Feynman Diagrams of the QED Vacuum Polarization

Richard J. Mathar*
Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany
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The Feynman diagrams of Quantum Electrodynamics are assembled from vertices where three edges meet: an incoming fermion, an outgoing fermion and an interaction line. If all vertices are of degree 3, the graphs are 3-regular (cubic), defining the vacuum polarization diagrams. Cutting an edge—a fermion line or an interaction line—generates *fairly* cubic graphs where two vertices have degree 1. These emerge in the perturbation theory for the Green's function (self energy) and for the effective interaction (polarization). The manuscript plots these graphs for up to 8 internal vertices.

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I. CUBIC GRAPHS

A. Zoology

The core element of the Feynman diagrams of QED is the vertex at which in incoming fermion is destroyed, an outgoing fermion is created, and a line of the bare interaction represents the "trigger" that causes that "scattering." In the nomenclature of graph theory these two fermions and the interaction are half-edges of a graph where the degree (total of edges) at that vertex is 3. If all vertices have degree 3, we call these graphs 3-regular (or cubic). By the handshake lemma, cubic graphs with odd numbers of vertices don't exist.

The Feynman diagrams are a subclass of cubic graphs by refining the following properties:

- they have signed edges in as far as there are two types of edges, fermions and interactions;
- the edges of the fermions are directed whereas the interaction lines are not (at least in the realm of this manuscript related to non-retarded, symmetric Coulomb interactions);
- at each vertex the in-degree of the fermion lines equals one, the out-degree of the fermion lines equals one, and the degree of the interaction line equals one.

This manuscript proceeds by pointing shortly at the connected cubic graphs without these properties, then switching on the two signs on the edges in Sections II–IV and then adding directions/orientations to the fermion lines in Sections V–VII. Another aspect of this "engineering" of the Feynman graphs is that breaking up (splitting) interaction edges in the cubic (vacuum polarization) graphs leads to the graphs relevant to the perturbation expansion of the effective interaction, and that breaking up fermion edges in the cubic graphs leads to the graphs of the Dyson expansion of the Green's function, which we do by moving on from Section II to Section IV and by moving on from Section V to Section VII.

B. Connected Undirected Unsigned Cubic Graphs

There are 1, 0, 1, 2, 5, 19,... unlabeled, non-oriented, connected cubic graphs without loops and without multi-edges on 0, 2, 4, 6,... vertices [1, A002851][2, 3].

Because the Feynman diagrams allow *bubble* diagrams where two vertices are connected by two fermion lines (for example in lowest order of the random phase approximation of the effective interaction), we drop the requirement that the cubic graphs have no multi-edges. So a more appropriate starting set are the 1, 1, 2, 6, 20,... unlabeled, non-oriented, connected cubic graphs on 0, 2, 4, 6, ... vertices [1, A000421][4][5, Table 1 column L]. For the purpose of this script the one graph with zero vertices contains a single edge which is a closed loop, a ring that closes in itself. The graph with two vertices is the graph where 2 vertices are connected by 3 edges; for more than two vertices

* mathar@mpia-hd.mpg.de; https://www.mpia-hd.mpg.de/~mathar

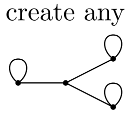
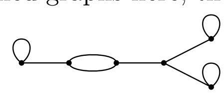
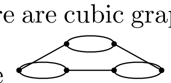
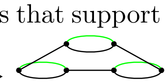
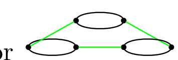
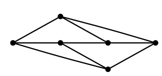
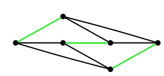
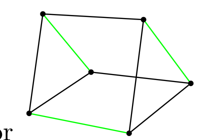
triple edges cannot occur (because then the vertex count is saturated with degree 3 at the pair of vertices, it becomes disconnected from the rest of the graph). So for more than two vertices the multi-edges can at most be double-edges.

Feynman diagrams also allow *tadpole* diagrams where a vertex is joining an interaction line and a fermion line that loops back to the same vertex. To include these, we also drop the requirement that the graphs have no loops and arrive at 1, 2, 5, 17, 71, 388, 2592,... unlabeled, undirected connected cubic multigraphs (allowing loops) on 0, 2, 4, 6, 8 ... vertices [6, D7][5, Table 1 column LM]. Palmer et al. provide the linear relation between the number of vertices, the number of single edges, the number of double edges and the number of loops [7]. The Online Encyclopedia of Integer Sequences offers a link to a file which plots all these graphs up 10 vertices [1, A005967].

II. VACUUM POLARIZATION (UNDIRECTED FERMIONS)

Colorizing the edges of the 1, 2, 5, 17,... cubic graphs mentioned in Section I B generates 1, 2, 5, 19, 88, 553, ... non-oriented vacuum polarization diagrams on 0, 2, 4, 6,... vertices which are plotted below. Fermion lines are colored black, interaction lines green. Because exactly two colors are used, one might also label edges with plusses and minuses; the graphs are also known as *signed* graphs for that reason.

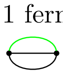
Sticking to the constraint that two fermion lines and one interaction line meet at each vertex, the fermion lines must actually congegrate as non-overlapping closed cycles in the graphs, and the interaction lines ensure that the graphs remain connected: We are in essence discussing multi-loop chord diagrams [8] where the interaction lines take the roles of the chords.

There are cubic graphs that do not create any colored/signed graphs here; the cubic graphs with a common footer of two or more tadpoles like this one  or this one  are generic examples of graphs that cannot be colored with that constraint. On the other hand there are cubic graphs that support more than one coloring. One generic class of that case are the chains of ring graphs like  →  or  which have two colorings. Another example is the 3-prism  →  or  .

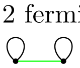
The graphs up to 6 vertices have been plotted by Kleinert et al. [9, Table IV]. The subset of graphs with one fermion is the number of ways of running chords through a convex polygon 1, 2, 5, 17, 79... [1, A054499][10, 11].

A. 2 vertices

1 fermion 1 bubble 0 tadpoles (1 graph):



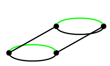
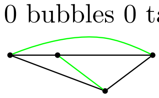
2 fermions 0 bubbles 2 tadpoles (1 graph):



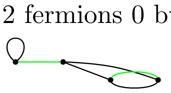
Total: 2 graphs on 2 vertices, 1 without tadpoles, 0 with neither bubbles nor tadpoles.

B. 4 vertices

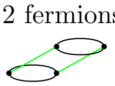
1 fermion 0 bubbles 0 tadpoles (2 graphs):

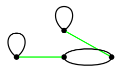
2 fermions 0 bubbles 1 tadpole (1 graph):



2 fermions 2 bubbles 0 tadpoles (1 graph):



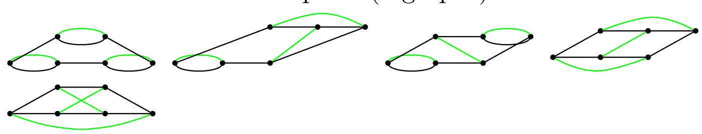
3 fermions 1 bubble 2 tadpoles (1 graph):



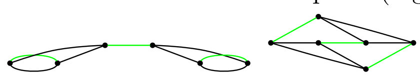
Total: 5 graphs on 4 vertices, 3 without tadpoles, 2 with neither bubbles nor tadpoles.

C. 6 vertices

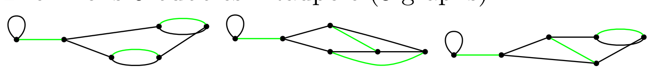
1 fermion 0 bubbles 0 tadpoles (5 graphs):



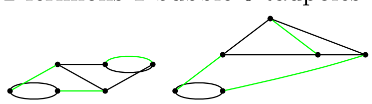
2 fermions 0 bubbles 0 tadpoles (2 graphs):



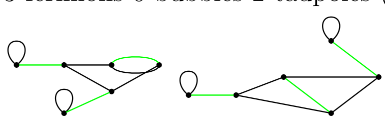
2 fermions 0 bubbles 1 tadpole (3 graphs):



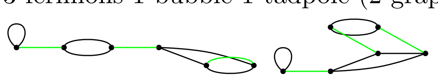
2 fermions 1 bubble 0 tadpoles (2 graphs):



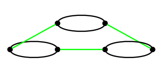
3 fermions 0 bubbles 2 tadpoles (2 graphs):



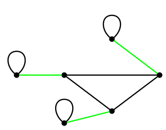
3 fermions 1 bubble 1 tadpole (2 graphs):



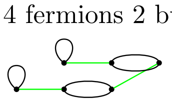
3 fermions 3 bubbles 0 tadpoles (1 graph):



4 fermions 0 bubbles 3 tadpoles (1 graph):



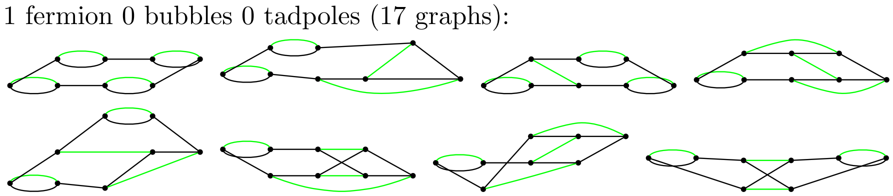
4 fermions 2 bubbles 2 tadpoles (1 graph):

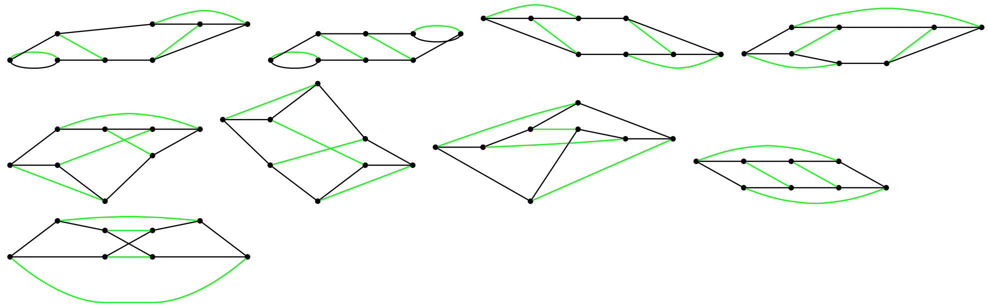


Total: 19 graphs on 6 vertices, 10 without tadpoles, 7 with neither bubbles nor tadpoles.

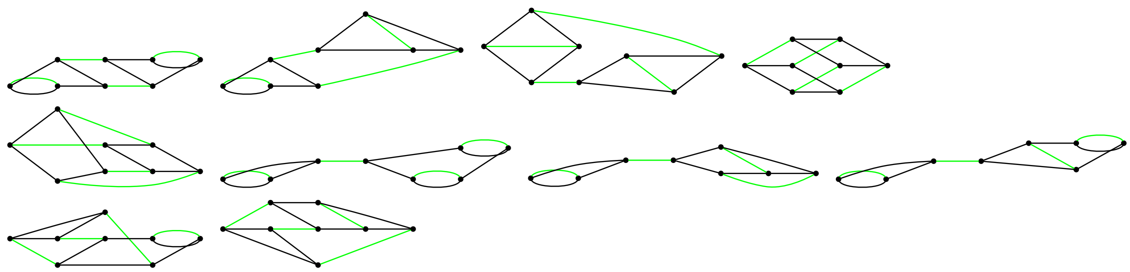
D. 8 vertices

1 fermion 0 bubbles 0 tadpoles (17 graphs):

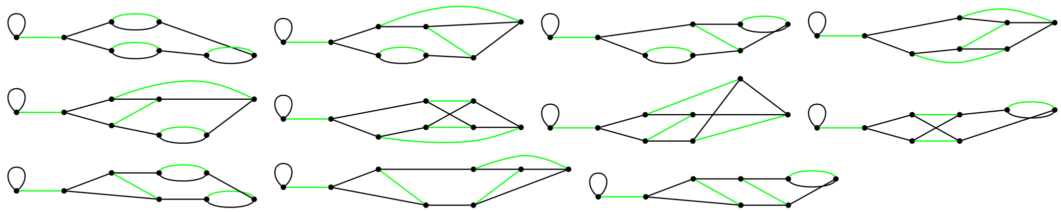




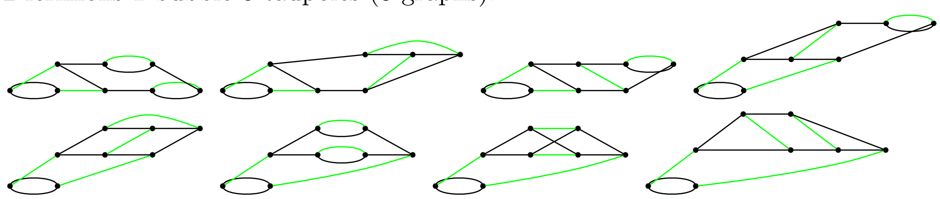
2 fermions 0 bubbles 0 tadpoles (10 graphs):



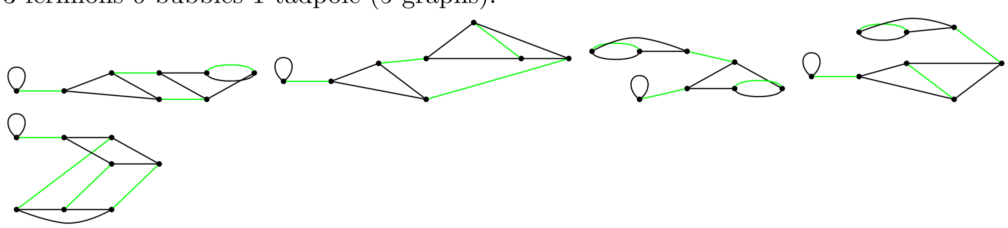
2 fermions 0 bubbles 1 tadpole (11 graphs):



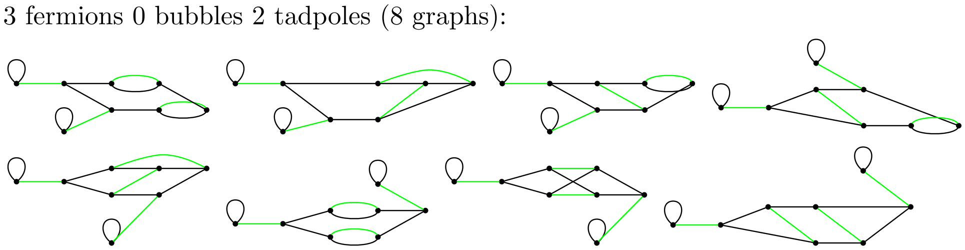
2 fermions 1 bubble 0 tadpoles (8 graphs):



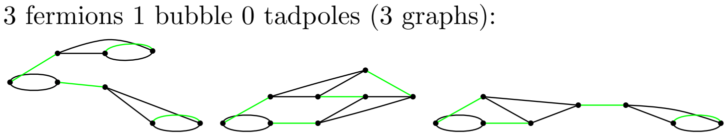
3 fermions 0 bubbles 1 tadpole (5 graphs):



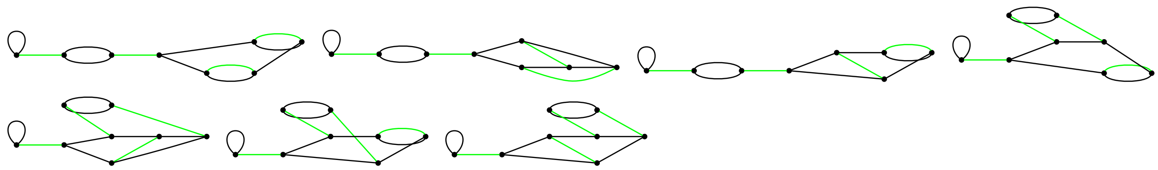
3 fermions 0 bubbles 2 tadpoles (8 graphs):



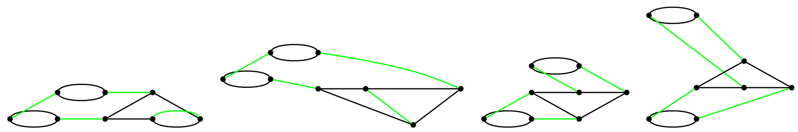
3 fermions 1 bubble 0 tadpoles (3 graphs):



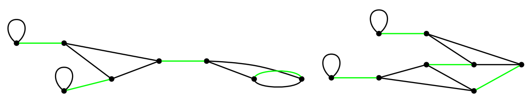
3 fermions 1 bubble 1 tadpole (7 graphs):



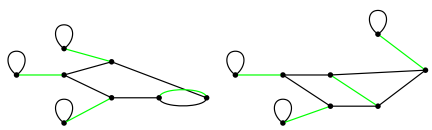
3 fermions 2 bubbles 0 tadpoles (4 graphs):



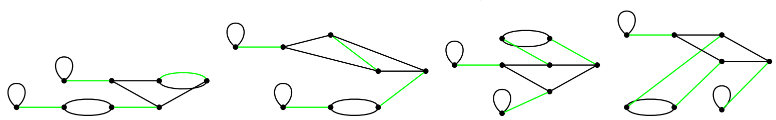
4 fermions 0 bubbles 2 tadpoles (2 graphs):



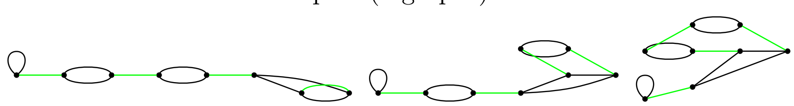
4 fermions 0 bubbles 3 tadpoles (2 graphs):



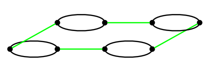
4 fermions 1 bubble 2 tadpoles (4 graphs):



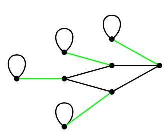
4 fermions 2 bubbles 1 tadpole (3 graphs):



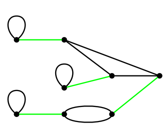
4 fermions 4 bubbles 0 tadpoles (1 graph):



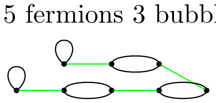
5 fermions 0 bubbles 4 tadpoles (1 graph):



5 fermions 1 bubble 3 tadpoles (1 graph):



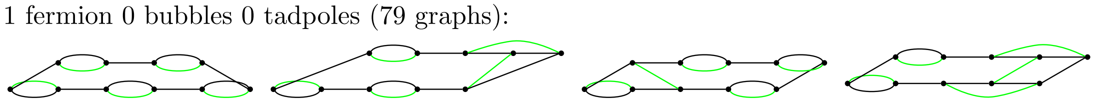
5 fermions 3 bubbles 2 tadpoles (1 graph):

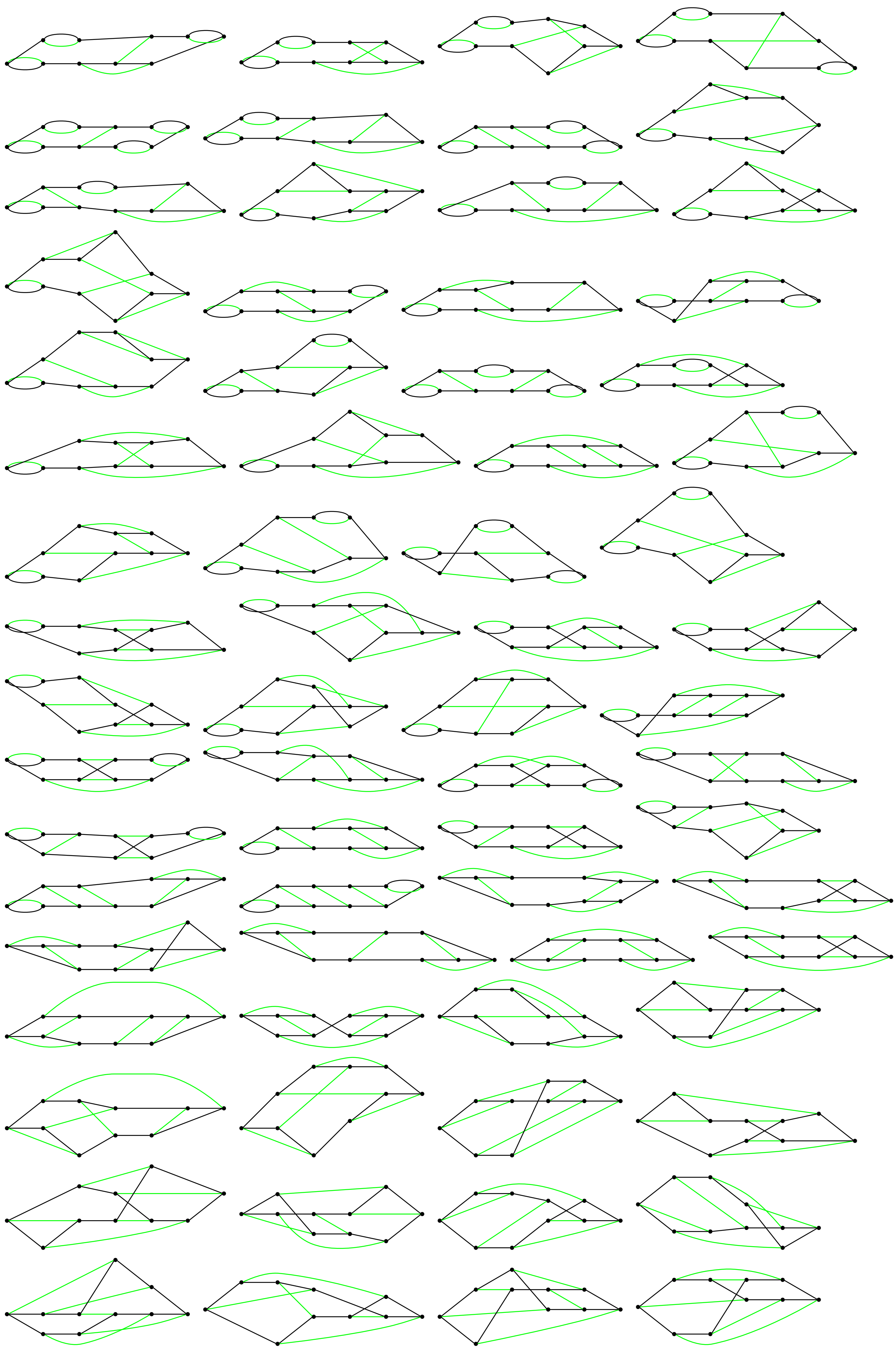


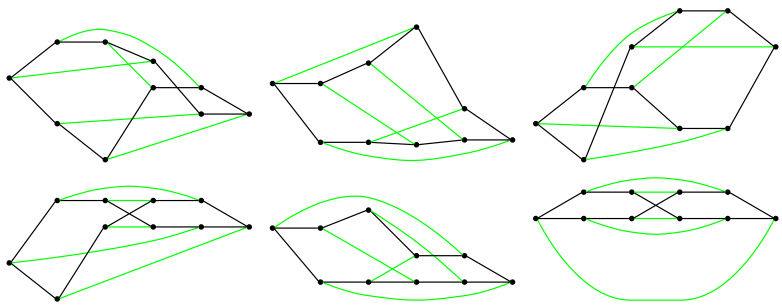
Total: 88 graphs on 8 vertices, 43 without tadpoles, 27 with neither bubbles nor tadpoles.

