DIRAC'S EQUATION (and the alleged fourth dimension)

(proof of an oscillating universe and of the three dimensional interpretation of the relativistic fourth dimension)

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January 2019

Abstract : this is a proof that the d'Alembert's Wave Equation, that of Schrodinger, of Klein-Gordon and of Dirac are all related one another and show the oscillation of the universe. Moreover, the Klein-Gordon's Equation gives us a three dimensional interpretation of either all relativistic fourth components or the rest energy.

We know from the relativity that the total energy E is:

$$E^2 = p^2c^2 + m_0^2c^4 \tag{1.1}$$

This is the most general formula we have for the energy and is suitable for a relativistic particle indeed. On this purpose, please see the following link on page 52:
https://scienzaufficialeattendibilita.weebly.com/uploads/1/3/9/1/13910584/la-teoria-della-relativit%C3%80-generale.pdf

Now, for a photon (a particle whose rest mass is equal to zero), we have: $E^2 = p^2c^2$, and:

$$E = pc \tag{1.2}$$

For a non relativistic particle, we know its kinetic energy is: $E_k = \frac{1}{2}m_0v^2$, but this is hidden in (1.1), which is more general, indeed. In fact, (1.1) can be rewritten in this way:

$$E = m_0c^2(1+\frac{p^2}{m_0^2c^2})^{1/2} \tag{1.3}$$

and for the developments of Taylor, we have:

$f(x) = \sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{1}{2}x$, from this, for the (1.3):

$E = m_0c^2(1+\frac{p^2}{m_0^2c^2})^{1/2} \approx m_0c^2(1+\frac{p^2}{2m_0^2c^2}) = m_0c^2 + \frac{p^2}{2m_0}$ and, for the kinetic energy, we have:

$$E_k = E - m_0c^2 = \frac{p^2}{2m_0} = \frac{1}{2}m_0v^2 \text{ qed.}$$

Now, let's take the general expression for a wave:

$$\Psi = A \cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)} = A \cdot e^{i(\frac{2\pi}{\lambda}\hat{k}\cdot\vec{x}-\frac{2\pi}{\lambda}vt)}, \tag{1.4}$$

as: $\vec{k} = \frac{2\pi}{\lambda}\hat{k}$, $\omega = \frac{2\pi}{T} = 2\pi f = 2\pi\frac{v}{\lambda}$.

Such a wave simultaneously propagates in space (x) and oscillates in time t; in fact, if we fix t=0, we see we have an oscillation along x ($\Psi = A \cdot e^{i(\vec{k}\cdot\vec{x})}$) and if we fix x=0 we have an oscillation in time ($\Psi = A \cdot e^{-i(\omega t)}$).

We also know that:

$$E = hf = \frac{h}{2\pi}2\pi f = \hbar\omega \tag{1.5}$$

and being (1.2) standing, we have:

$pc = \hbar\omega$, from which :

$$p = \hbar\frac{\omega}{c} = \hbar\frac{2\pi}{\lambda} = \hbar k = p \tag{1.6}$$

and (1.4) becomes:

$$\Psi = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E}{\hbar}t)} \tag{1.7}$$

By simply putting such a Ψ in the following equations:

$$(i\hbar\frac{\partial}{\partial t})\Psi = E\Psi = (\hbar\omega)\Psi \tag{1.8}$$

$$(\frac{\hbar}{i}\nabla)\Psi = \vec{p}\Psi = (\hbar\vec{k})\Psi \; ; \tag{1.9}$$

we have that they give identities, sot they are correct.

In one dimension: $(\frac{\hbar}{i}\frac{\partial}{\partial x})\Psi = p\Psi = (\hbar k)\Psi$; (∇ gradient)

So, we can deduce the following operatorial identities:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \tag{1.10}$$

$$\vec{p} \rightarrow \frac{\hbar}{i}\nabla \tag{1.11}$$

As (1.2) stands: $E^2 = p^2c^2$, we have:

$$(i\hbar\frac{\partial}{\partial t})^2\Psi = c^2(\frac{\hbar}{i}\nabla)^2\Psi, \tag{1.12}$$

that is:

$$\boxed{\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0} \tag{1.13}$$

or also ($\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, laplacian, divergence of a gradient): $\Delta\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$,

which is the **d'Alembert's Wave Equation**.

