n+1}/F_n) = \tau$.

One can learn many other things from the above eqs-(2-6). Upon defining

$$\phi \;=\; \tau^{-1} \;\Leftrightarrow\; (1+\phi) \;=\; \tau \;\Leftrightarrow\; \phi\;(1+\phi) = 1$$

let us ask *what* are the pairs of *integers* (d,n) obeying the equation

$$(1+\phi)^n \;=\; d \;+\; \phi^n \;\Rightarrow\; (1+\phi)^n \;-\; \phi^n \;=\; d \tag{7}$$

The powers $(1+\phi)^n$, for positive and negative integers, are instrumental in the generation of the fractal dimensions associated with the fractal Cantorian spacetime $\mathcal{E}_\infty$ formulation by El Naschie, see [2] and references therein. A careful examination of the relations in eqs-(5,6) leads to the answer. For $n \neq 0$, the values of n are given by

$$n \ = \ odd \ = \ 2k \ + \ 1; \quad k = 0, 1, 2, \cdots \tag{8}$$

and those for d are

$$d \ = \ d(k) \ = \ F_{2k} \ + \ F_{2k+2} \ , \quad k = 0, 1, 2, \cdots \tag{9}$$

A list of the first few values for the pairs (d, n) obeying eq-(7) are given by

$$(d, n) \ = \ (0, 0); \quad (1, 1); \quad (4, 3), \quad (11, 5), \quad (29, 7), \quad (76, 9), \quad \cdots \tag{10a}$$

where $(0, 0)$ and $(1, 1)$ are the trivial values. Note that eq-(7) is invariant under $n \rightarrow -n, d \rightarrow -d$. Therefore one may include the *reflected* values of (10a) as well

$$(d', n') \ = \ (-d, -n) \ = \ (0, 0); \ (-1, -1); \ (-4, -3), \ (-11, -5), \ (-29, -7), \ (-76, -9), \ \cdots \tag{10b}$$

What is so particular of eq-(7) ? The answer lies in the following *continuous* fraction expansion associated with eq-(7), and obtained by a simple recursion as follows

$$(1+\phi)^n \ = \ d \ + \ \phi^n \ = \ d \ + \ \frac{1}{\phi^{-n}} \ = \ d \ + \ \frac{1}{(1+\phi)^n} \ = \ d \ + \ \frac{1}{d+\phi^n} \tag{11}$$

Iterating this process over and over yields the continuous fraction expansion of

$$(1+\phi)^n \ = \ (d, d, d, d, \cdots) \tag{12}$$

For example

$$(1+\phi) \ = \ (1, 1, 1, 1, \cdots); \ \ (1+\phi)^3 \ = \ 4 \ + \ \phi^3 \ = \ (4, 4, 4, 4, \cdots);$$
$$(1+\phi)^5 \ = \ 11 \ + \ \phi^5 \ = \ (11, 11, 11, 11, \cdots); \quad \cdots \tag{13}$$

In a seminal paper Hardy [3] gave a proof of nonlocality for two particles that only requires a total of four dimensions in Hilbert space, like Bell's proof, but does not require inequalities. By choosing appropriate basis states $\pm >_i$, for particle i with $i = 1, 2$ (as Hardy emphasized, these states do not necessarily have to be associated with spin-they could be associated with any other appropriate physical quantity), any two-particle entangled state can be written as

$$\Psi \ = \ \alpha \ |+>_1 \ \ |+>_2 \ \ - \beta \ |->_1 \ \ |->_2, \ \ \alpha^2 + \beta^2 = 1 \tag{14}$$

where α, β are two real constants. Hardy chose a $-\beta$ sign for later convenience and considered two-particle states for which each particle lives in a two dimensional Hilbert space. He pointed out that for particles living in a higher number of dimensions of their Hilbert space one could perform a measurement that projects the state of the two particles onto an appropriate four-dimensional subspace and preserves the entanglement, and proceed from there.

He showed that it is possible to demonstrate nonlocality for two particles *without* using inequalities for all entangled states (except maximally entangled states such as the singlet state). For example, it is possible to demonstrate nonlocality without using Bell inequalities for two spin-$\frac{1}{2}$ particles prepared in *nonmaximally* entangled states. He found that the *maximum* nonlocal effect is when the following fraction is maximum

$$\gamma \ = \ \left(\frac{(|\alpha| - |\beta|) \ |\alpha\beta|}{1 - |\alpha\beta|} \right)^2, \quad \alpha^2 + \beta^2 \ = \ 1 \tag{15}$$

The maximum occurs when

$$\gamma \ = \ \frac{1}{2} \ (5 \ \sqrt{5} \ - \ 11) \simeq 0.090169 \tag{16}$$

and which happens to be precisely equal to

$$\gamma \ = \ \phi^5 \ = \ F_5 \ \phi \ - \ F_4 \ = \ 5 \ (\frac{\sqrt{5} \ - \ 1}{2}) \ - \ 3 \ = \ \frac{1}{2} \ (5 \ \sqrt{5} \ - \ 11) \tag{17}$$

Hardy's argument has been generalized to two spin-s particles and N spin-$\frac{1}{2}$ particles [4].

Based on the findings in eqs-(16-17), it now begs the question if there is another physical realization of the other *odd* powers $\phi^n, n = 1, 3, 7, 9, 11, \cdots$ *pertaining* the non-maximally entangled states of three, four, $\cdots, N$ particles.

Zheng [5] showed that three-particle nonmaximally entangled states (the so-called W states) revealed quantum nonlocality without using inequalities, and the maximal probability of obtaining the nonlocality proof was found to be 0.25. Hence, such probability turned out to be much larger than the one found by Hardy given earlier by $\phi^5(0.090169)$ using the two-particle nonmaximally entangled states.