E. 10 vertices

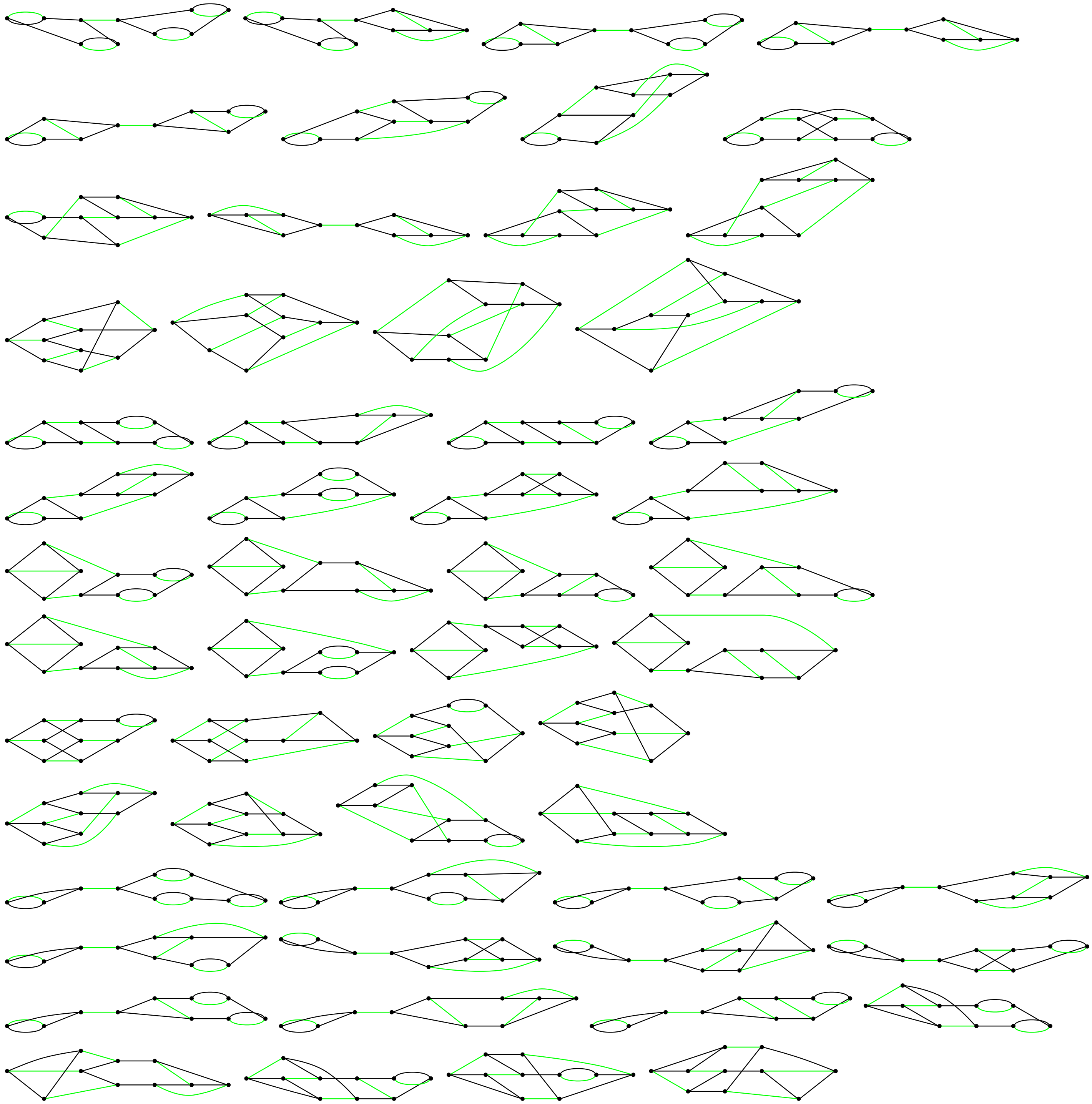
1 fermion 0 bubbles 0 tadpoles (79 graphs):

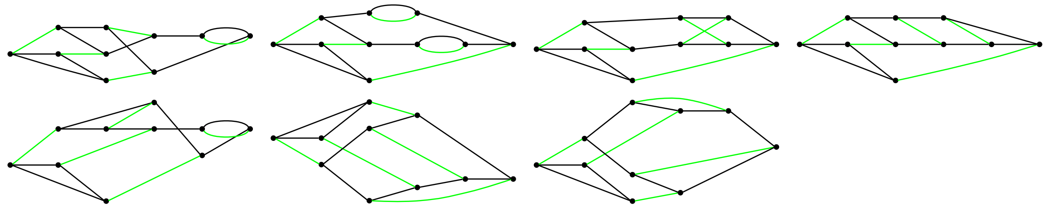




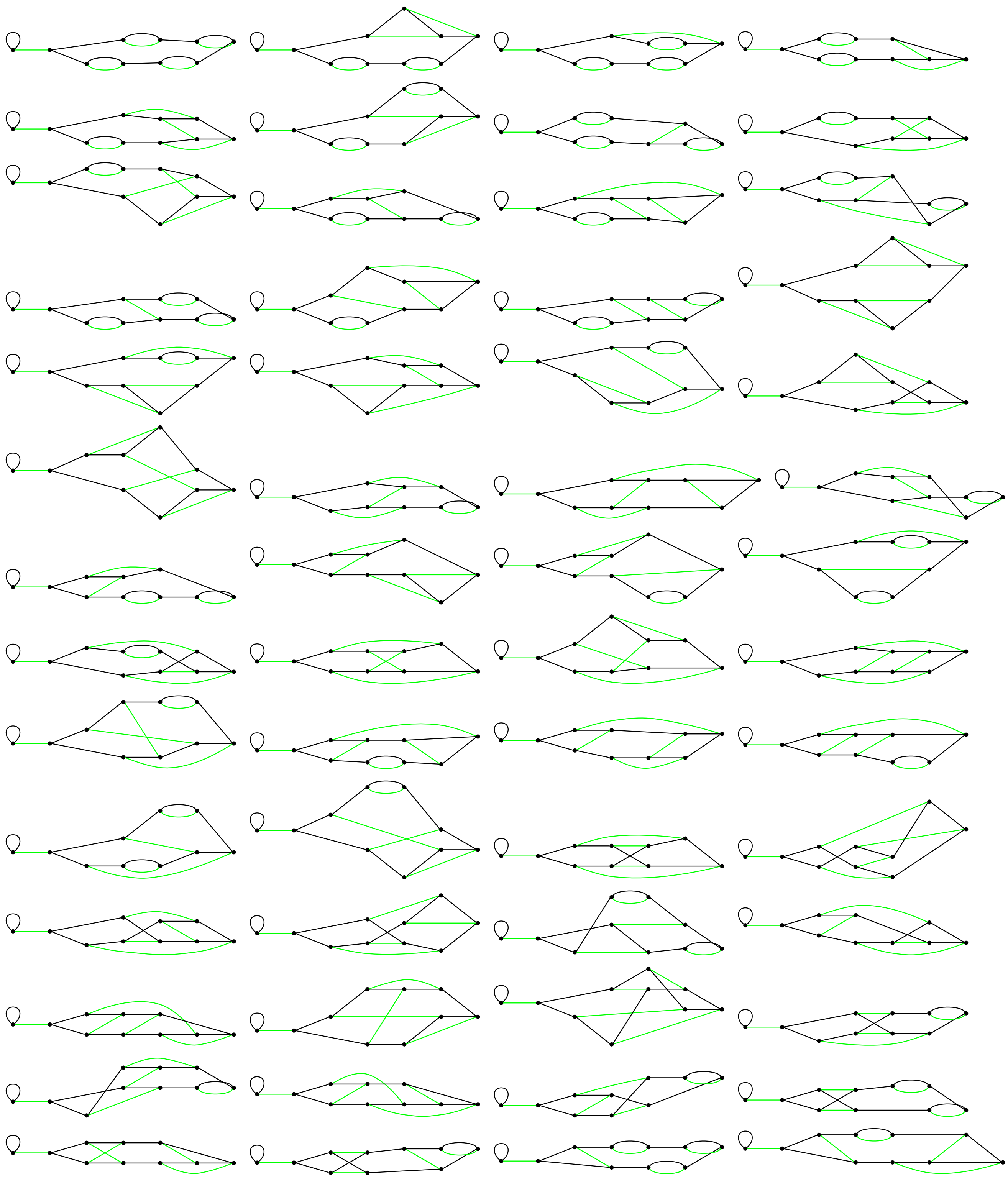


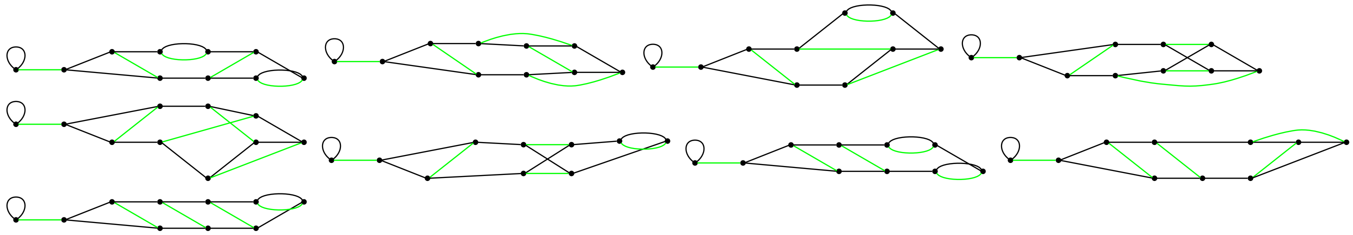
2 fermions 0 bubbles 0 tadpoles (63 graphs):



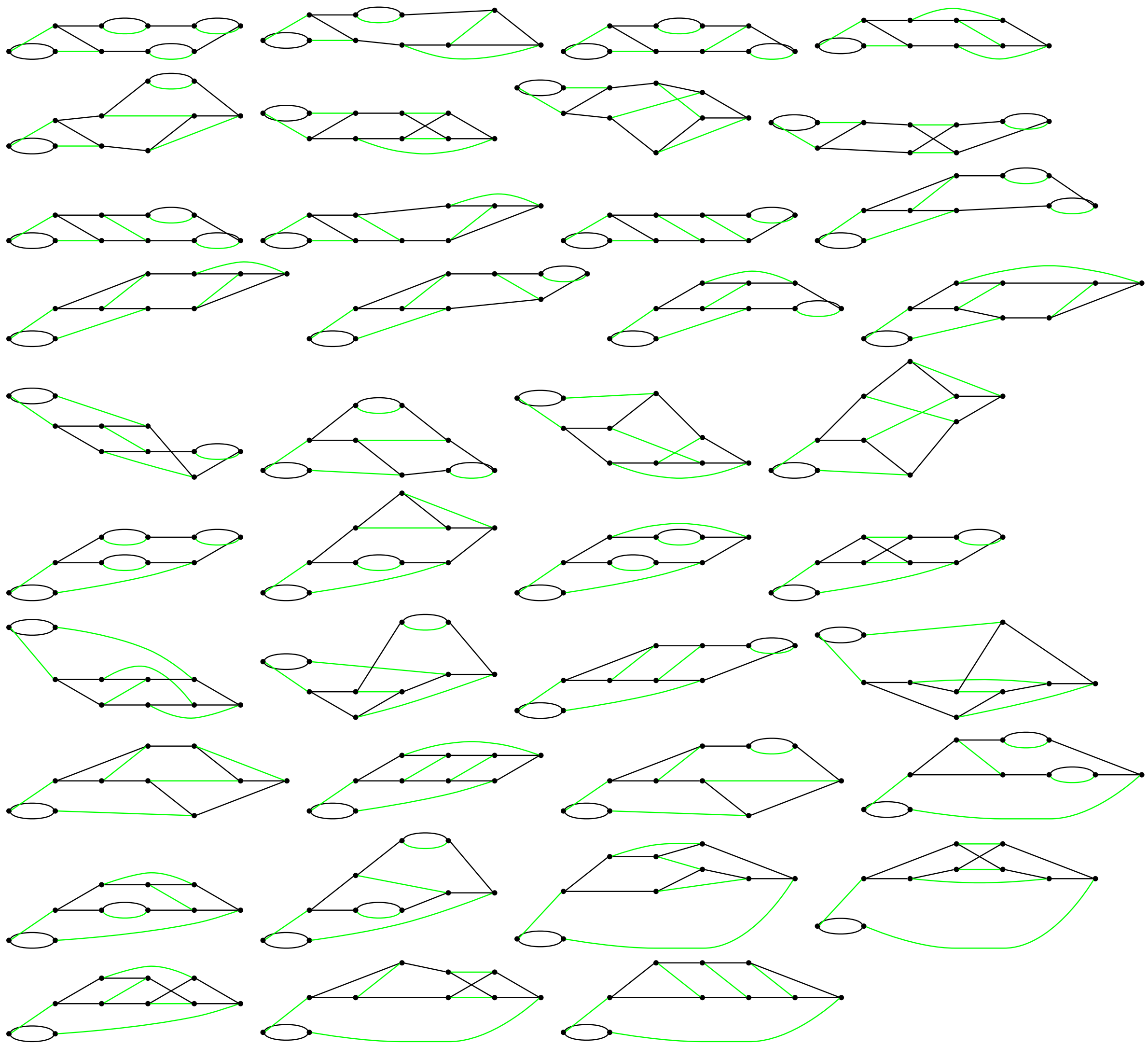


2 fermions 0 bubbles 1 tadpole (65 graphs):

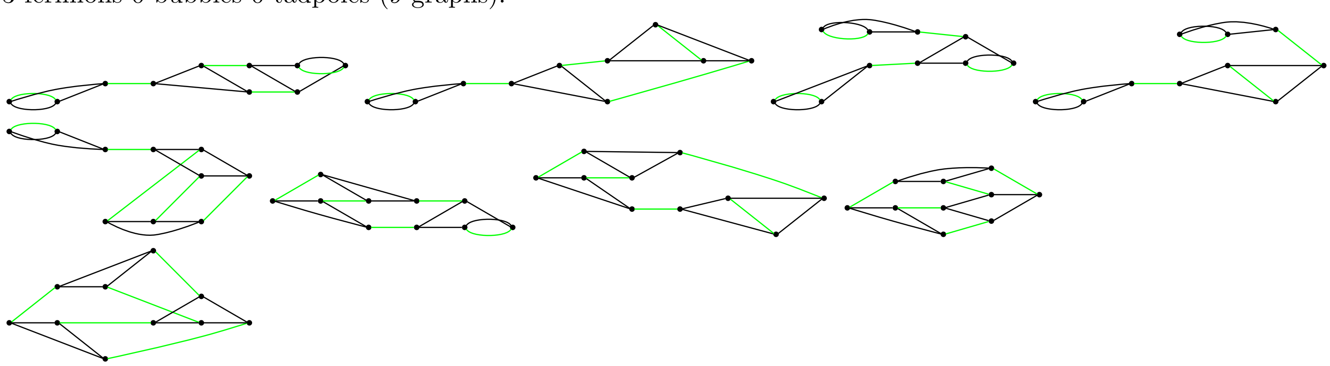




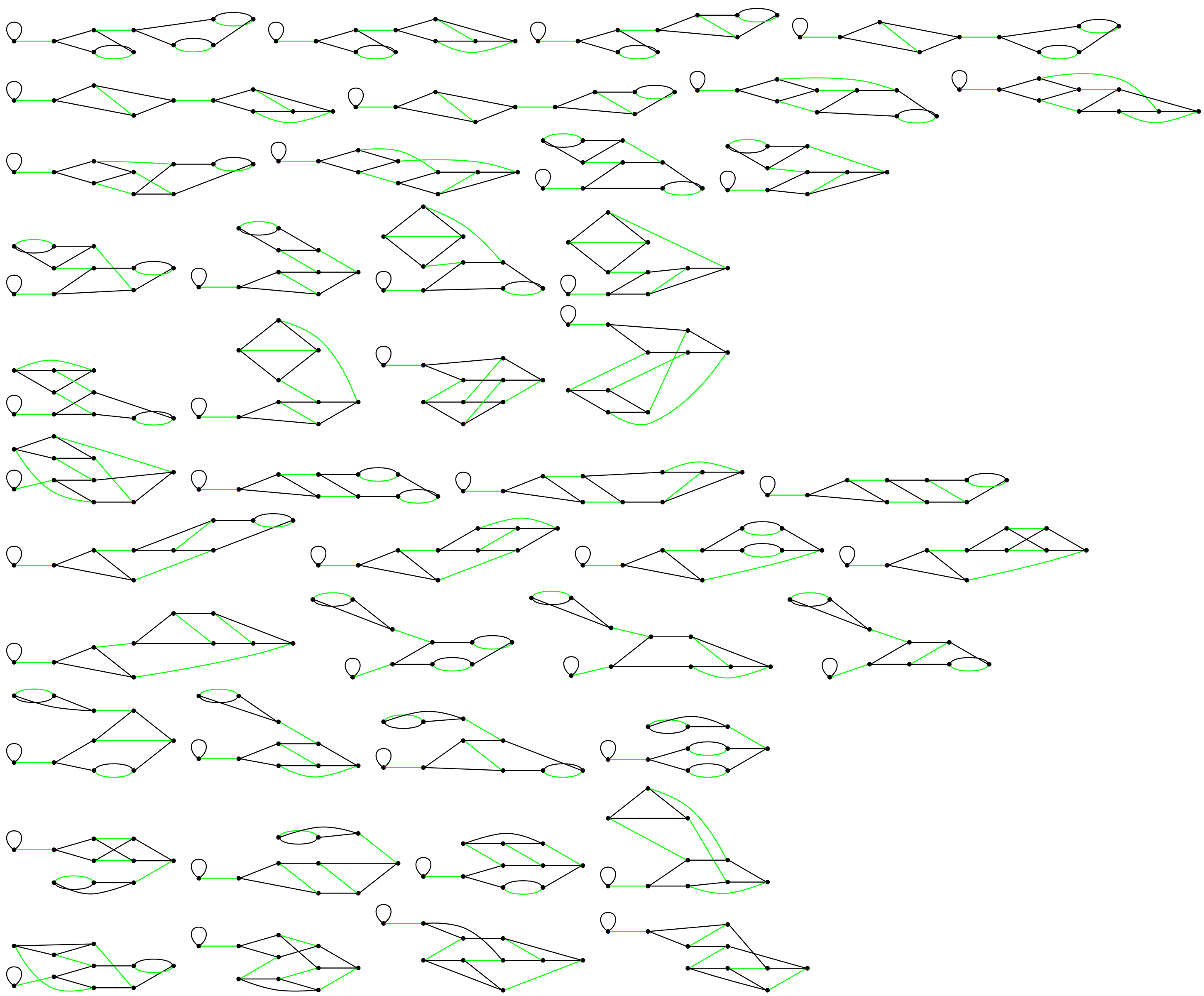
2 fermions 1 bubble 0 tadpoles (39 graphs):