Please notice such an equation, derived in a '' relativistic'' environment (photon, i.e. a particle propagating by speed c and with a zero rest mass) is invariant under a Lorentz's Transformation. Please, see also the following link on page 55:

https://scienzaufficialeattendibilita.weebly.com/uploads/1/3/9/1/13910584/la-teoria-della-relativit%C3%80-generale.pdf

If now we consider non relativistic particles (atoms are like that, ordinarily), we will get a non relativistic ''wave'' equation, which is the Schrodinger's Equation. In fact, if in (1.7) we no longer consider $E = pc$, but $E_k = \frac{1}{2}m_0v^2$ (a non relativistic equation, indeed), we get:

$$\Psi = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E_k}{\hbar}t)} = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{p^2}{2m_0\hbar}t)} \tag{1.14}$$

and as well as we got (1.12), by a direct use of (1.14) in the following equation:

$$(i\hbar\frac{\partial}{\partial t})\Psi=(-\frac{\hbar^2}{2m_0}\nabla^2)\Psi \qquad (1.15)$$

($(i\hbar\frac{\partial}{\partial t})\Psi=(-\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2})\Psi$, in one dimension)

we get an identity. Therefore, (1.15) is true. Please notice that in (1.14) we have no longer used a total E, but just an Ek, and we are going to take that into account.

The left side of (1.15) is $(i\hbar\frac{\partial}{\partial t})\Psi=E_k\Psi$, but we know that Ek=H-V, so, still in force of the (1.15):

$-\frac{\hbar^2}{2m_0}\Delta\Psi=(H-V)\Psi$, that is:

$$\boxed{\Delta\Psi+\frac{2m_0}{\hbar^2}(H-V)\Psi=0} \qquad (1.16)$$

which is the **Schrodinger's Equation**.

Let' get into a more general situation, where we have a relativistic particle with a rest mass not equal to zero.

As well as we did before, as for (1.1) we have: $E=\sqrt{p^2c^2+m_0^2c^4}$, then, by using such an E still in (1.7) $\Psi=A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x}-\frac{E}{\hbar}t)}$, we will have:

$$\Psi=A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x}-\frac{\sqrt{p^2c^2+m_0^2c^4}}{\hbar}t)} \qquad (1.17)$$

and, as usual, still by introduction of an equation into another, we see that such a Ψ is a solution for the following:

$$\boxed{(\nabla^2\Psi-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2})-\frac{m_0^2c^2}{\hbar^2}\Psi=0} \qquad (1.18)$$

which is nothing but the **Klein-Gordon's Equation** and it is similar to that of d'Alembert, but has an item more.

Let's really carry out the introduction of (1.17) in (1.18), to see that all this really stands. We have:

$\nabla^2\Psi=(i)^2\frac{p^2}{\hbar^2}\Psi=-\frac{p^2}{\hbar^2}\Psi$ and

$-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}=-\frac{1}{c^2}(-i)^2\frac{E^2}{\hbar^2}\Psi=\frac{1}{c^2\hbar^2}(p^2c^2+m_0^2c^4)\Psi$ and so:

$-\frac{p^2}{\hbar^2}\Psi+\frac{1}{c^2\hbar^2}(p^2c^2+m_0^2c^4)\Psi-\frac{m_0^2c^2}{\hbar^2}\Psi=0$, that is 0=0.