One may note that $0.25 \sim \phi^3 = 0.236$, consequently in this three-particle case there is *no precise* agreement as the one found before in eqs-(16-17) for the two-particle case. Although correlation is *not* causation, we should explore further whether or not the result ϕ^5 found by Hardy was a mere numerical coincidence or it reveals something deeper. Nottale has advocated long ago the fractal spacetime origins behind Quantum Mechanics [6].

We finalize with a discussion of the dimensions $d(k)$ found in eq-(9) and the connection with the two-parameter (p, q) deformed quantum calculus when $p = \tau; q = \tilde{\tau}$.

The (p, q) number is defined for any number n as

$$[n]_{p,q} \ = \ [n]_{q,p} \ \equiv \ \frac{p^n - q^n}{p - q} \ = \ p^{n-1} \ + \ p^{n-2} q \ + \ \ldots \ + \ p q^{n-2} \ + \ q^{n-1} \tag{18}$$

which is a natural generalization of the q-number

$$[n]_q \ \equiv \ \frac{1 - q^n}{1 - q} \ = \ 1 \ + \ q \ + \ \ldots \ + \ q^{n-2} \ + \ q^{n-1} \tag{19}$$

In [1] we found that when p, q are given by the Golden Mean, and its Galois conjugate, respectively

$$p \;=\; \tau \;=\; \frac{1+\sqrt{5}}{2}, \;\; q \;=\; \tilde{\tau} \;=\; -\; 1/\tau \;=\; \frac{1-\sqrt{5}}{2} \tag{20}$$

the p, q *deformations* of the integers $[n]_{p,q}$ *coincide* precisely with the Fibonacci numbers (also integers) as a result of Binet's formula

$$[n]_{p,q} \;=\; [n]_{q,p} \;\equiv\; \frac{\tau^n \;-\; (-1)^n \tau^{-n}}{\sqrt{5}} \;=\; F_n \tag{21}$$

Therefore, the values of the dimensions d found in eq-(9) can be rewritten also

$$d(k) \;=\; F_{2k} \;+\; F_{2k+2} \;=\; =\; [2k]_{q,p} \;+\; [2k+2]_{q,p}, \;\; p = \tau; q = \tilde{\tau} \tag{22}$$

Since

$$[2k]_{q,p} \;+\; [2k+2]_{q,p} \;\neq\; [2k+2k+2]_{q,p} \;=\; [4k+2]_{q,p} \;=\; F_{4k+2} \tag{23}$$

one deduces that

$$F_{2k} \;+\; F_{2k+2} \;\neq\; F_{4k+2} \tag{24}$$

After writing the number $4k+2 = 2(2k+1)$, and replacing $k \to s$, gives $2(2s+1)$ which *is* the Hilbert space dimension of two-particle spin-s states ($s =$ integer). Therefore, the algebraic relations (22-24) could be telling us that the *odd* powers ϕ^{2k+1} may contain some relevant information of the quantum entanglement properties between two particles of spin-s that have not been revealed yet.

To conclude, we may ask what other Galois-conjugate pairs $(p, q = \tilde{p})$, besides $p = \tau; q = \tilde{\tau}$, yield integer values for $[n]_{q,p} = N$ in eq-(18) ?; are $p = \tau; q = \tilde{\tau}$ special or are there an infinite number of Galois-conjugate pairs of solutions ?

Given the Galois-conjugate pairs $\frac{1}{2}(1 \pm \sqrt{m})$, where m is a square-free integer, the Binet formula (6) can be *generalized* to [7]

$$\frac{(\frac{1+\sqrt{m}}{2})^n \;-\; (\frac{1-\sqrt{m}}{2})^n}{\sqrt{m}} \;=\; G_n \tag{25}$$

and where $\frac{1}{2}(1 \pm \sqrt{m})$ are the roots to the quadratic equation $x^2 - x - (\frac{m-1}{4}) = 0$. However we must emphasize that *not* all values of G_n corresponding to the square-free integers $m = 2, 3, 5, 7, 11, 13, 14, 15, 17, \cdots$ are *integers*. The numbers G_n belong to a generalized Fibonacci sequence given by [7]

$$G_{n+2} \;=\; G_{n+1} \;+\; (\frac{m-1}{4})\; G_n, \;\; G_1 \;=\; G_2 \;=\; 1 \tag{26}$$

Therefore, by forcing all the G_n to be integers leads to $m = 4k+1, k = 1, 2, \cdots$ subject to the condition that m must be square-free. Two specific examples for the values of m where all the G_n are integers are $m = 13, 17$, their Galois-conjugate pairs are then given by $\frac{1}{2}(1 \pm \sqrt{13})$ and $\frac{1}{2}(1 \pm \sqrt{17})$, and whose integer sequences (26) are respectively

$$\{\ 1,\ 4,\ 7,\ 19,\ \cdots\ \} \quad \{\ 1,\ 5,\ 9,\ 29,\ \cdots\ \} \tag{27}$$

The Golden mean is associated with the 5-fold symmetry of the Penrose tiling (quasicrystal) of the two-dim plane, and can be obtained via the cut-and-projection method of the cubic lattice in $5D$ onto the two-dim plane. Quasicrystals with $10, 12$ and 18-fold symmetry are known. We were unable to find examples of quasicrystals with a $13, 17$-fold symmetry in the literature. We believe the Galois-conjugate pairs $\frac{1}{2}(1 \pm \sqrt{m})$, with $m = 4k + 1$, and square-free which generalize Binet's formula (6), *and* which generate *integer*-values for *all* the G_n's deserves further scrutiny, in particular, in their role in the physics of quasicrystals.

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