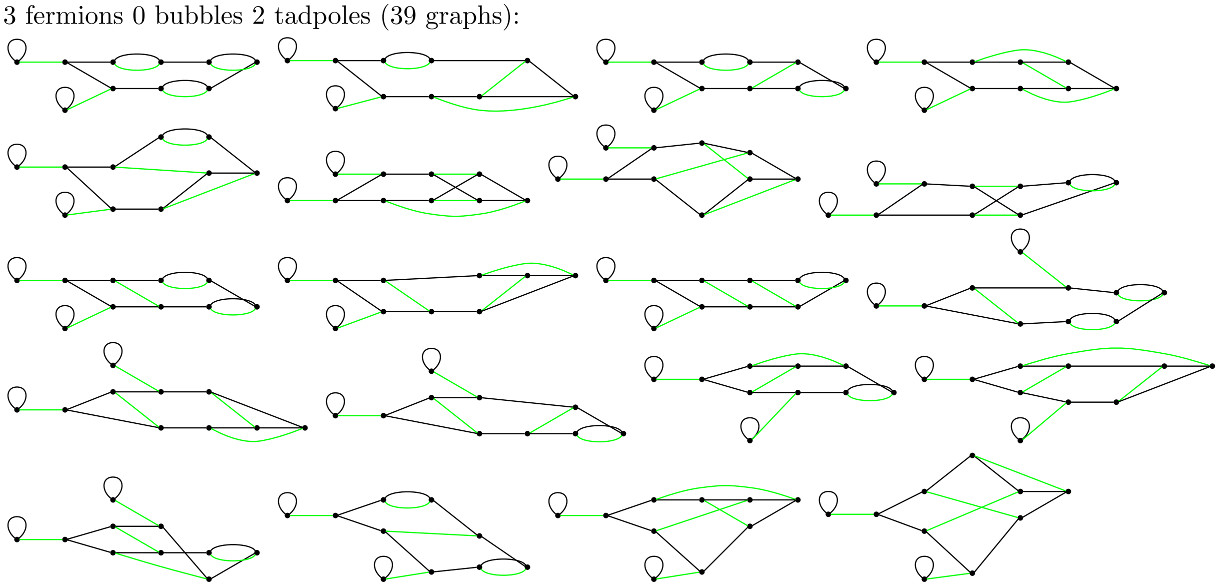
3 fermions 0 bubbles 0 tadpoles (9 graphs):

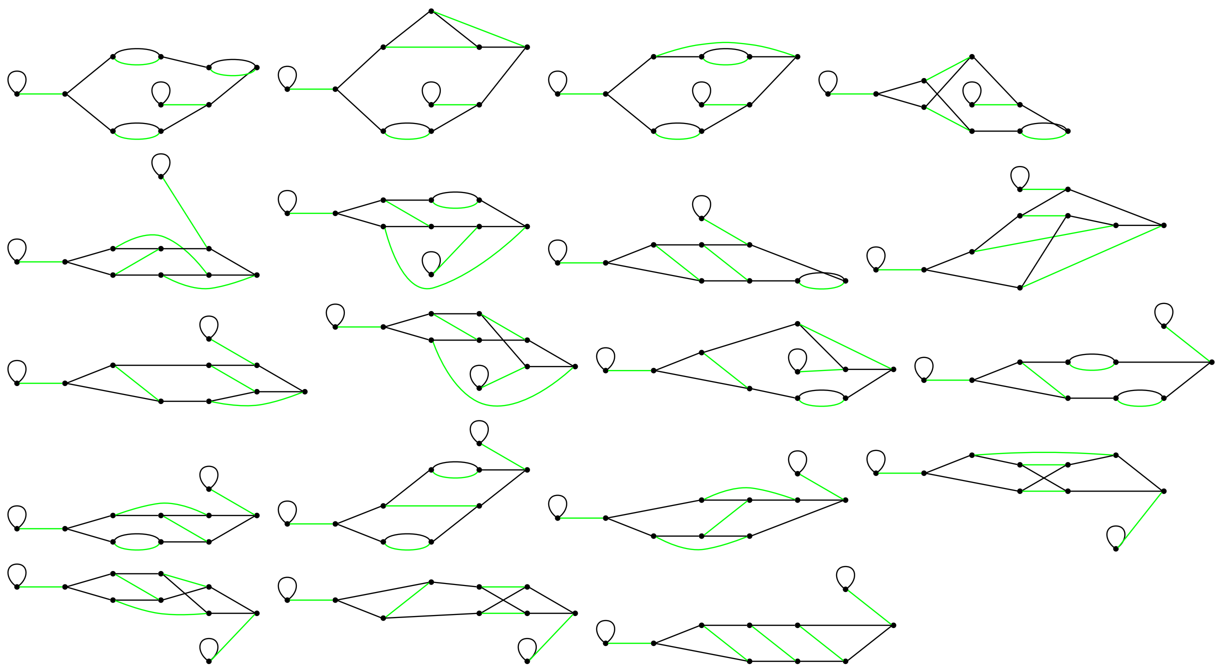


3 fermions 0 bubbles 1 tadpole (44 graphs):

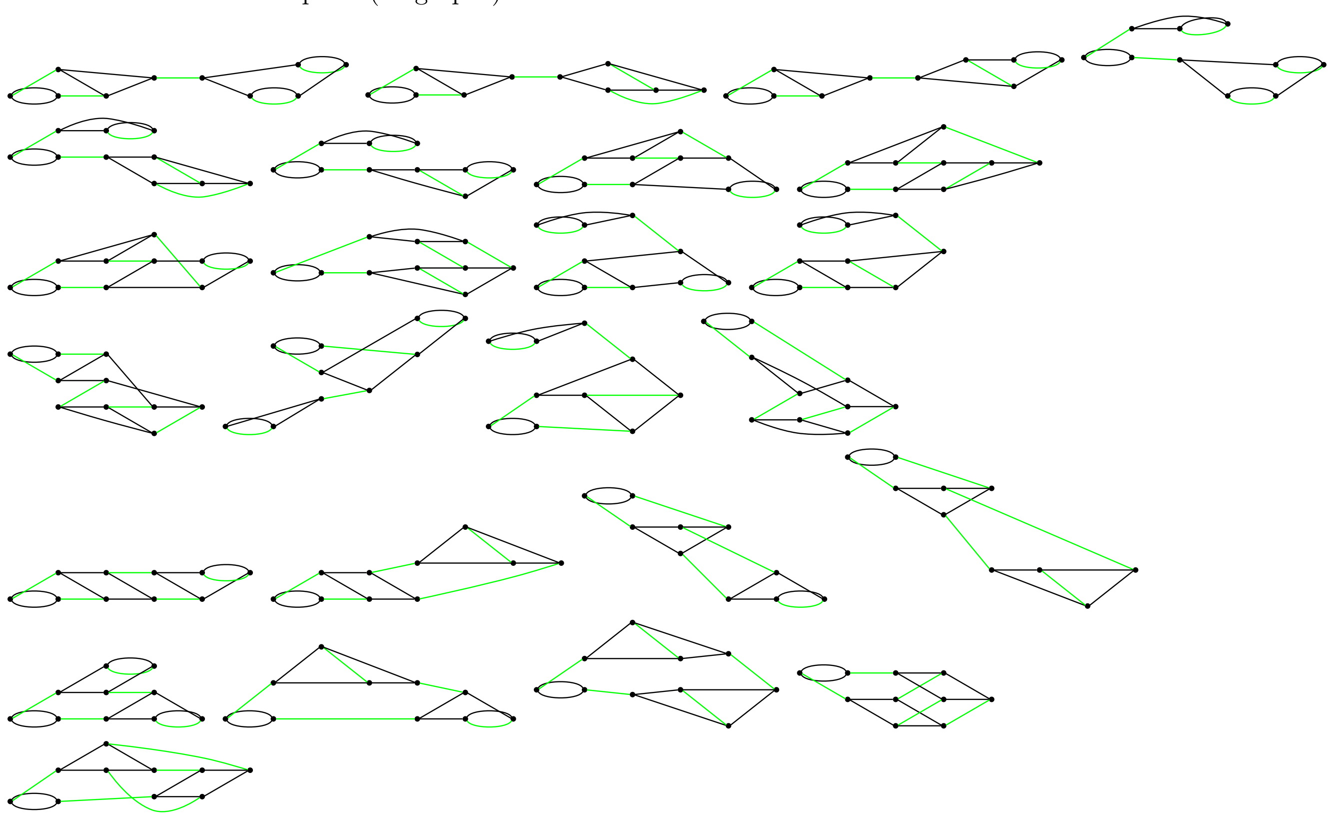


3 fermions 0 bubbles 2 tadpoles (39 graphs):

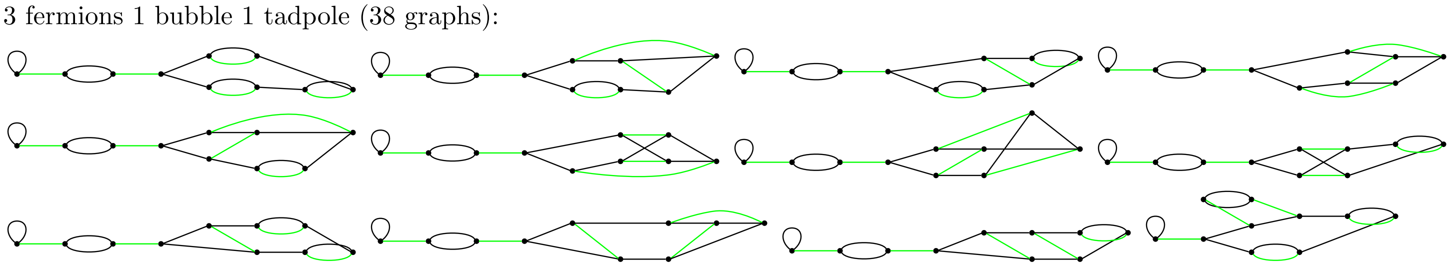


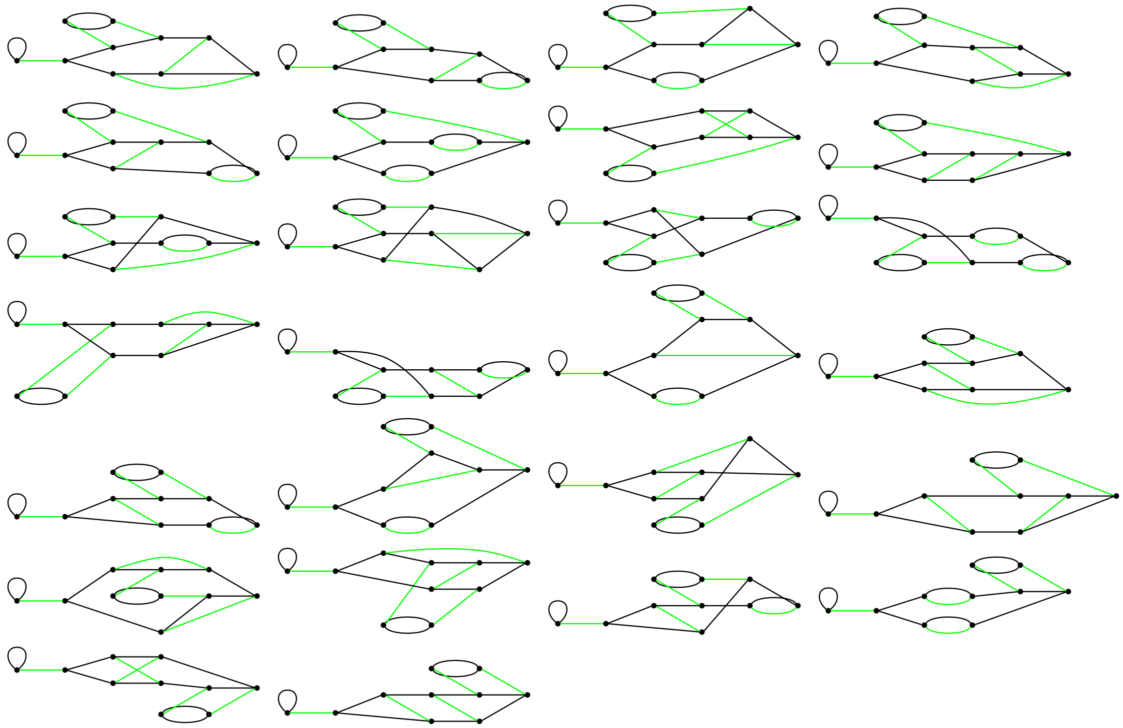


3 fermions 1 bubble 0 tadpoles (25 graphs):

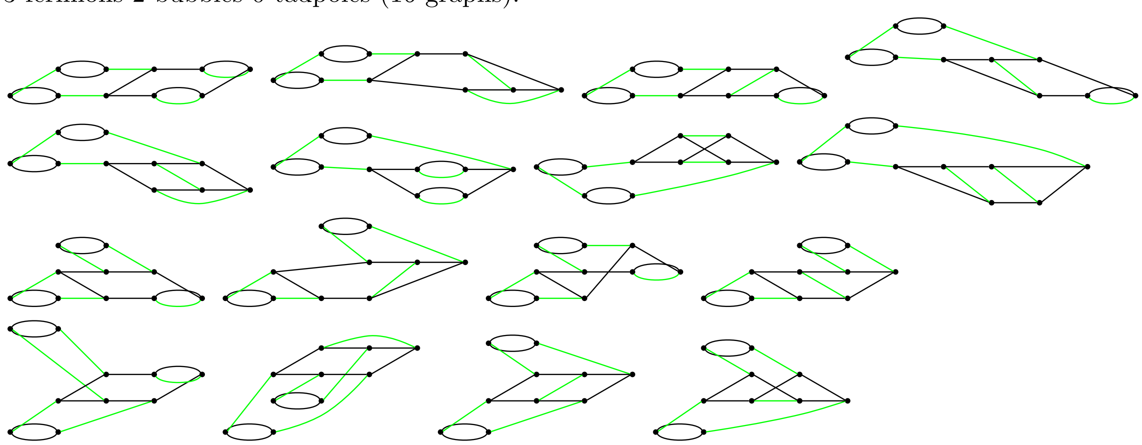


3 fermions 1 bubble 1 tadpole (38 graphs):

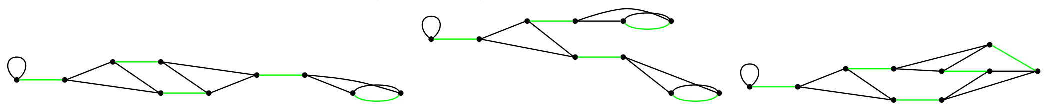




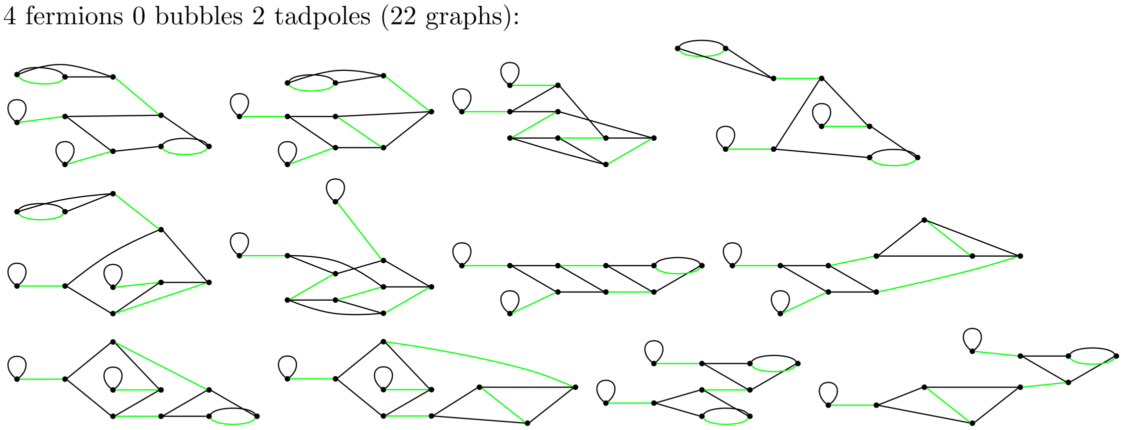
3 fermions 2 bubbles 0 tadpoles (16 graphs):

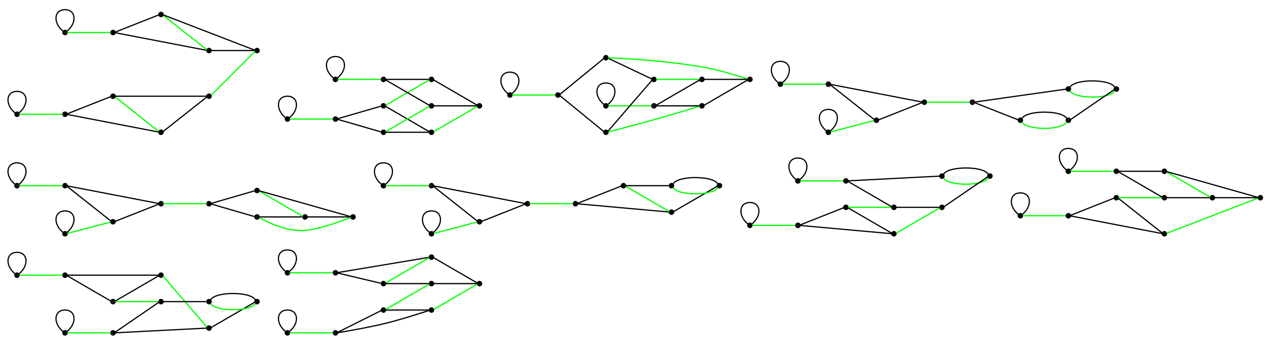


4 fermions 0 bubbles 1 tadpole (3 graphs):

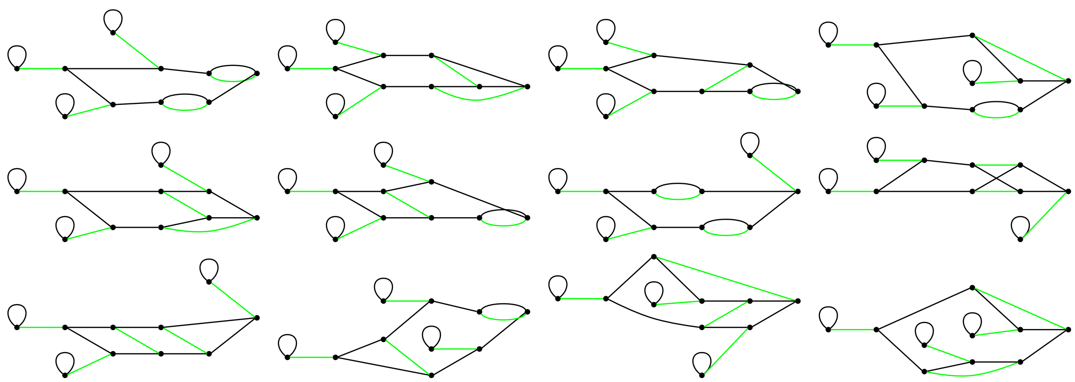


4 fermions 0 bubbles 2 tadpoles (22 graphs):

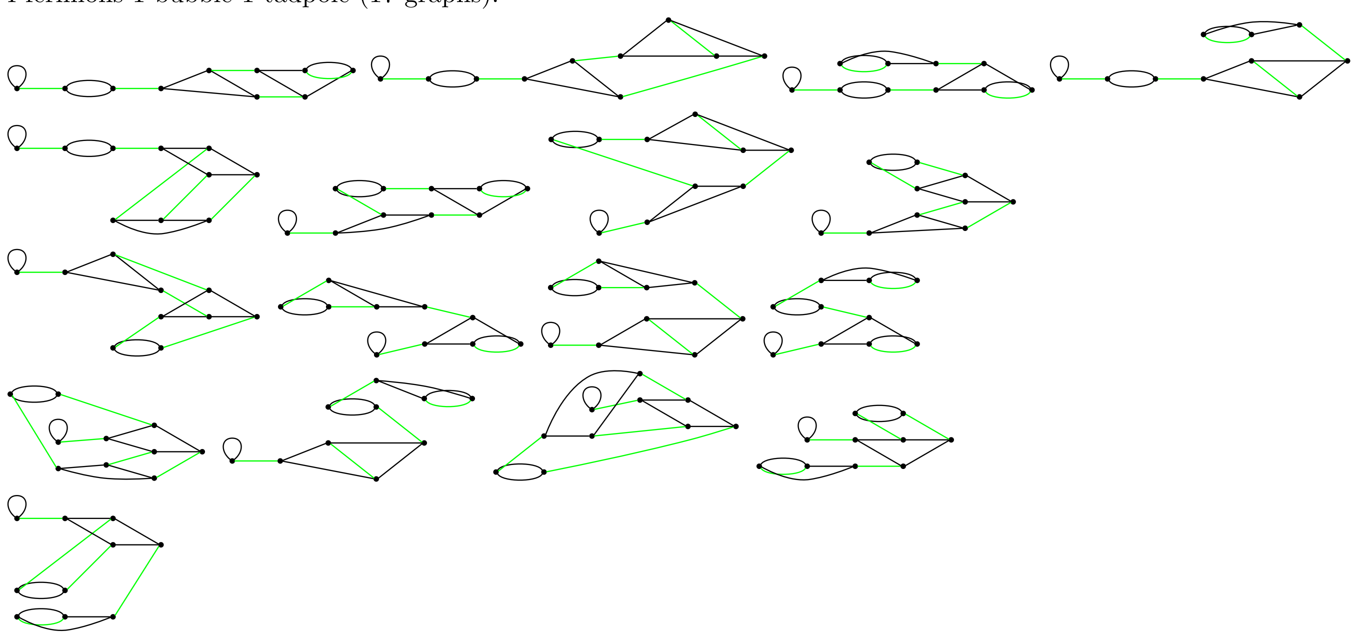




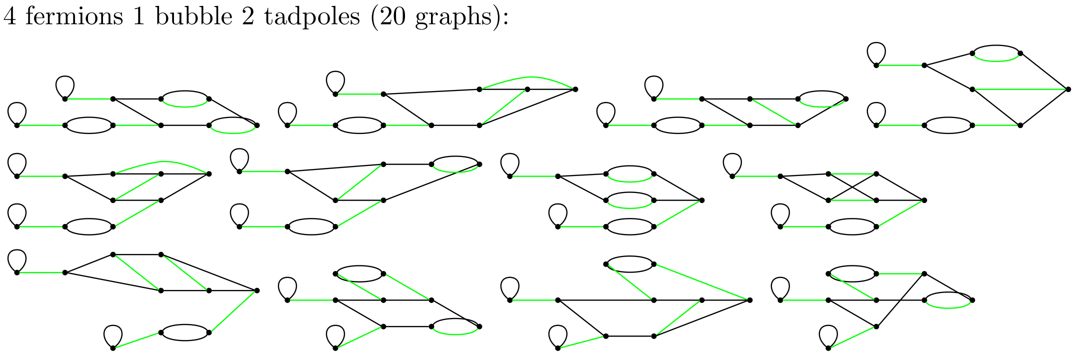
4 fermions 0 bubbles 3 tadpoles (12 graphs):