Let's set $l=\frac{m_0c}{\hbar}$; such an l is dimensionally like the wave vector k. By such an l, we have that (1.17) and (1.18) can be rewritten as follows:

$$\boxed{\Psi=A\cdot e^{i(\vec{k}\cdot\vec{x}-\sqrt{(k^2+l^2)}ct)}=A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega' t)}} \qquad (1.19)$$

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - l^2\Psi = 0 \tag{1.20}$$

where $\boxed{\omega' = \sqrt{(k^2+l^2)}c}$.

Relativity says that a body with a zero speed, with respect to us, has, on the othe hand, a spatial fourth component ct, a fourth 4-momentum component mc and an intrinsic rest energy m_0c^2. Hence, in jumping from a photon, whose m0 is zero, to a relativistic particle with a rest mass m0 , the wave equation jumps from the d'Alembert's (1.13) to the Klein-Gordon's (1.20), with a wave function (1.19), instead of the (1.4) and the difference is that the rest mass component m0 , which causes the existence of a "rest" energy m_0c^2 (whose essence is "four-dimensional" and shows up with Relativity and with the energy-momentum vector) is nothing but an increase of time oscillation, where we go from an angular frequency ω to $\omega' = \sqrt{(k^2+l^2)}c$ higher! This is a three-dimensional interpretation of an entity whose nature is allegedly four-dimensional. More objections to the existence of an alleged fourth dimension can be found at the following link, on page 23: http://vixra.org/pdf/1303.0074v1.pdf

Let's rewrite the Klein-Gordon's Equation (1.20) in the following way:

$$-\frac{\partial^2\Psi}{\partial t^2} + c^2\nabla^2\Psi - l^2c^2\Psi = 0 \tag{1.21}$$

and after taking into account that $i^2 = -1$ and $(a-b)(a+b) = a^2 - b^2$, we have that such an equation can be rewritten like this:

$$[i\frac{\partial}{\partial t} - (i\alpha\cdot\nabla - \beta m_0)][i\frac{\partial}{\partial t} + (i\alpha\cdot\nabla - \beta m_0)] = 0, \tag{1.22}$$

or also:

$$\begin{cases} [i\dfrac{\partial}{\partial t} - (i\alpha\cdot\nabla - \beta m_0)]\Psi = 0 \\ \\ [i\dfrac{\partial}{\partial t} + (i\alpha\cdot\nabla - \beta m_0)]\Psi = 0 \end{cases} \tag{1.23}$$

and (1.22) can be developed as:

$$[-\frac{\partial^2}{\partial t^2} + (\alpha\cdot\nabla)^2 + im_0\beta(\alpha\cdot\nabla) + im_0(\alpha\cdot\nabla)\beta - \beta^2 m_0^2]\Psi = 0 \tag{1.24}$$

This equation is equal to the (1.21) if:

$\beta^2 = \frac{c^4}{\hbar^2}$, $\alpha\beta + \beta\alpha = 0$, $\alpha_i\alpha_j = c^2$ if i=j and $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$ if $i \neq j$

The last two conditions on alphas make us have only ∇^2 and not mixed terms in ∇. (1.23), here reported:

$$\boxed{(i\frac{\partial}{\partial t} + i\alpha\cdot\nabla - \beta m_0)\Psi = 0} \tag{1.25}$$

can be considered as the **Dirac's Equation**, which is usually provided in the following form, in natural units ($\hbar = c = 1 \rightarrow \beta = 1$):

$$\boxed{(i\gamma^{\mu}\partial_{\mu} - m_0)]\Psi = 0} \ , \qquad (1.26)$$

where $i\gamma^{\mu}\partial_{\mu} = i\gamma^{\mu}\,\partial/\partial x^{\mu}$, which contains a summation under the Einstein convention, gives, under the values of μ, the derivative under the time $\frac{\partial}{\partial t}$ and under x, y and z of $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$:

$$i\gamma^{\mu}\partial_{\mu} \rightarrow i\frac{\partial}{\partial t} + i\alpha \cdot \nabla$$

Further developments of the Dirac's Equation will not be carried out, here.
Thank you for your attention.
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