# NECESSARY AND SUFFICIENT CONDITIONS FOR A FACTORABLE MATRIX TO BE HYPONORMAL

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ABSTRACT. Necessary and sufficient conditions are given for a special subclass of the factorable matrices to be hyponormal operators on $\ell^2$. Three examples are then given of polynomials that generate hyponormal weighted mean operators, and one example that does not. Paired with the main result presented here, various computer software programs may then be used as an aid for classifying operators in that subclass as hyponormal or not.

## 1. Introduction

A lower triangular infinite matrix $M = M$ $(\{a_i\},\{c_j\})$, acting through multiplication to give a bounded linear operator on $\ell^2$, is *factorable* if its entries are

$$m_{ij} = \begin{cases} a_i c_j & if \quad j \leq i \\ 0 & if \quad j > i \end{cases}$$

where $a_i$ depends only on $i$ and $c_j$ depends only on $j$. A *hyponormal* operator $M$ satisfies

$$\langle (M^*M - MM^*)f, f\rangle \geq 0$$

for all $f$ in $\ell^2$.

## 2. Main Result

We begin by noting that an earlier version of the following result can be found in [2].

**Theorem 2.1.** *Suppose $M = M(\{a_i\},\{c_j\})$ is a factorable matrix that acts as a bounded operator on $\ell^2$ and that the following conditions are satisfied:*

(1) *both $\{a_n\}$and $\{\frac{a_n}{c_n}\}$ are positive (strictly) decreasing sequences that converge to 0, and*
(2) *the operator $B = [b_{ij}]$ defined by*

$$b_{ij} = \begin{cases} c_i(\frac{1}{c_j} - \frac{1}{c_{j+1}}\frac{a_{j+1}}{a_j}) & if \quad i \leq j \\ -\frac{a_{j+1}}{a_j} & if \quad i = j+1 \\ 0 & if \quad i > j+1 \end{cases}$$

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*is bounded on* $\ell^2$.
*The operator* $M$ *is hyponormal if and only if* (*see Remark* 2.1) *the following recursively defined sequence is positive for all* $n$*:*

$$\delta_0 = \frac{c_1^2(a_0^2-a_1^2)-(c_1a_0-c_0a_1)^2}{c_1^2a_0^2} \text{ and for } n \geq 1,$$

$$\delta_n = 1-\frac{a_{n+1}^2}{a_n^2}-\frac{(c_n^2-c_{n-1}^2)a_{n-1}^2(c_{n+1}a_n-c_na_{n+1})^2}{c_{n+1}^2a_n^2(c_na_{n-1}-c_{n-1}a_n)^2}-\frac{\frac{(c_{n+1}a_n-c_na_{n+1})^2(c_na_n-c_{n-1}a_{n-1})^2}{c_{n+1}^2a_n^2(c_na_{n-1}-c_{n-1}a_n)^2}}{\delta_{n-1}}.$$

*Proof.* The proof of [2, Theorem 2.1] actually justifies this stronger result. □

**Remark 2.1.** *There is a situation that must be avoided in the "only if" part of the theorem. It must not be true that* $\delta_N = 0$ *for some* $N \geq 1$ *while* $\delta_n > 0$ *for all* $n \leq N-1$. *In this case it is still possible for* $M$ *to be hyponormal even though* $\delta_{N+1}$ *is undefined, as* [2, *Example* 2.4] *illustrates.*

The next few operators qualify for the theorem since the operator $B$ is the sum of a (bounded) weighted unilateral shift and a positive constant multiple of the adjoint of the Cesàro matrix, the factorable matrix with constant row segments given by $a_n = 1/(n+1)$, which is known to be a bounded operator on $\ell^2$ (see [1]). Similarly, $M$ itself is also bounded.

**Example 2.1.** *(Weighted mean operator generated by the positive integers) Consider the operator with weight sequence* $c_n = n+1$ *and*

$$a_n = \frac{1}{\sum_{l=0}^{n} c_k} = \frac{2}{(n+1)(n+2)}.$$

*Straightforward computations yield*

$$\delta_0 = \frac{7}{36}$$

*and for* $n \geq 1$,

$$\delta_n = \frac{18n^3+51n^2+32n+7}{(n+3)^2(3n+2)^2} - \frac{n^2(3n+5)^2}{(n+2)^2(n+3)^2(3n+2)^2} \cdot \frac{1}{\delta_{n-1}}.$$

*It can be verified using induction that*

$$\delta_n = \frac{(n+1)(3n+7)}{(n+3)^2(3n+4)}$$

*for all* $n$, *so this operator is hyponormal. The hyponormality of this operator was earlier noted in* [3].

**Example 2.2.** *Consider the weighted mean operator generated by the weight sequence* $c_n = \binom{n+2}{2}$, *so*

$$a_n = \frac{1}{\sum_{l=0}^{n} c_k} = \frac{1}{\binom{n+3}{3}}.$$

*It is straightforward to compute that*

$$\delta_0 = \frac{7}{72}$$

*and for $n \geq 1$,*

$$\delta_n = \frac{(10n+7)(5n^2+16n+8)}{(n+4)^2(5n+6)^2} - \frac{n^2(5n+11)^2}{(n+3)^2(n+4)^2(5n+6)^2} \cdot \frac{1}{\delta_{n-1}}.$$

*It can then be verified that*

$$\delta_n = \frac{(n+1)(5n+14)}{(n+4)^2(5n+9)}$$

*for all $n$, so this operator is hyponormal.*

**Example 2.3.** *(Weighted mean) Consider the weighted mean operator with weight sequence $c_n = \binom{n+3}{3}$. so*

$$a_n = \frac{1}{\sum_{l=0}^{n} c_k} = \frac{1}{\binom{n+4}{4}}.$$

*Straightforward computations give*

$$\delta_0 = \frac{23}{400}$$

*and for $n \geq 1$,*

$$\delta_n = \frac{98n^3+483n^2+624n+207}{(n+5)^2(7n+12)^2} - \frac{n^2(7n+19)^2}{(n+4)^2(n+5)^2(7n+12)^2} \cdot \frac{1}{\delta_{n-1}}.$$

*It can then be verified that*

$$\delta_n = \frac{(n+1)(7n+23)}{(n+5)^2(7n+16)}$$

*for all $n$, so this operator is hyponormal.*

**Example 2.4.** *Consider the weighted mean operator generated by the weight sequence $c_n = 2n^2 + n + 3$. Since $\delta_0 = 7/36$ and*

$$\delta_1 = -\frac{697}{52052} < 0,$$

*this operator is not hyponormal on $\ell^2$.*

## 3. Concluding remark

For those with knowledge of infinite matrices and access to premier computational software, this theorem could be an aid for further classification of factorable matrices as hyponormal or not.

## References

[1] A. Brown, P. R. Halmos, and A. L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged) **26** (1965), 125-137.

[2] H. C. Rhaly Jr. and B. E. Rhoades, *Posinormal factorable matrices with a constant main diagonal*, Rev. Un. Mat. Argentina **55** (2014), no. 1, 19–24.

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