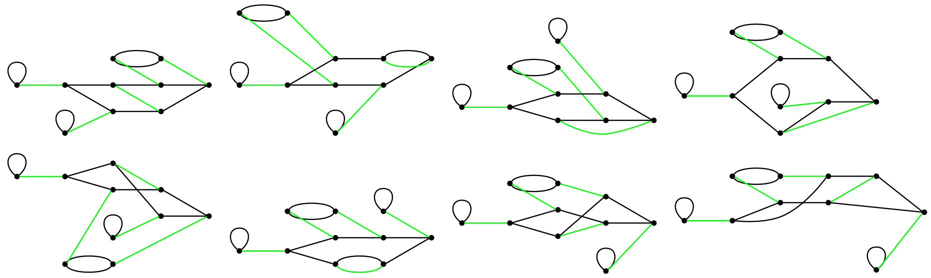


4 fermions 1 bubble 1 tadpole (17 graphs):

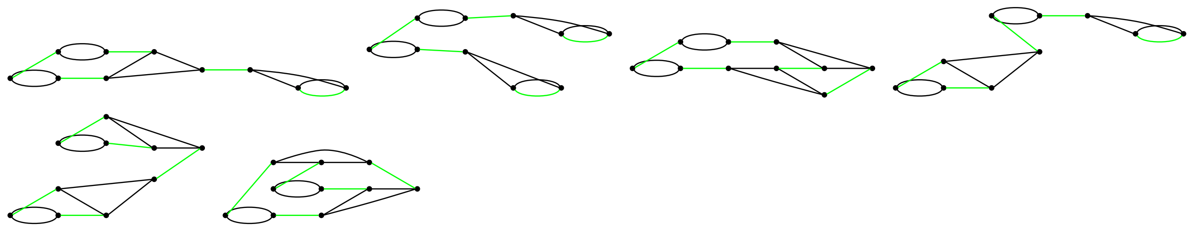


4 fermions 1 bubble 2 tadpoles (20 graphs):

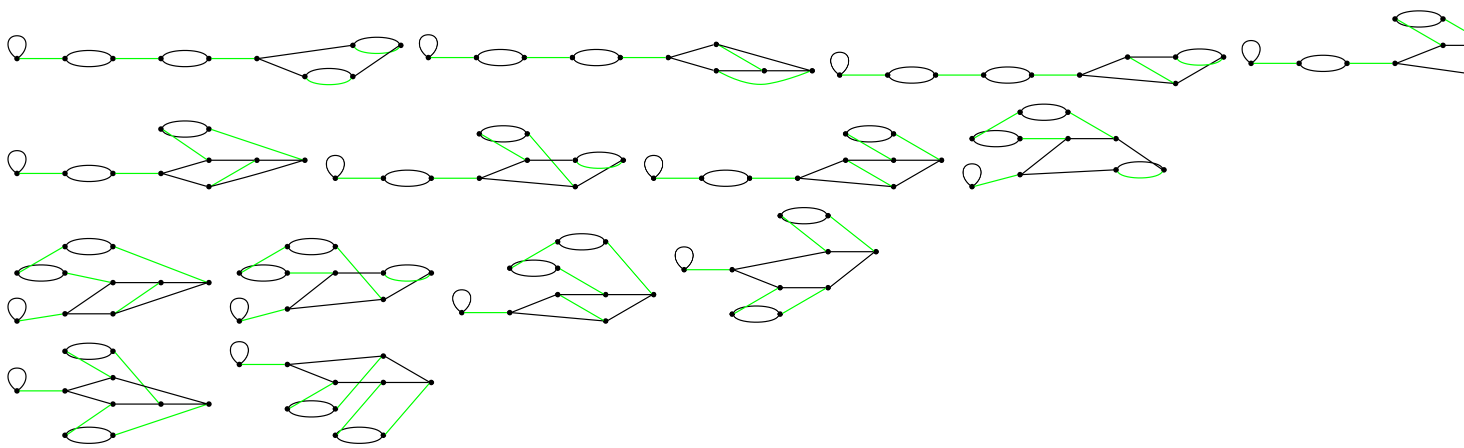




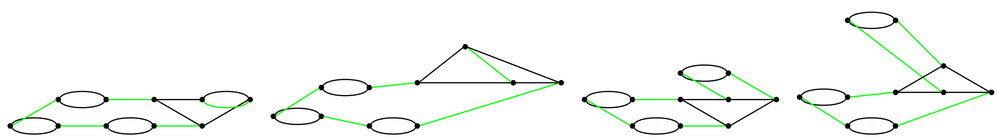
4 fermions 2 bubbles 0 tadpoles (6 graphs):



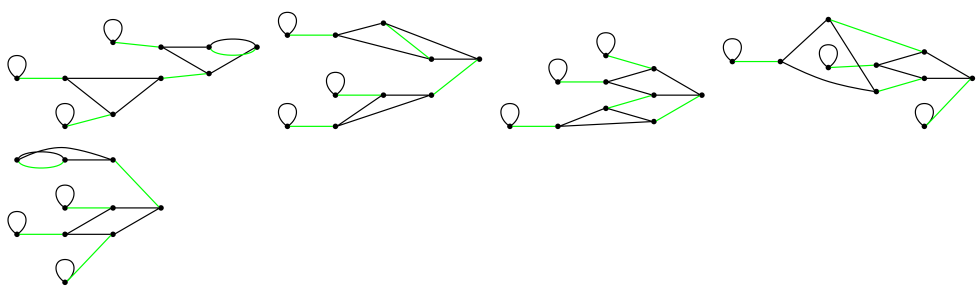
4 fermions 2 bubbles 1 tadpole (14 graphs):



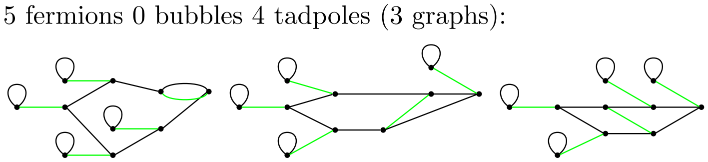
4 fermions 3 bubbles 0 tadpoles (4 graphs):



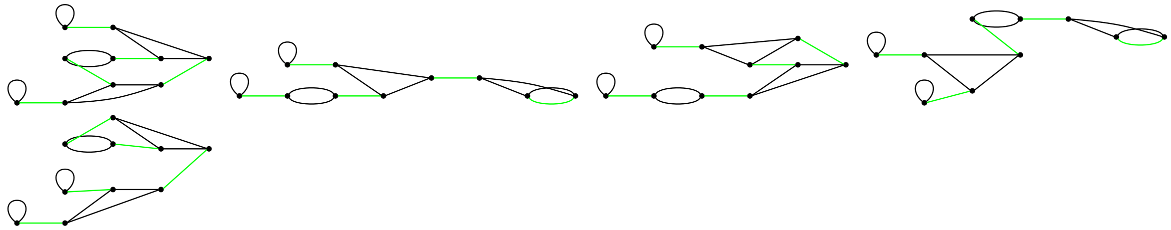
5 fermions 0 bubbles 3 tadpoles (5 graphs):



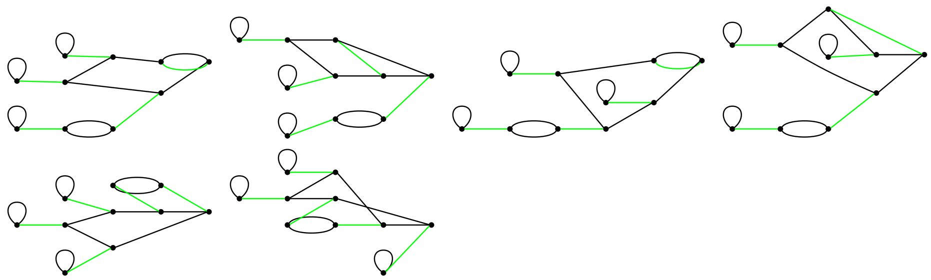
5 fermions 0 bubbles 4 tadpoles (3 graphs):



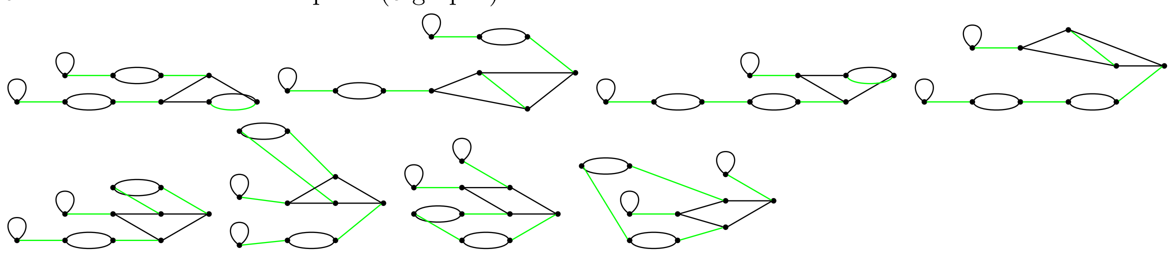
5 fermions 1 bubble 2 tadpoles (5 graphs):



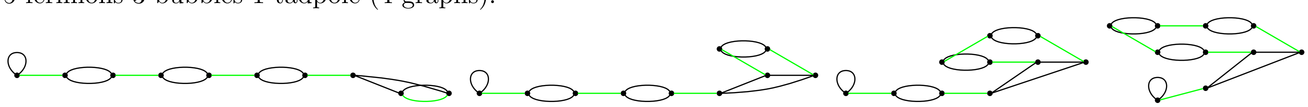
5 fermions 1 bubble 3 tadpoles (6 graphs):



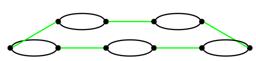
5 fermions 2 bubbles 2 tadpoles (8 graphs):



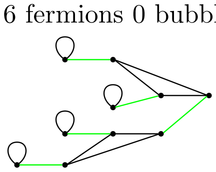
5 fermions 3 bubbles 1 tadpole (4 graphs):



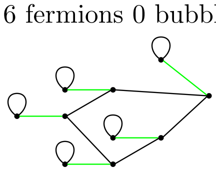
5 fermions 5 bubbles 0 tadpoles (1 graph):



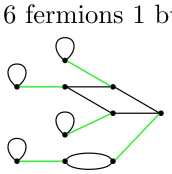
6 fermions 0 bubbles 4 tadpoles (1 graph):



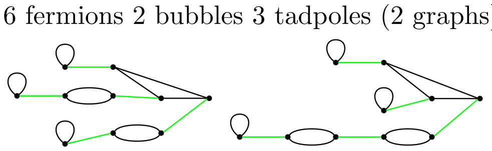
6 fermions 0 bubbles 5 tadpoles (1 graph):



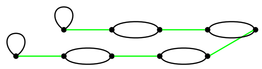
6 fermions 1 bubble 4 tadpoles (1 graph):



6 fermions 2 bubbles 3 tadpoles (2 graphs):



6 fermions 4 bubbles 2 tadpoles (1 graph):



Total: 553 graphs on 10 vertices, 242 without tadpoles, 151 with neither bubbles nor tadpoles.

III. EFFECTIVE INTERACTION (UNDIRECTED FERMIONS)

The diagrams of Section II are less important in the relevant perturbation theory because vacuum diagrams effectively cancel out for the calculation of the Green's function [12, 13]. But they may be used as precursor diagrams to generate the Green's function diagrams of Dyson's equations and the diagrams of the effective interaction of dielectric function by cutting either a fermion line or an interaction line, and either regarding the cut point as a new vertex of degree 1 or removing that line (and considering the subgraph as a *fairly* cubic graph with two vertices of degree 2 [14, 15]).

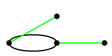
We generate 1, 1, 4, 23, 169, 1613, . . . diagrams for non-directed fermions on 0, 2, 4, 6,. . . internal vertices by cutting through interaction lines of the vacuum polarization. If the interaction line was a *bridge*, the cut produces a disconnected diagram, which is not retained. Depending on the symmetry of the vacuum diagram, cuts may produce as many effective interaction diagrams as there are interaction lines in the vacuum diagram; only the set of non-isomorphic diagrams is retained.

The plots below show the diagrams up to 8 internal vertices; the 1613 diagrams with 10 internal vertices are not shown to reduce the manuscript size. Machine-readable edge-lists and Cycle Indices are available on request for all graphs of Sections II–VII up to 10 (internal) vertices.

If we consider only the subset of diagrams with a single fermion, we obtain 1, 2, 8, 39, 287, . . . , the number of chord diagrams with a marked chord [1, A322176]. The 8 graphs with 6 internal vertices and one fermion have been plotted by Kinoshita and Nio [16, Fig 4.].

A. 2 internal vertices

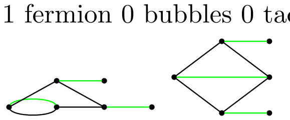
1 fermion 1 bubble 0 tadpoles (1 graph):



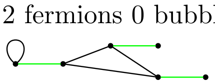
Total: 1 graph on 2 vertices, 1 without tadpoles, 0 with neither bubbles nor tadpoles.

B. 4 internal vertices

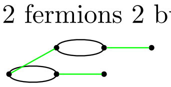
1 fermion 0 bubbles 0 tadpoles (2 graphs):



2 fermions 0 bubbles 1 tadpole (1 graph):



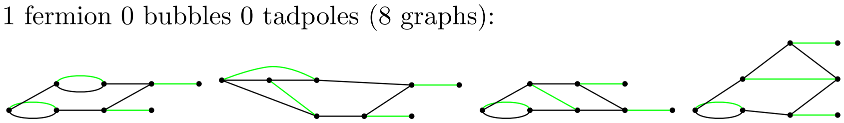
2 fermions 2 bubbles 0 tadpoles (1 graph):

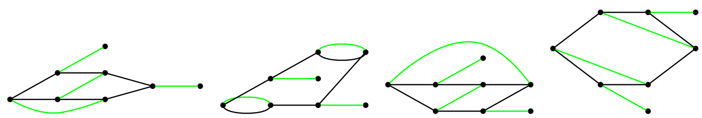


Total: 4 graphs on 4 vertices, 3 without tadpoles, 2 with neither bubbles nor tadpoles.

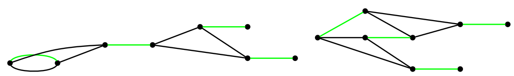
C. 6 internal vertices

1 fermion 0 bubbles 0 tadpoles (8 graphs):

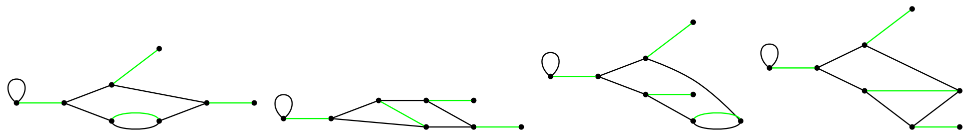




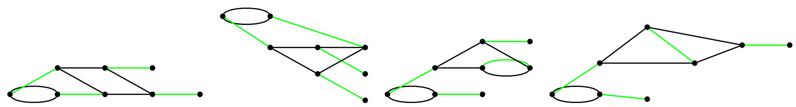
2 fermions 0 bubbles 0 tadpoles (2 graphs):



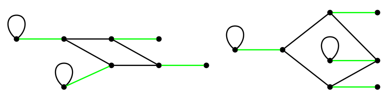
2 fermions 0 bubbles 1 tadpole (4 graphs):



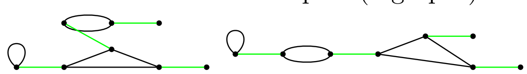
2 fermions 1 bubble 0 tadpoles (4 graphs):



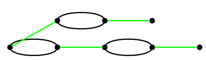
3 fermions 0 bubbles 2 tadpoles (2 graphs):



3 fermions 1 bubble 1 tadpole (2 graphs):



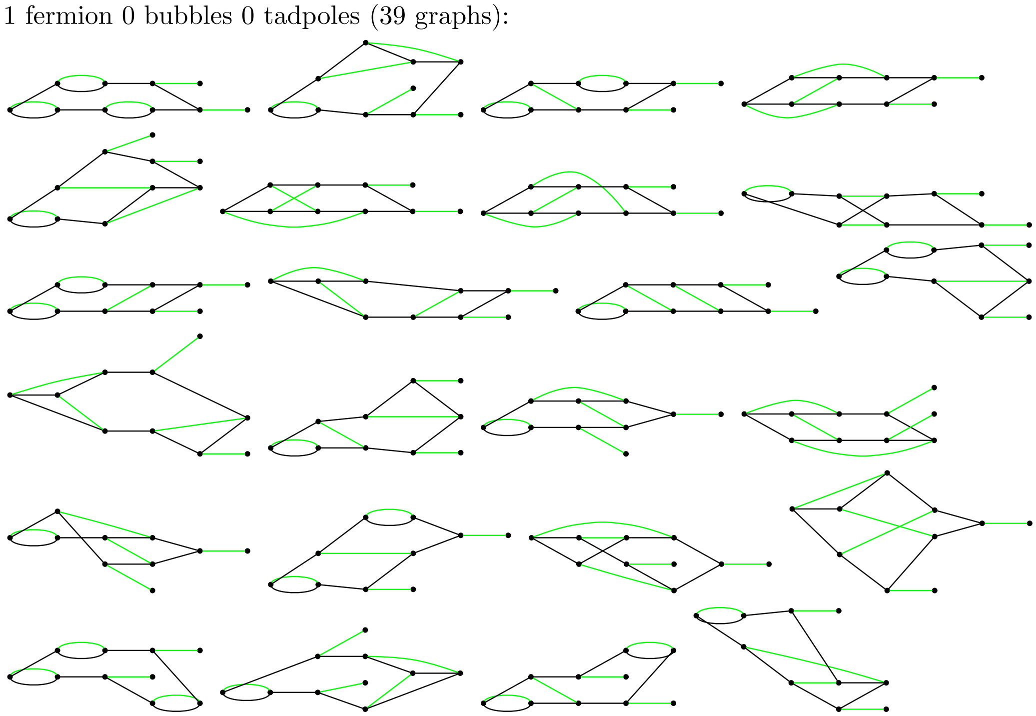
3 fermions 3 bubbles 0 tadpoles (1 graph):

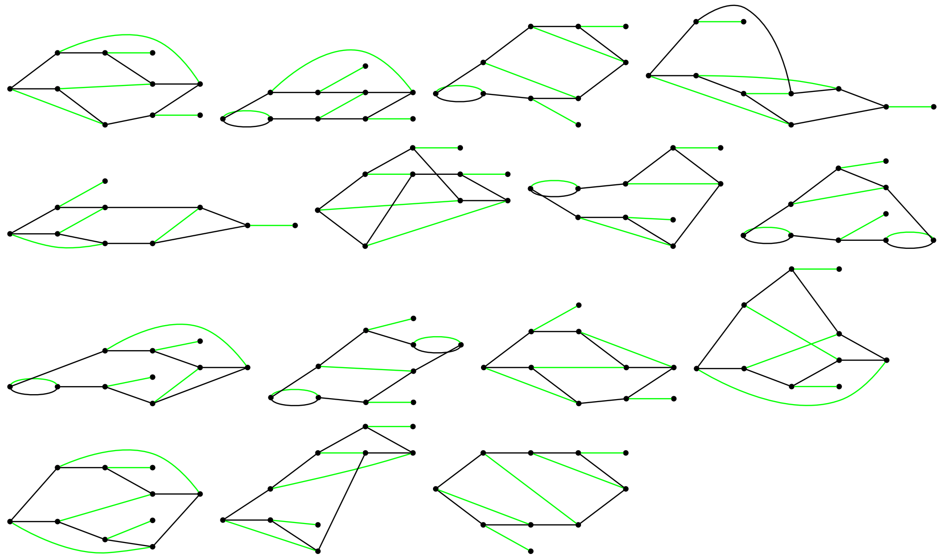


Total: 23 graphs on 6 vertices, 15 without tadpoles, 10 with neither bubbles nor tadpoles.

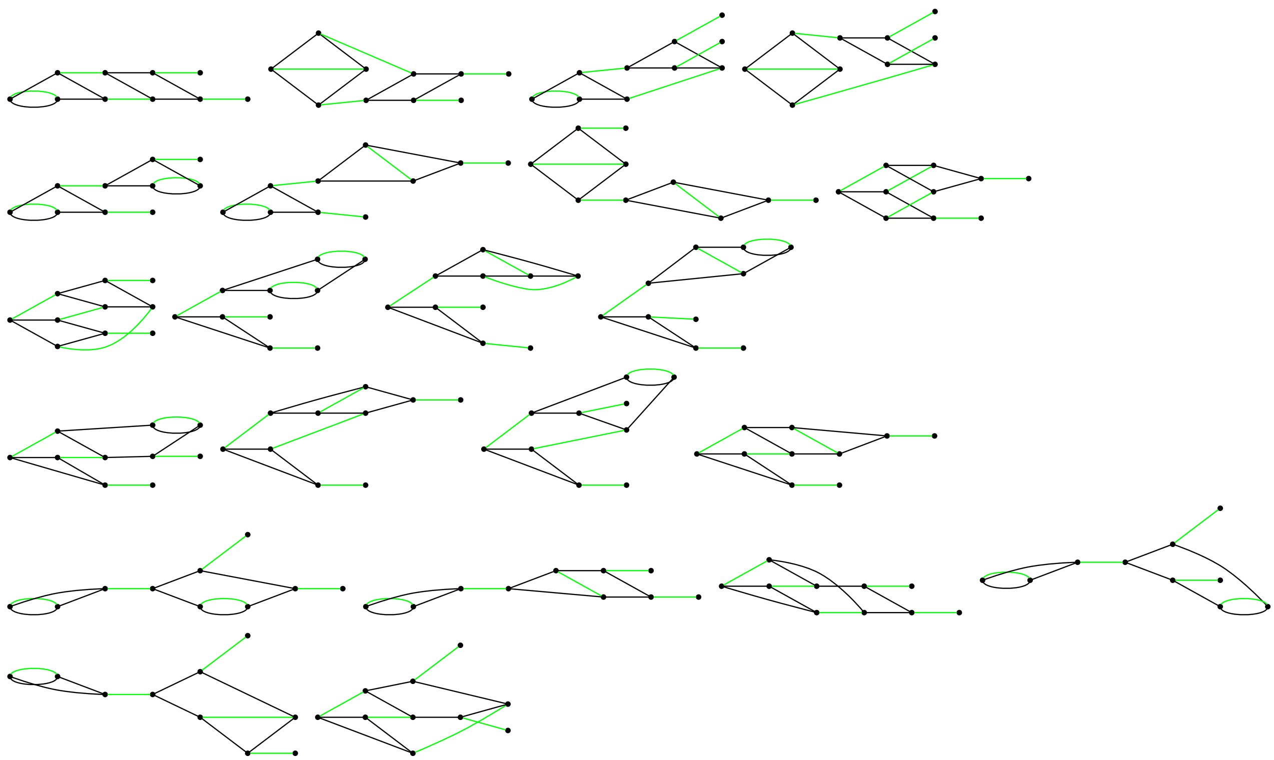
D. 8 internal vertices

1 fermion 0 bubbles 0 tadpoles (39 graphs):

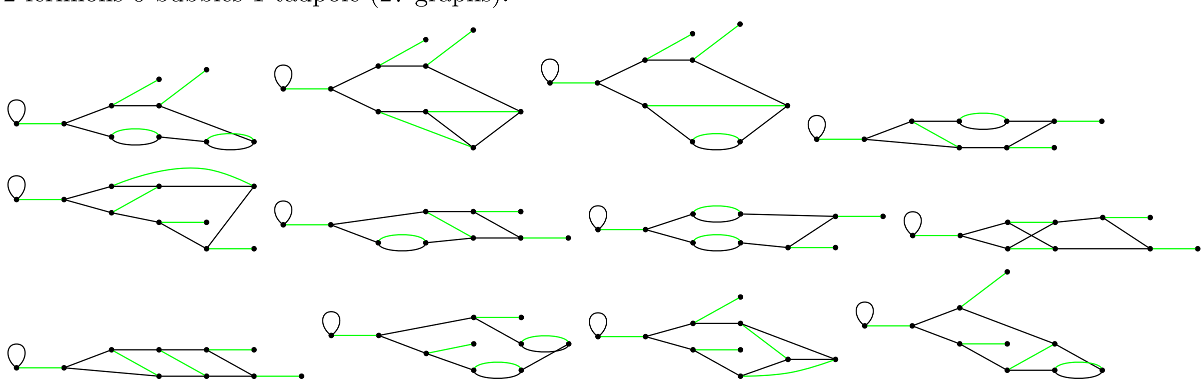


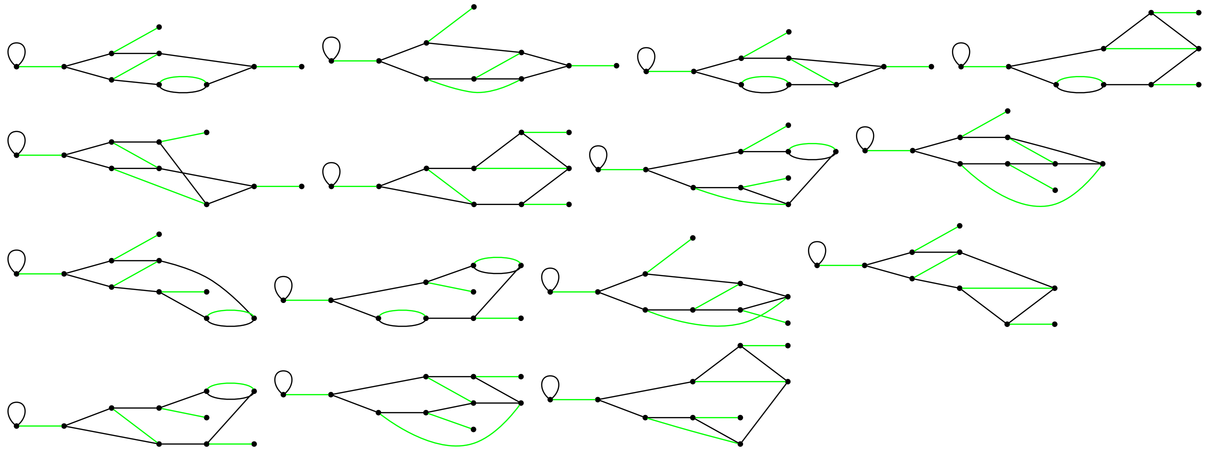


2 fermions 0 bubbles 0 tadpoles (22 graphs):

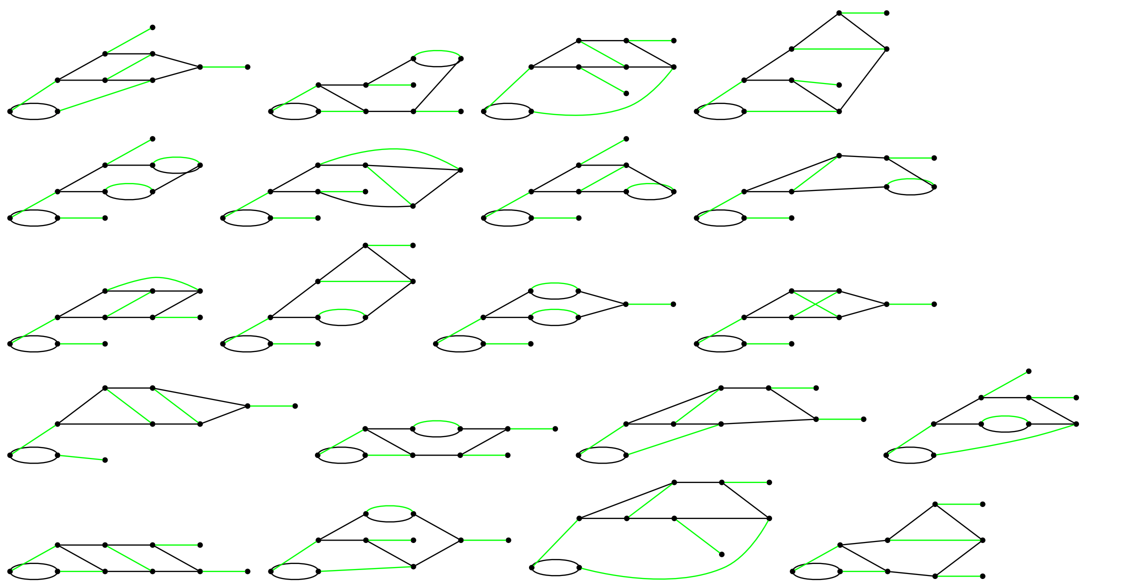


2 fermions 0 bubbles 1 tadpole (27 graphs):

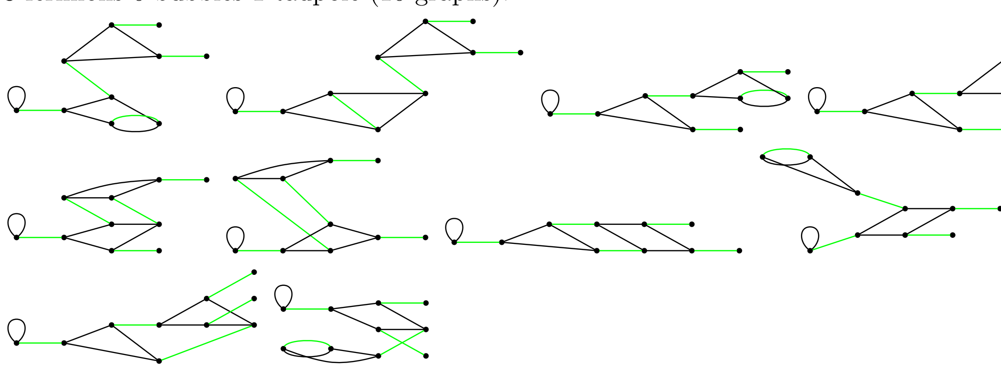




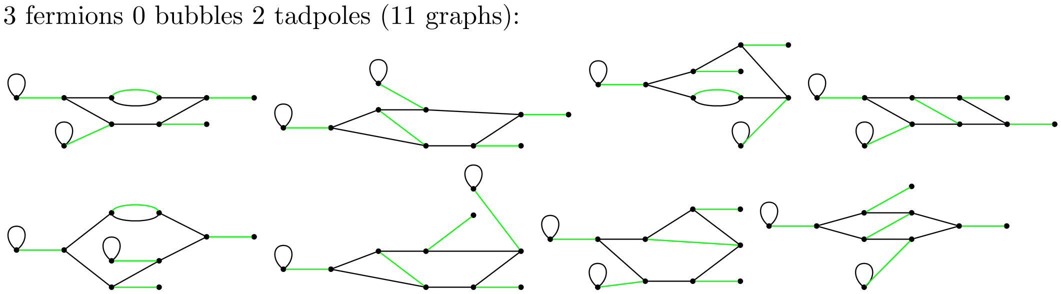
2 fermions 1 bubble 0 tadpoles (20 graphs):

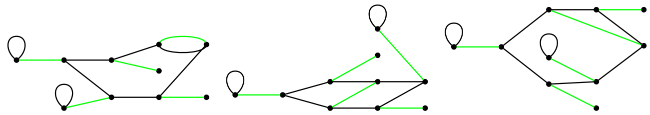


3 fermions 0 bubbles 1 tadpole (10 graphs):

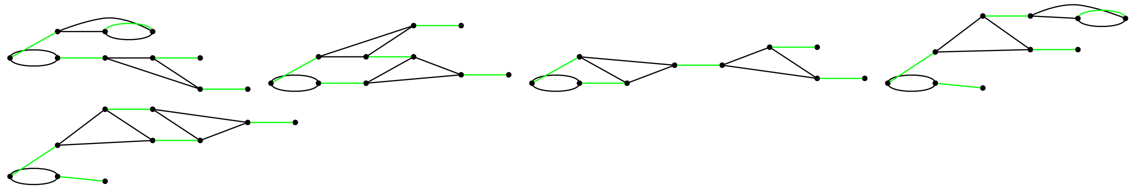


3 fermions 0 bubbles 2 tadpoles (11 graphs):

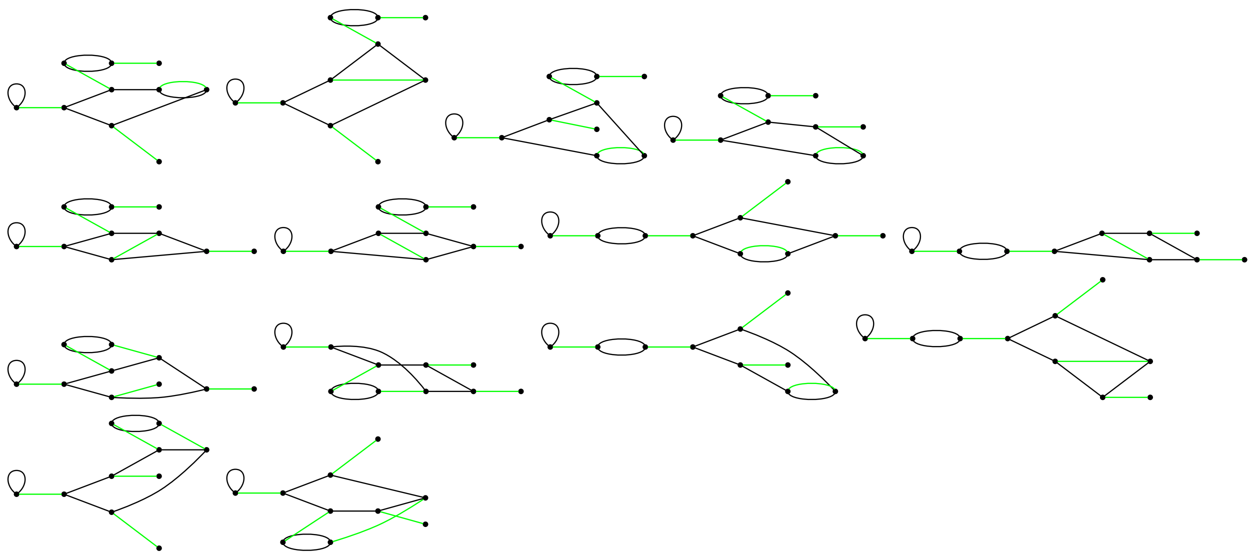




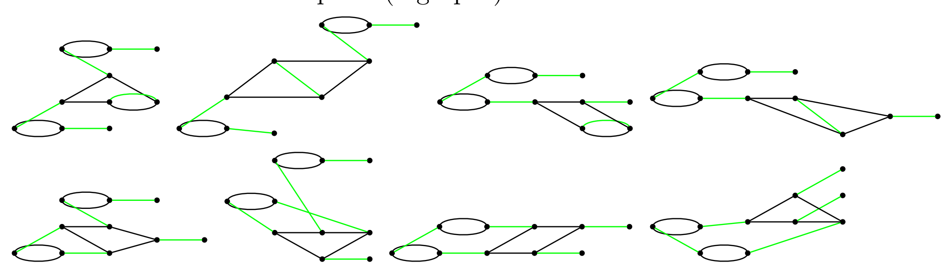
3 fermions 1 bubble 0 tadpoles (5 graphs):



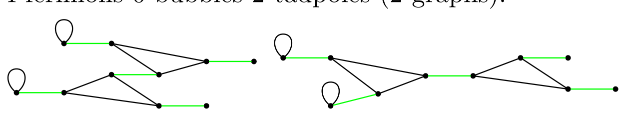
3 fermions 1 bubble 1 tadpole (14 graphs):



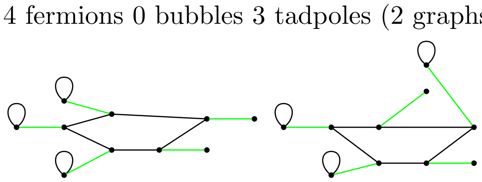
3 fermions 2 bubbles 0 tadpoles (8 graphs):



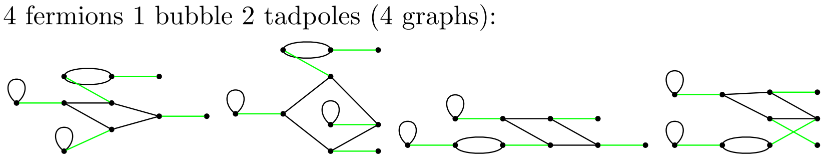
4 fermions 0 bubbles 2 tadpoles (2 graphs):



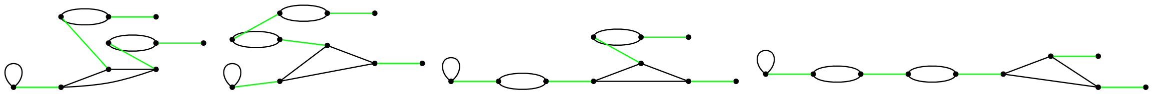
4 fermions 0 bubbles 3 tadpoles (2 graphs):



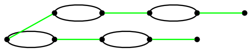
4 fermions 1 bubble 2 tadpoles (4 graphs):



4 fermions 2 bubbles 1 tadpole (4 graphs):



4 fermions 4 bubbles 0 tadpoles (1 graph):



Total: 169 graphs on 8 vertices, 95 without tadpoles, 61 with neither bubbles nor tadpoles.

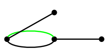
IV. GREENS FUNCTION (UNDIRECTED FERMIONS)

We generate 1, 2, 9, 51, 390, 3649, . . . non-oriented Greens Function diagrams with 0, 2, 4,. . . internal vertices by cutting a fermion line of the vacuum polarization graphs of Section II. Because the fermion lines in the vacuum diagrams were living in cycles, all diagrams remain connected by such cuts.

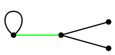
1, 1, 3, 11, 65, 513,. . . of these have one fermion. The 47 graphs (out of 65) with 8 internal vertices and one fermion (no bubbles nor tadpoles) which are not 1-connected (i.e., which cannot be decomposed in two graphs with more than one vertex by cutting a single fermion line) have been plotted by Kinoshita and Linquist [17, 18].

A. 2 internal vertices

1 fermion 0 bubbles 0 tadpoles (1 graph):



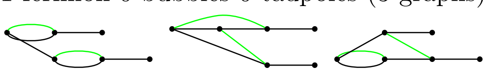
2 fermions 0 bubbles 1 tadpole (1 graph):



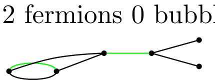
Total: 2 graphs on 2 vertices, 1 without tadpoles, 1 with neither bubbles nor tadpoles.

B. 4 internal vertices

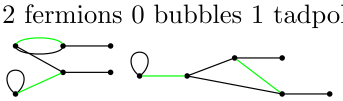
1 fermion 0 bubbles 0 tadpoles (3 graphs):



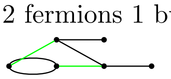
2 fermions 0 bubbles 0 tadpoles (1 graph):



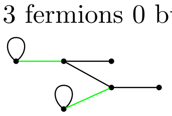
2 fermions 0 bubbles 1 tadpole (2 graphs):



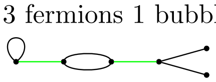
2 fermions 1 bubble 0 tadpoles (1 graph):



3 fermions 0 bubbles 2 tadpoles (1 graph):



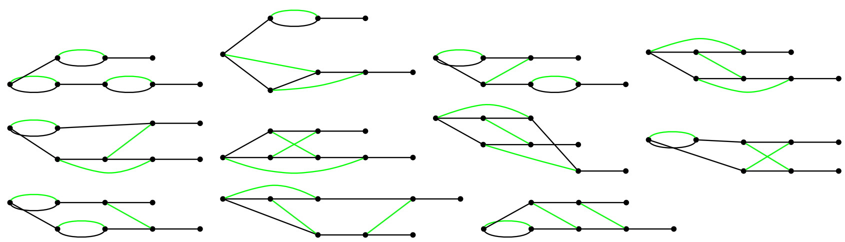
3 fermions 1 bubble 1 tadpole (1 graph):



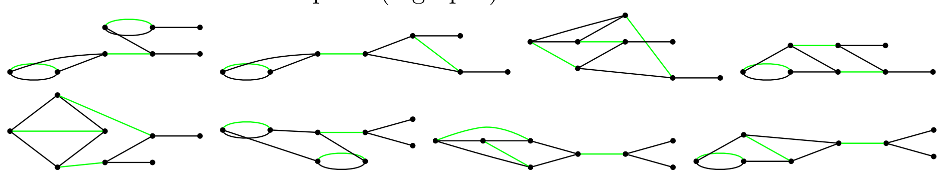
Total: 9 graphs on 4 vertices, 5 without tadpoles, 4 with neither bubbles nor tadpoles.

C. 6 internal vertices

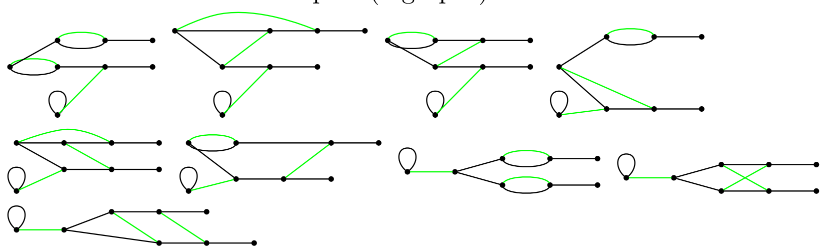
1 fermion 0 bubbles 0 tadpoles (11 graphs):



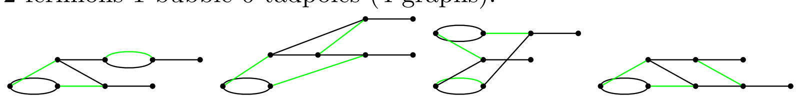
2 fermions 0 bubbles 0 tadpoles (8 graphs):



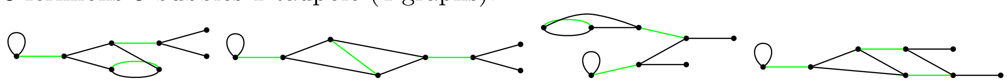
2 fermions 0 bubbles 1 tadpole (9 graphs):



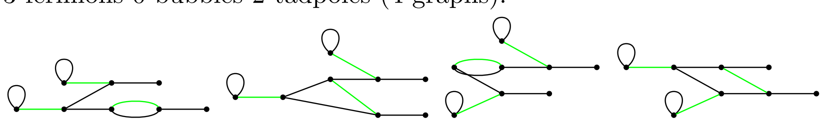
2 fermions 1 bubble 0 tadpoles (4 graphs):



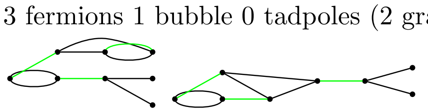
3 fermions 0 bubbles 1 tadpole (4 graphs):



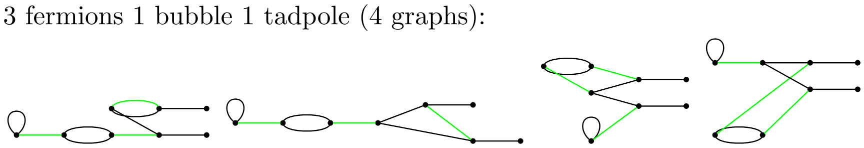
3 fermions 0 bubbles 2 tadpoles (4 graphs):



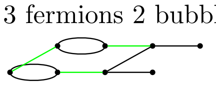
3 fermions 1 bubble 0 tadpoles (2 graphs):



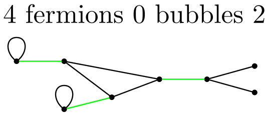
3 fermions 1 bubble 1 tadpole (4 graphs):



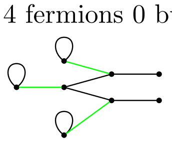
3 fermions 2 bubbles 0 tadpoles (1 graph):



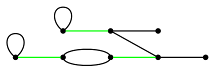
4 fermions 0 bubbles 2 tadpoles (1 graph):



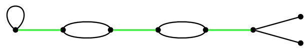
4 fermions 0 bubbles 3 tadpoles (1 graph):



4 fermions 1 bubble 2 tadpoles (1 graph):



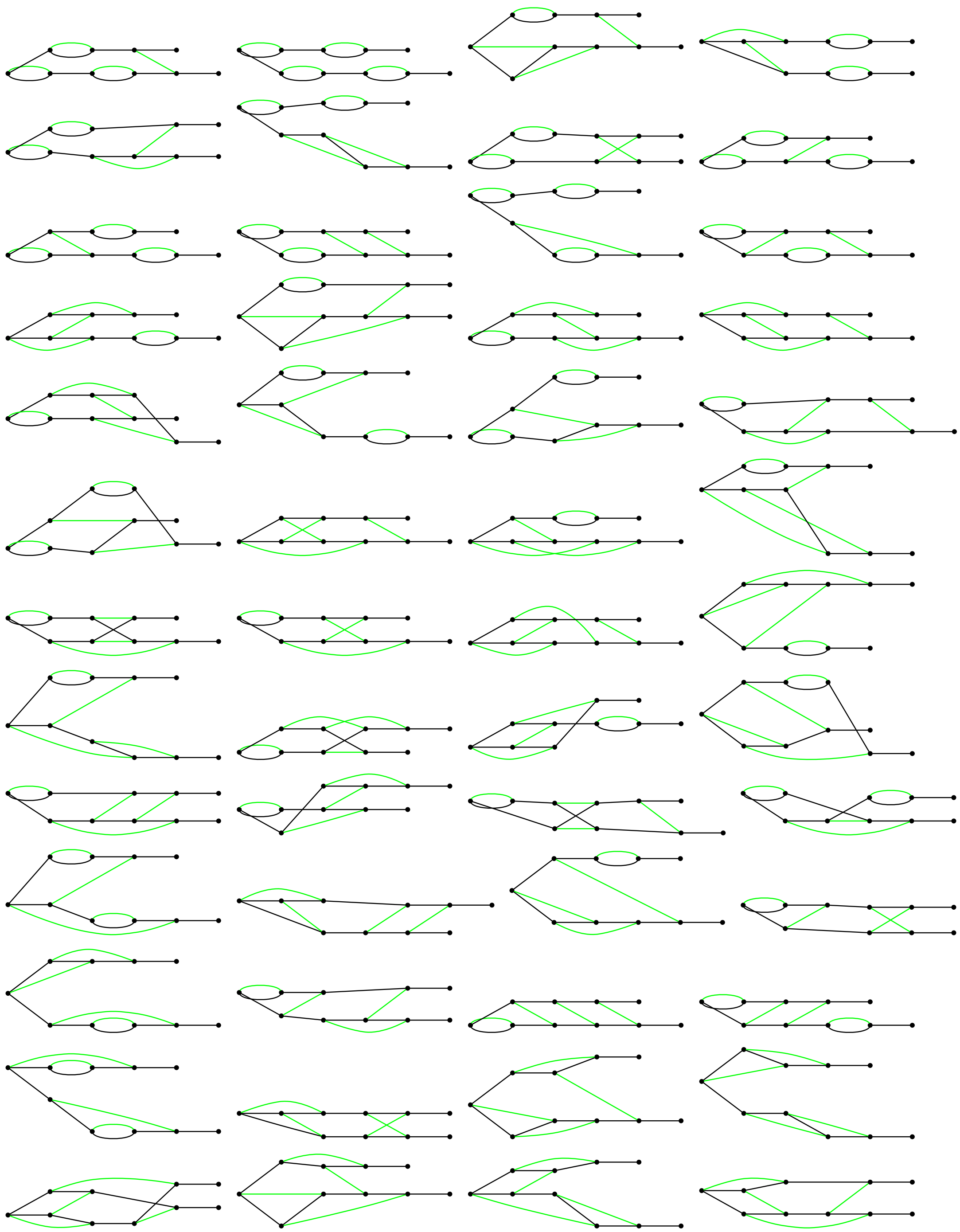
4 fermions 2 bubbles 1 tadpole (1 graph):

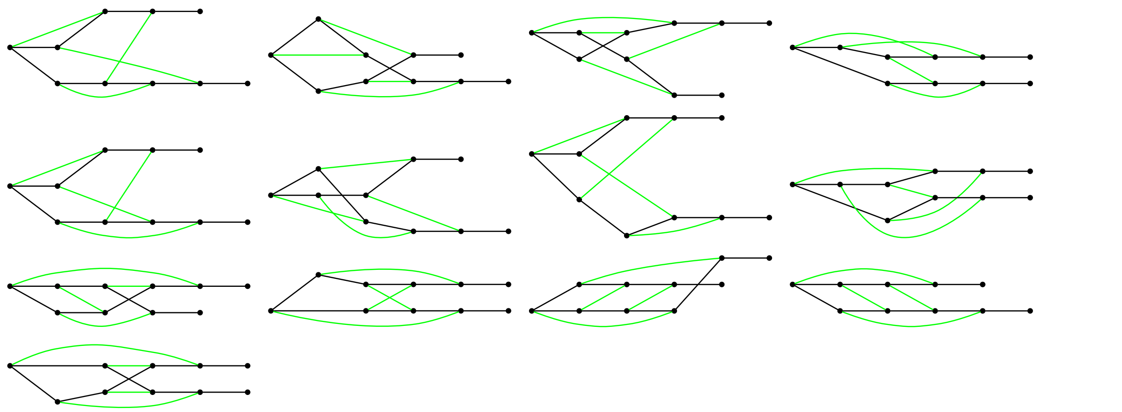


Total: 51 graphs on 6 vertices, 26 without tadpoles, 19 with neither bubbles nor tadpoles.

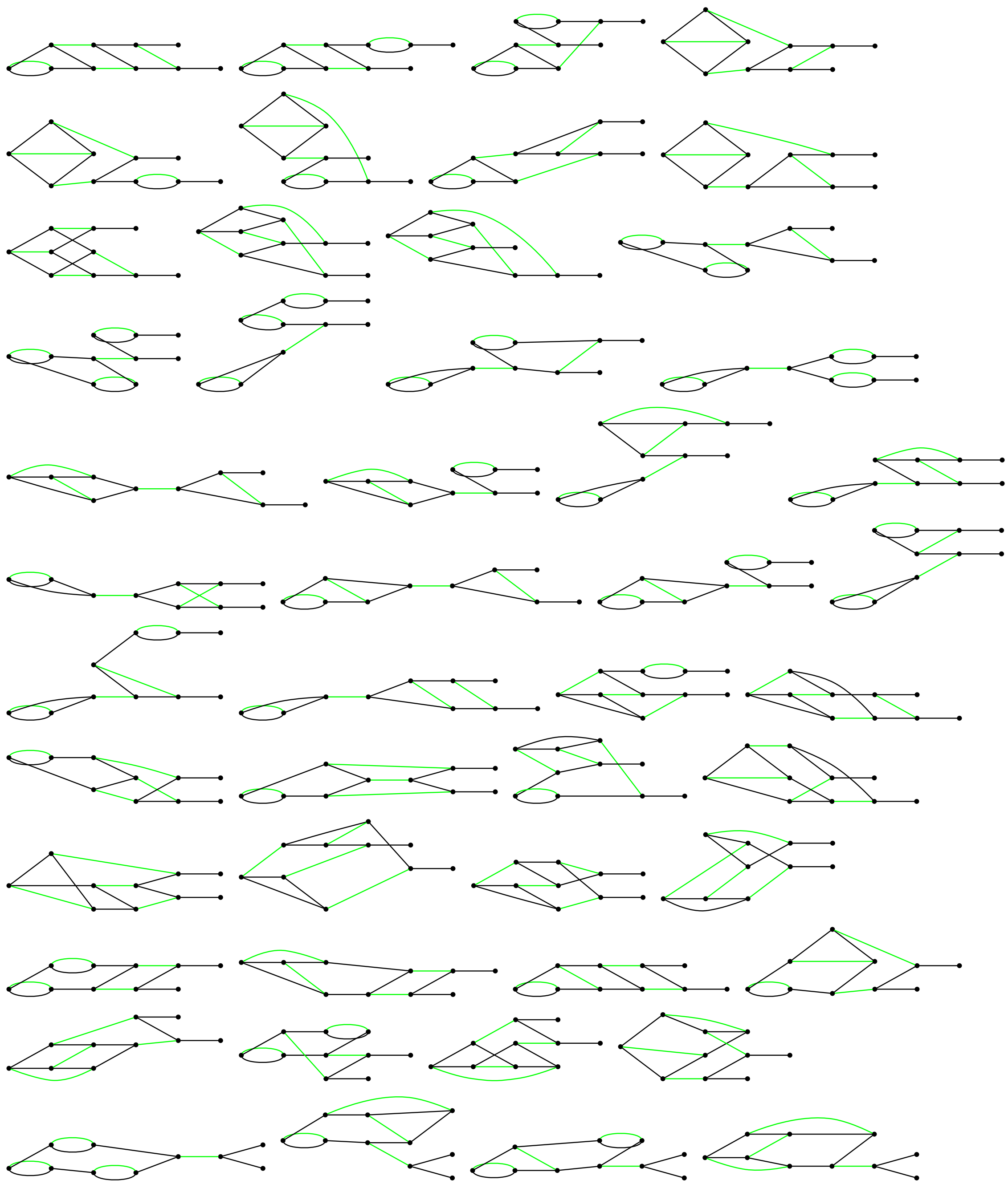
D. 8 internal vertices

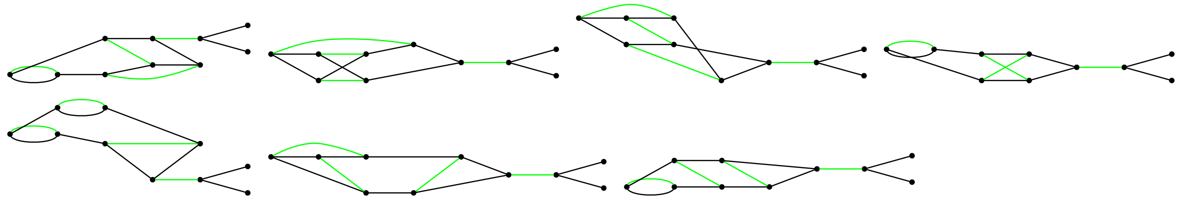
1 fermion 0 bubbles 0 tadpoles (65 graphs):



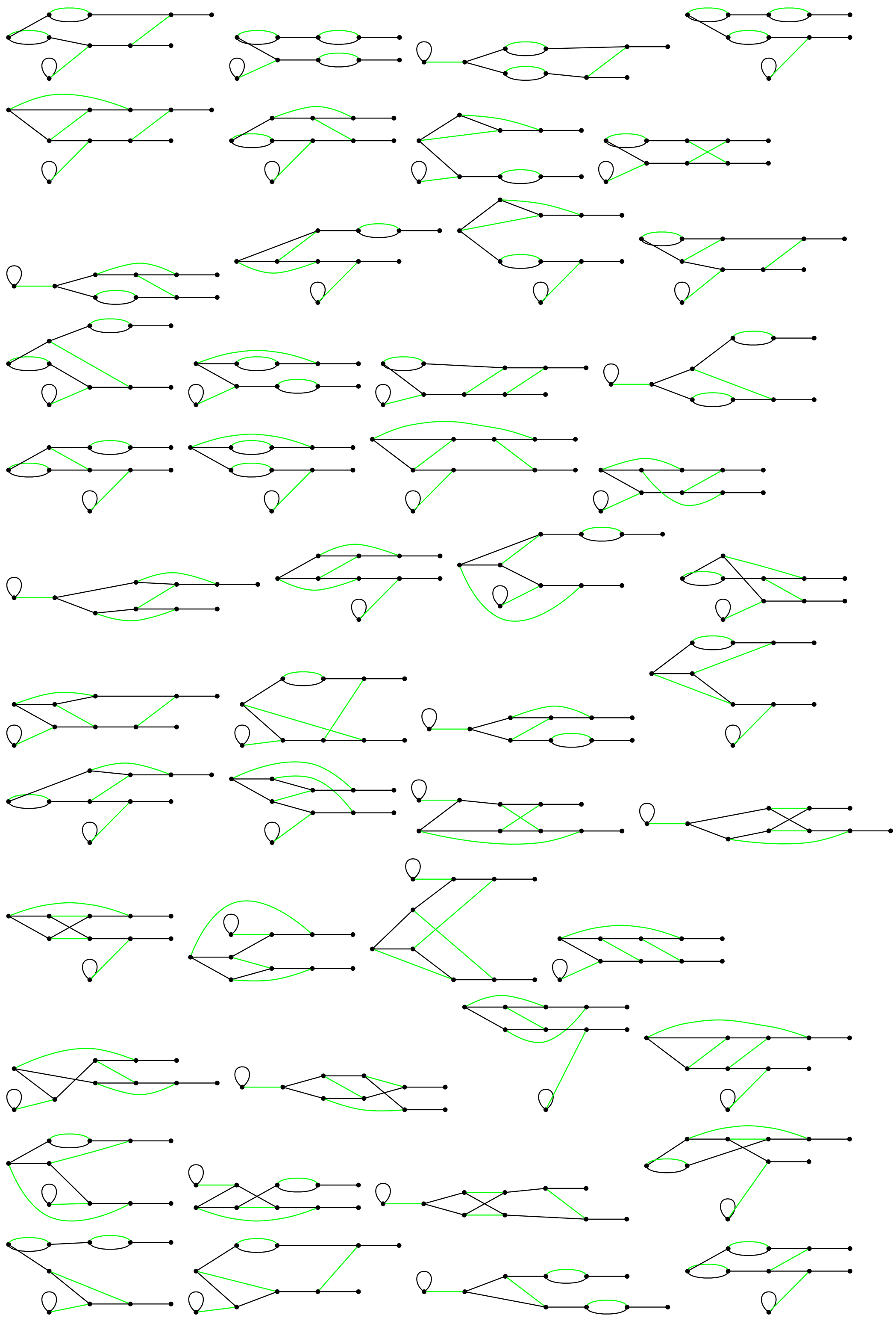


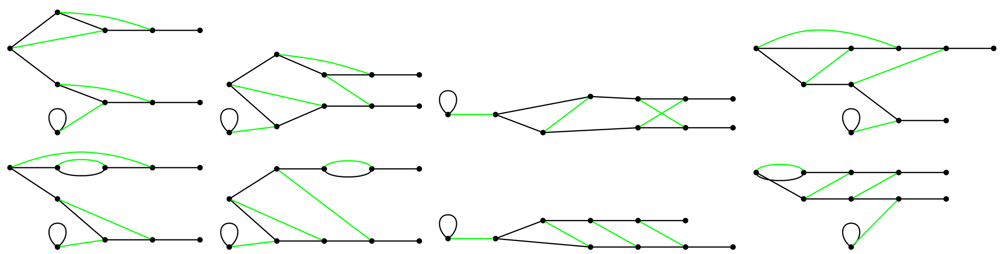
2 fermions 0 bubbles 0 tadpoles (55 graphs):



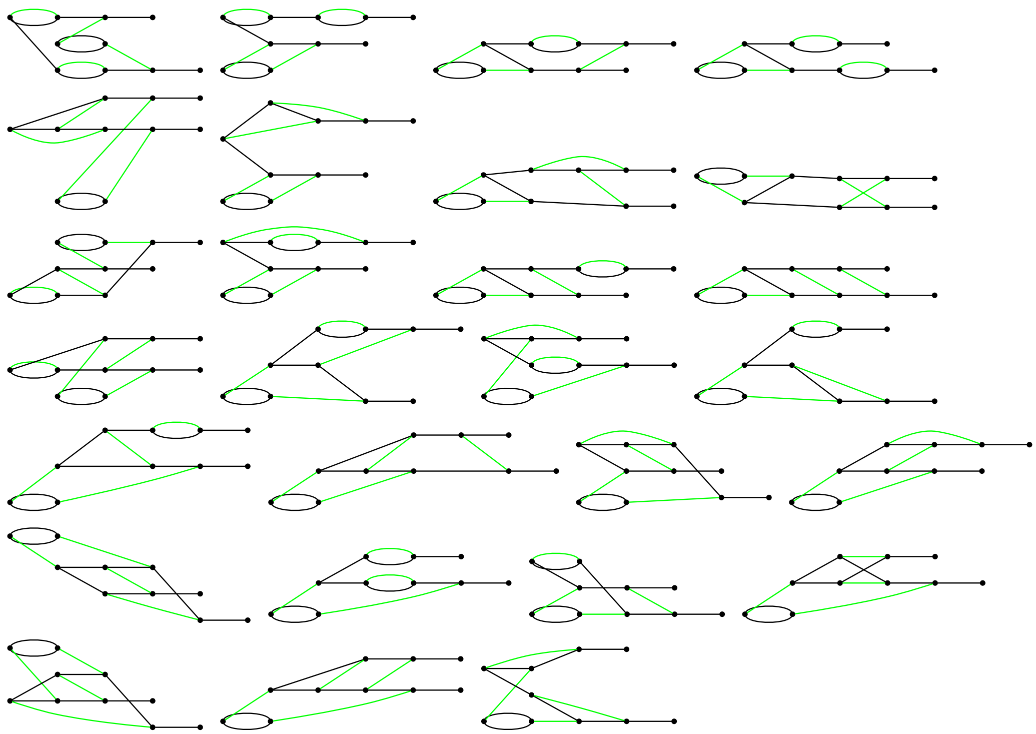


2 fermions 0 bubbles 1 tadpole (56 graphs):

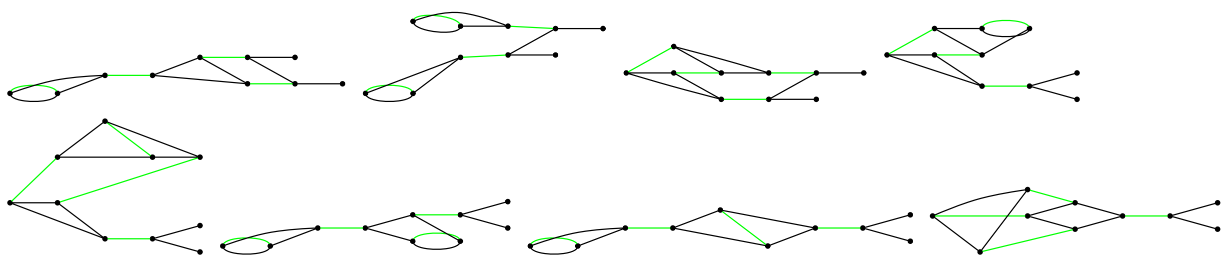




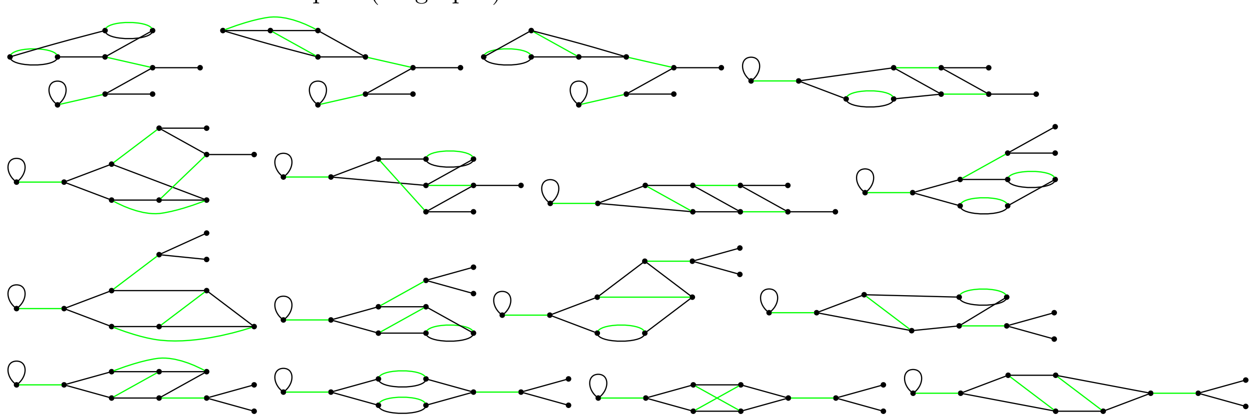
2 fermions 1 bubble 0 tadpoles (27 graphs):

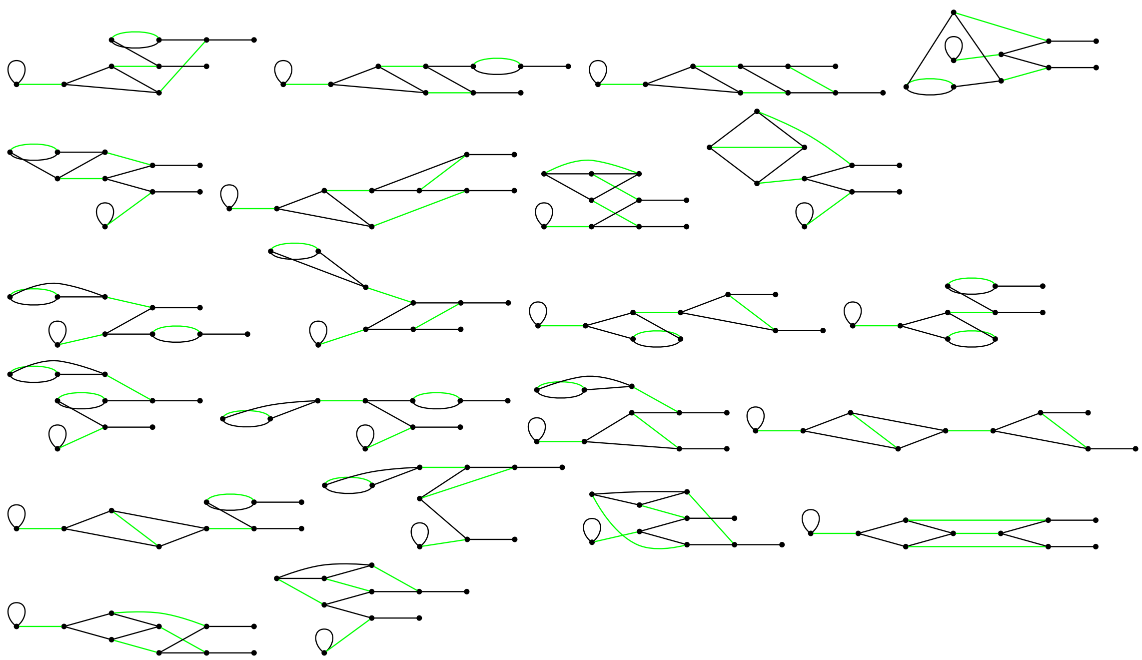


3 fermions 0 bubbles 0 tadpoles (8 graphs):

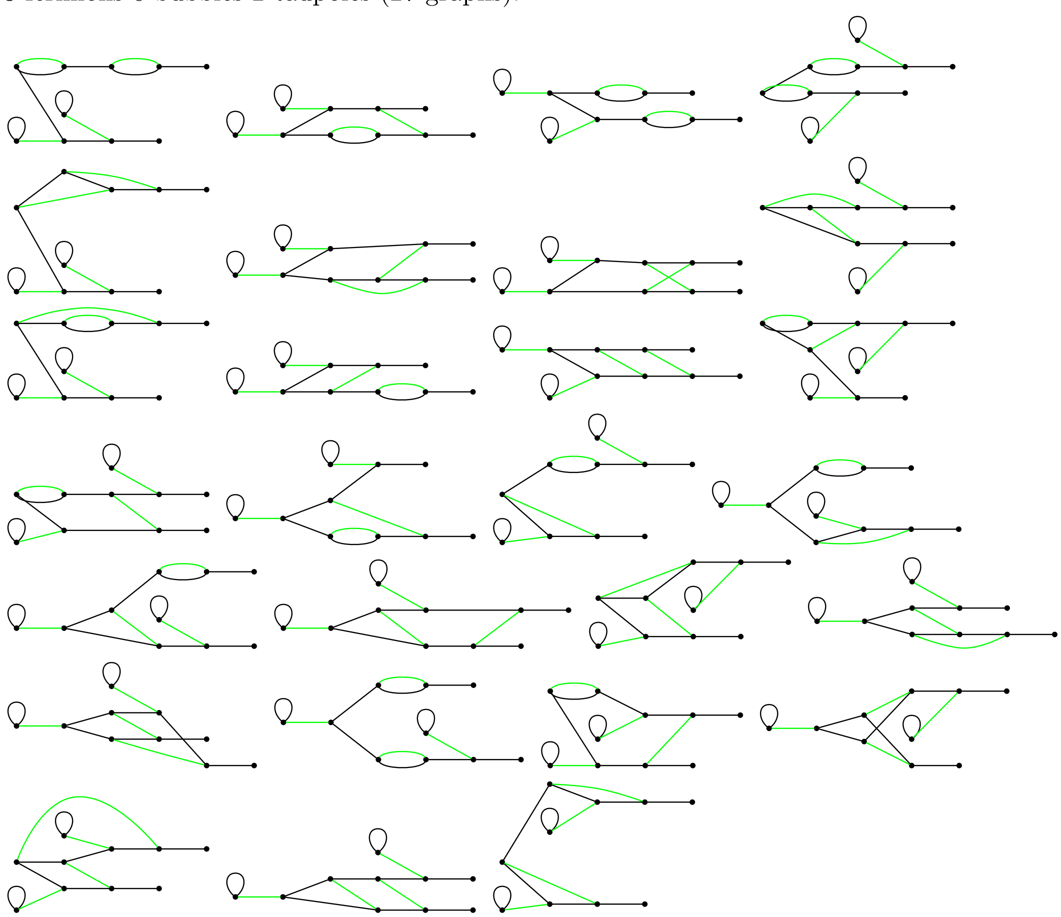


3 fermions 0 bubbles 1 tadpole (38 graphs):

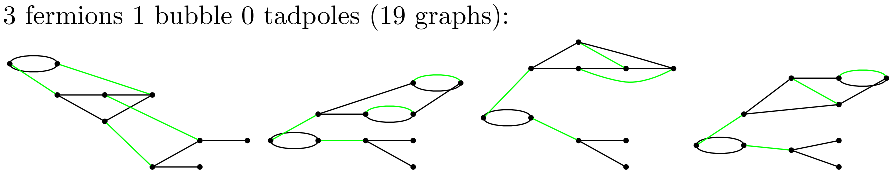


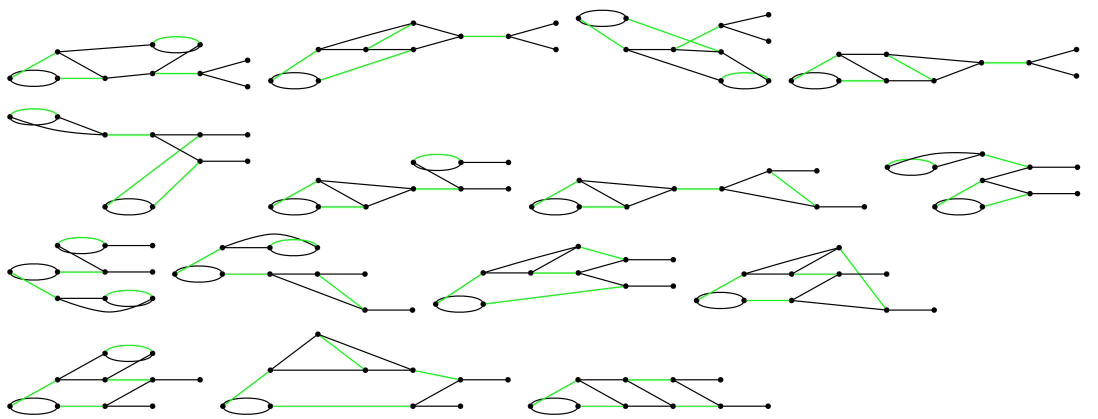


3 fermions 0 bubbles 2 tadpoles (27 graphs):

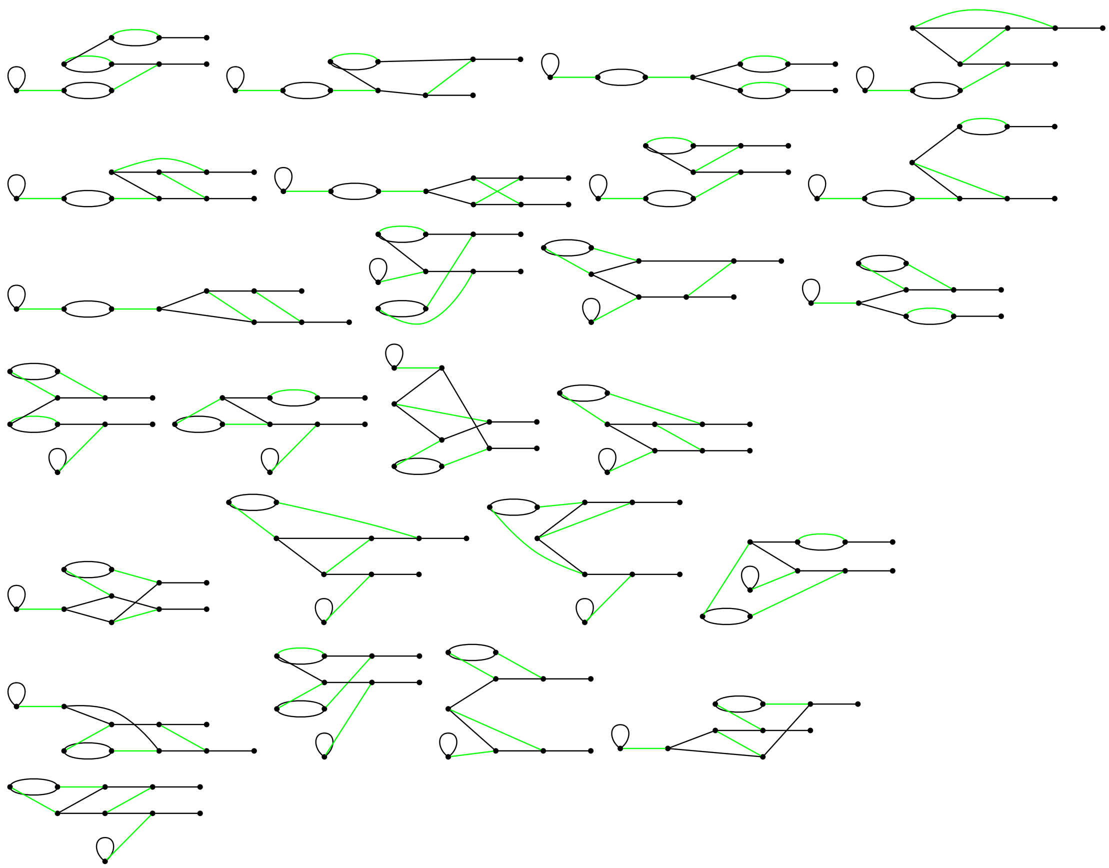


3 fermions 1 bubble 0 tadpoles (19 graphs):

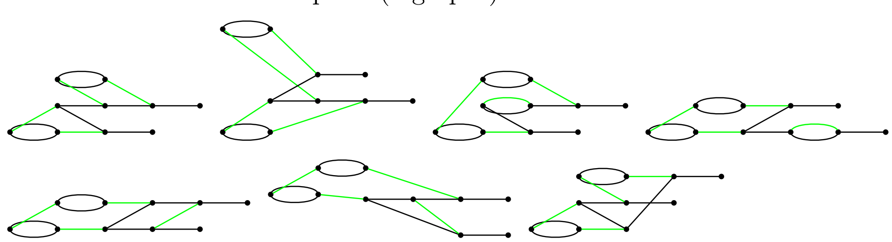




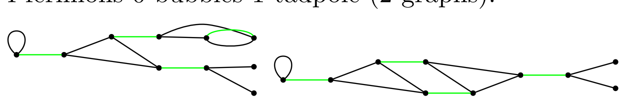
3 fermions 1 bubble 1 tadpole (25 graphs):



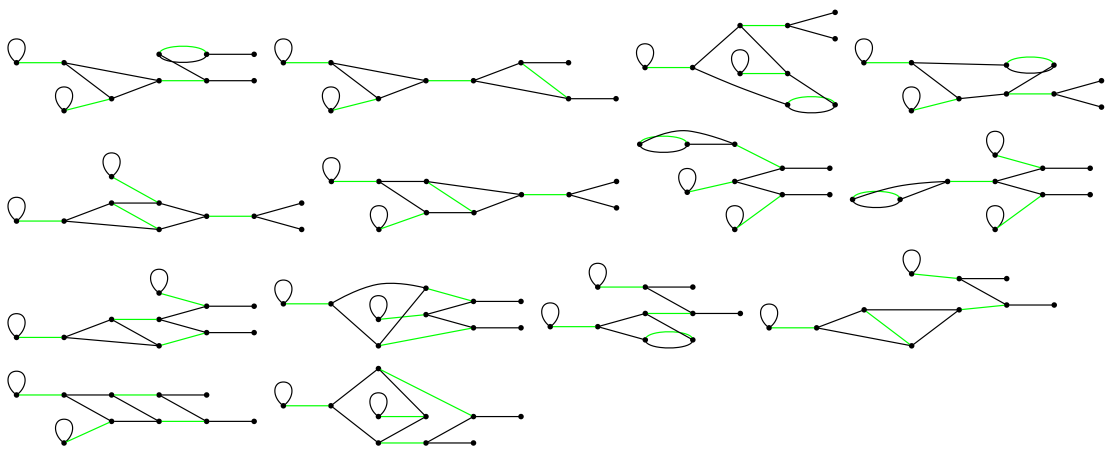
3 fermions 2 bubbles 0 tadpoles (7 graphs):



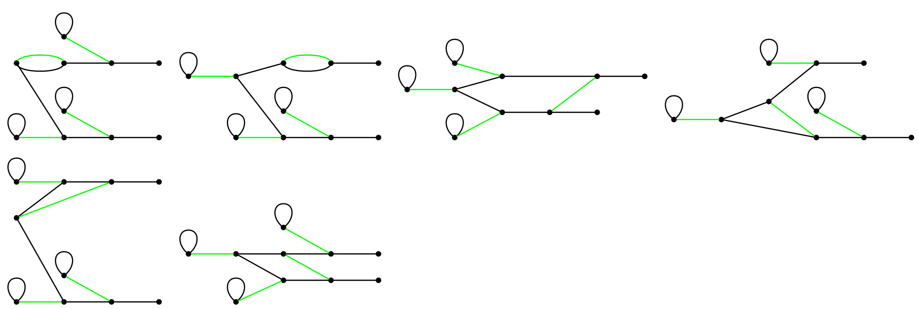
4 fermions 0 bubbles 1 tadpole (2 graphs):



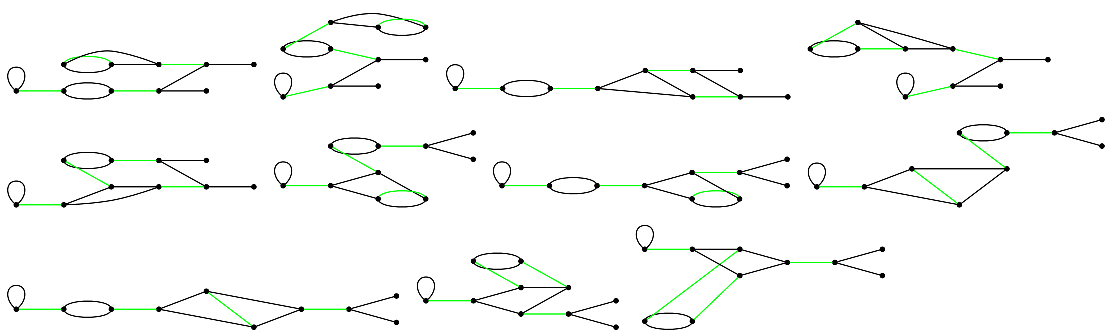
4 fermions 0 bubbles 2 tadpoles (14 graphs):



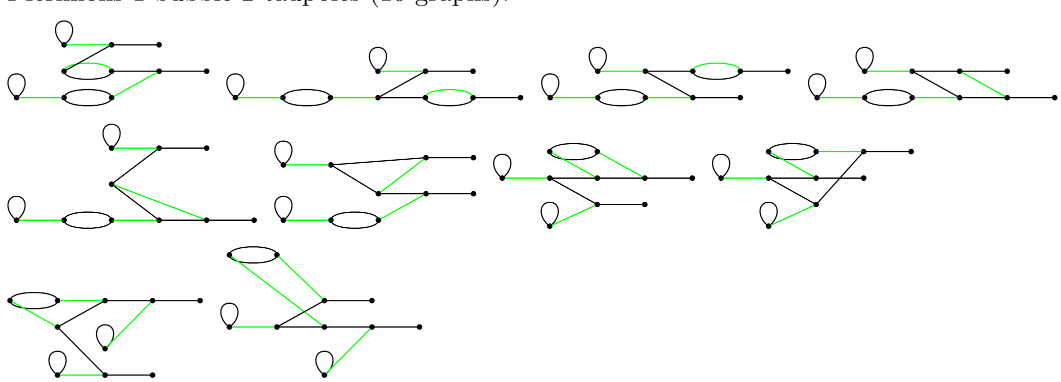
4 fermions 0 bubbles 3 tadpoles (6 graphs):



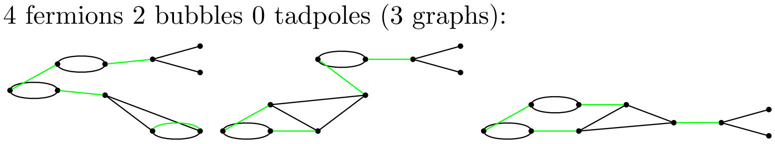
4 fermions 1 bubble 1 tadpole (11 graphs):



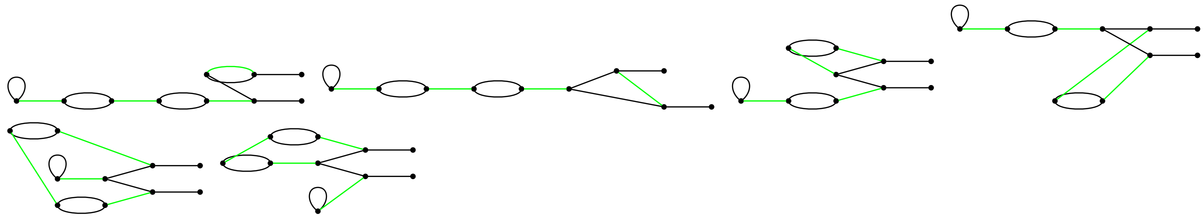
4 fermions 1 bubble 2 tadpoles (10 graphs):



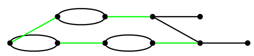
4 fermions 2 bubbles 0 tadpoles (3 graphs):



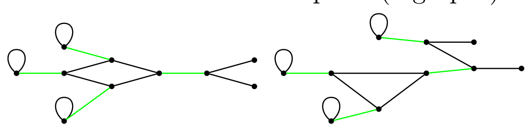
4 fermions 2 bubbles 1 tadpole (6 graphs):



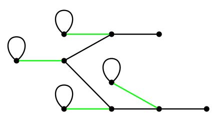
4 fermions 3 bubbles 0 tadpoles (1 graph):



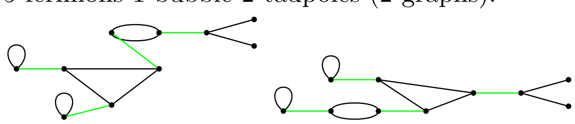
5 fermions 0 bubbles 3 tadpoles (2 graphs):



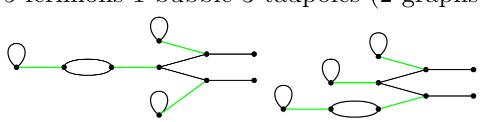
5 fermions 0 bubbles 4 tadpoles (1 graph):



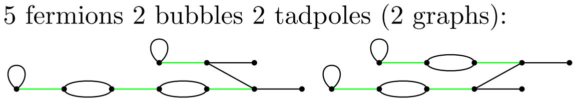
5 fermions 1 bubble 2 tadpoles (2 graphs):



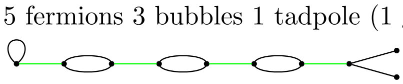
5 fermions 1 bubble 3 tadpoles (2 graphs):



5 fermions 2 bubbles 2 tadpoles (2 graphs):



5 fermions 3 bubbles 1 tadpole (1 graph):



Total: 390 graphs on 8 vertices, 185 without tadpoles, 128 with neither bubbles nor tadpoles.

V. VACUUM POLARIZATION (DIRECTED FERMIONS)

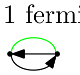
Adding orientations to the cycles of the fermion lines—then called Wilson loops—in the diagrams of Section II creates 1, 2, 5, 20, 107, 870, . . . vacuum polarization diagrams on 0, 2, 4, 6,. . . vertices [1, A170946][19, Table 1][6, D10]. The 20 diagrams on 6 vertices have been plotted by Pelster et al. [20].

1, 1, 2, 5, 18, 105,. . . of them have one fermion [1, A007769][10].

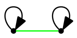
Depending on the symmetry of the non-directed vacuum polarization graphs, a graph of Section II may produce 2, 4, 8,. . . oriented graphs, where the multiplicity is a power of 2 limited by the number of non-trivial cycles (i.e., fermion cycles not in tadpoles or bubbles). One graph of the graphs with 2 fermions of which 0 are bubbles are 0 tadpoles is the lowest-order diagram where orienting the two fermion cycles (each of length 3) generates two non-isomorphic diagrams.

A. 2 vertices

1 fermion 1 bubble 0 tadpoles (1 graph):



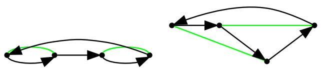
2 fermions 0 bubbles 2 tadpoles (1 graph):



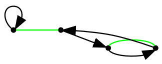
Total: 2 graphs on 2 vertices, 1 without tadpoles, 0 with neither bubbles nor tadpoles.

B. 4 vertices

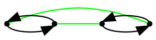
1 fermion 0 bubbles 0 tadpoles (2 graphs):



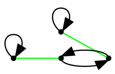
2 fermions 0 bubbles 1 tadpole (1 graph):



2 fermions 2 bubbles 0 tadpoles (1 graph):



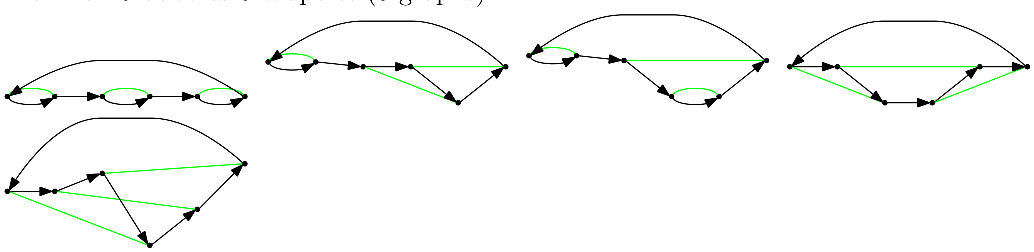
3 fermions 1 bubble 2 tadpoles (1 graph):



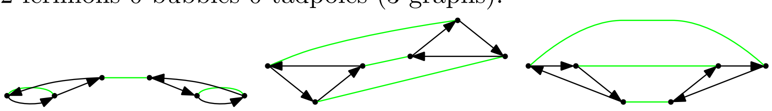
Total: 5 graphs on 4 vertices, 3 without tadpoles, 2 with neither bubbles nor tadpoles.

C. 6 vertices

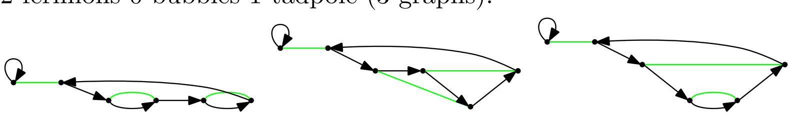
1 fermion 0 bubbles 0 tadpoles (5 graphs):



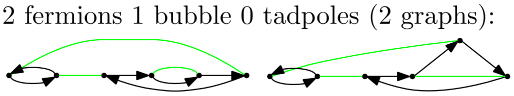
2 fermions 0 bubbles 0 tadpoles (3 graphs):



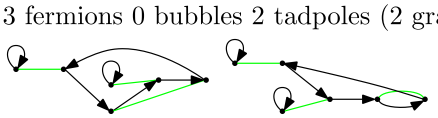
2 fermions 0 bubbles 1 tadpole (3 graphs):



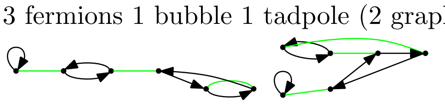
2 fermions 1 bubble 0 tadpoles (2 graphs):



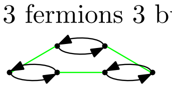
3 fermions 0 bubbles 2 tadpoles (2 graphs):



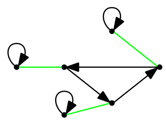
3 fermions 1 bubble 1 tadpole (2 graphs):



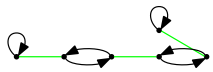
3 fermions 3 bubbles 0 tadpoles (1 graph):



4 fermions 0 bubbles 3 tadpoles (1 graph):



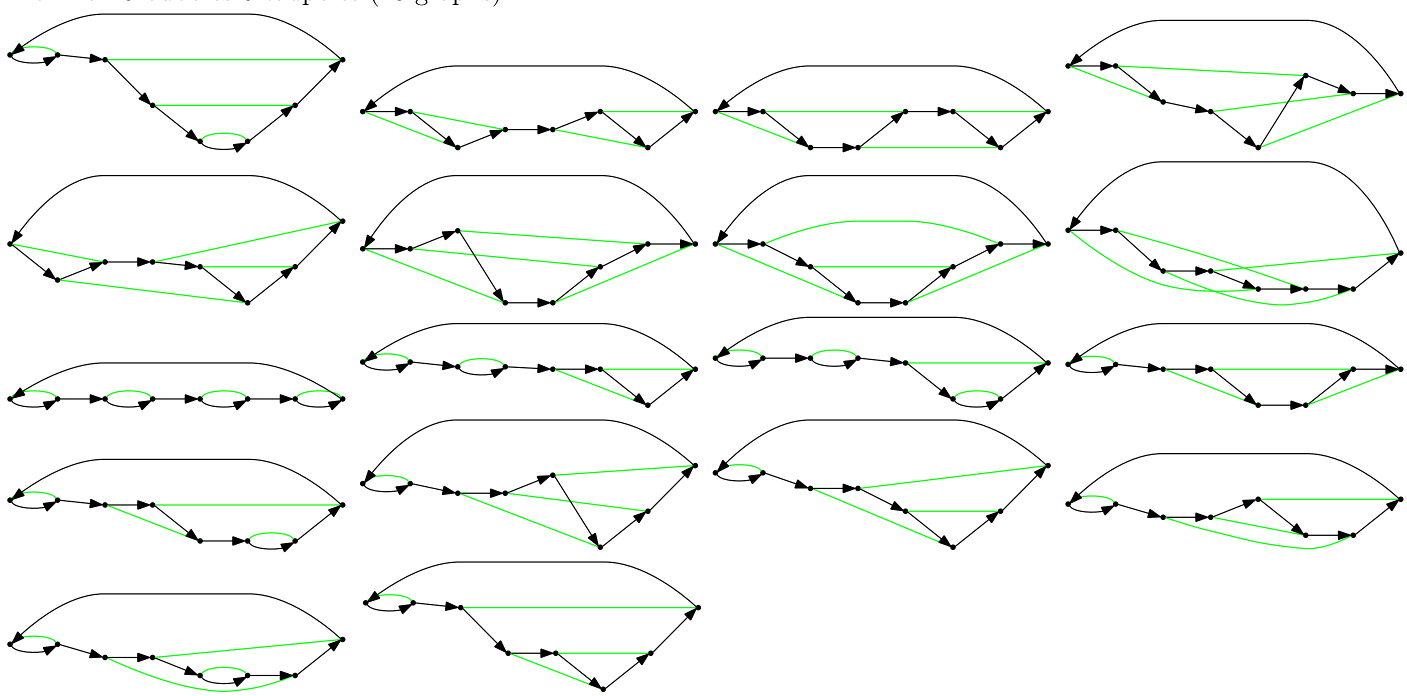
4 fermions 2 bubbles 2 tadpoles (1 graph):



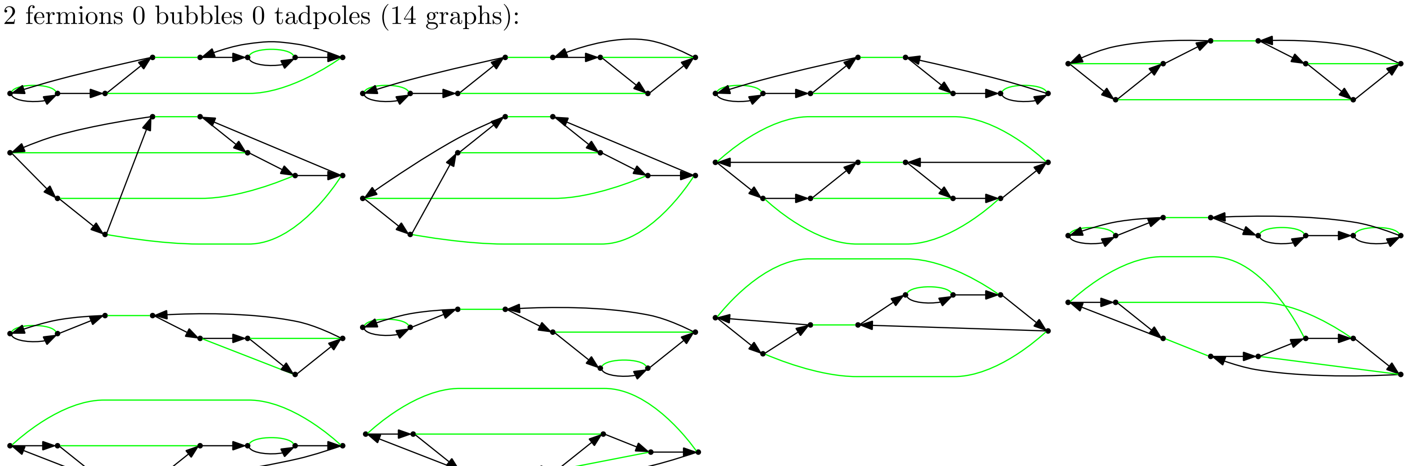
Total: 20 graphs on 6 vertices, 11 without tadpoles, 8 with neither bubbles nor tadpoles.

D. 8 vertices

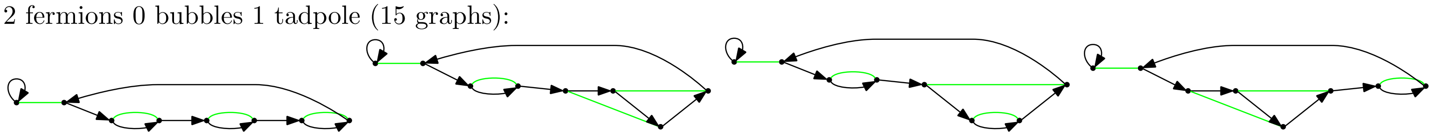
1 fermion 0 bubbles 0 tadpoles (18 graphs):

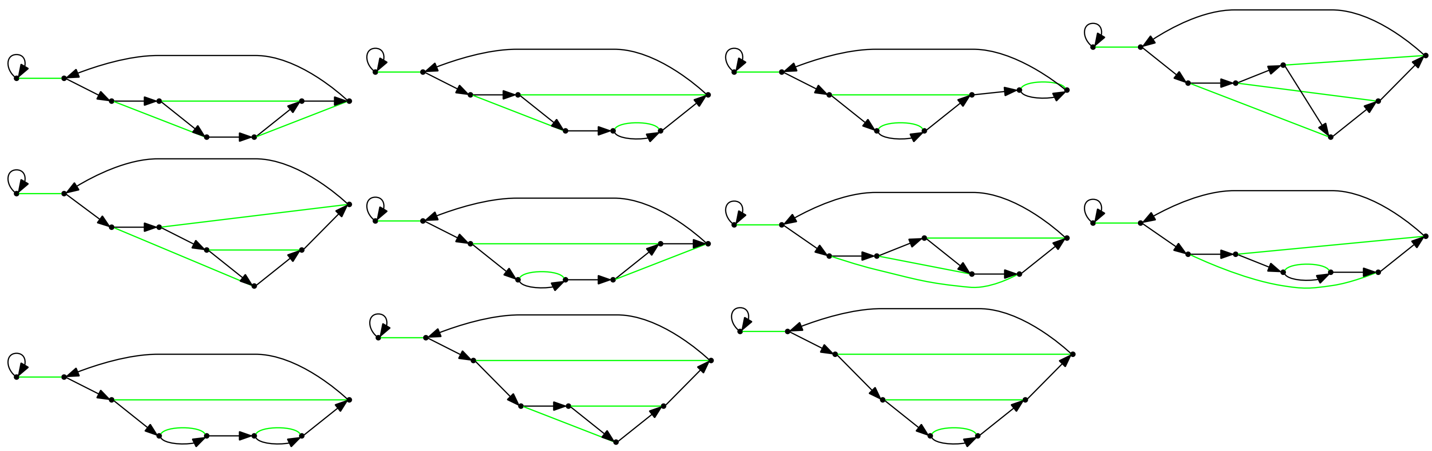


2 fermions 0 bubbles 0 tadpoles (14 graphs):

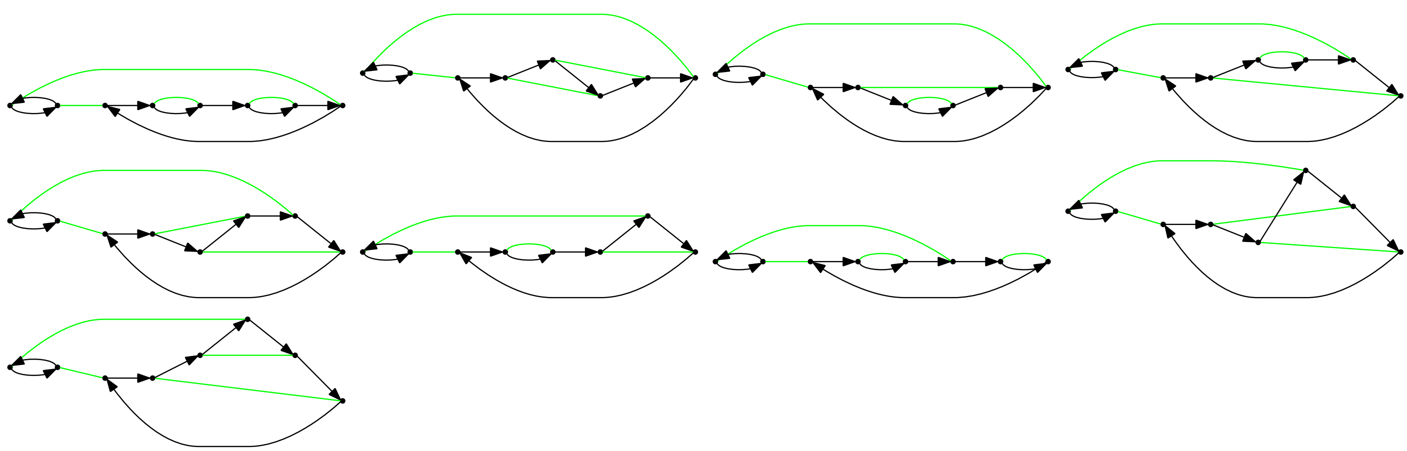


2 fermions 0 bubbles 1 tadpole (15 graphs):

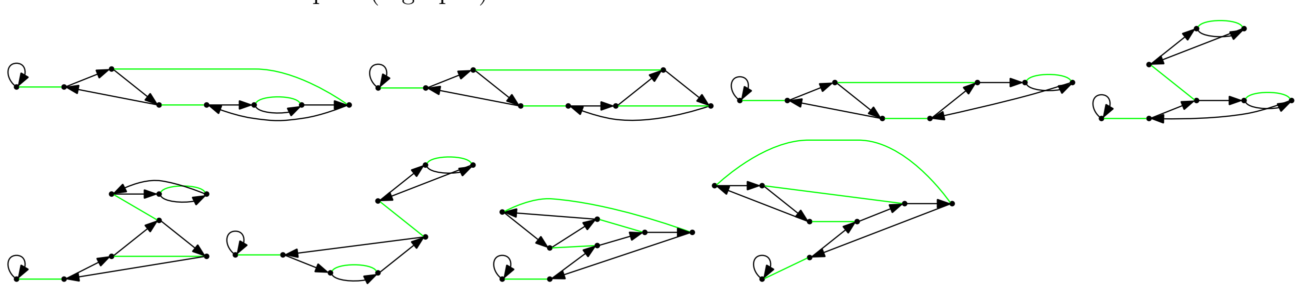




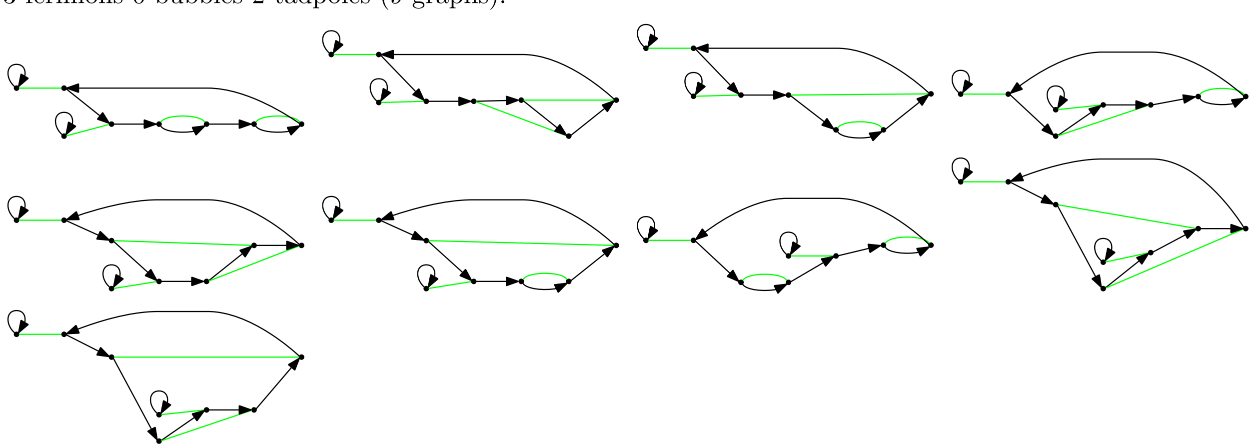
2 fermions 1 bubble 0 tadpoles (9 graphs):



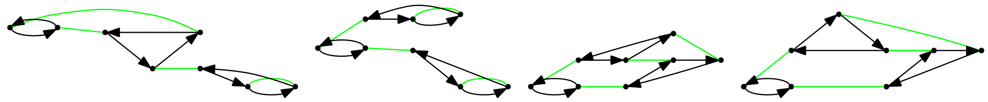
3 fermions 0 bubbles 1 tadpole (8 graphs):



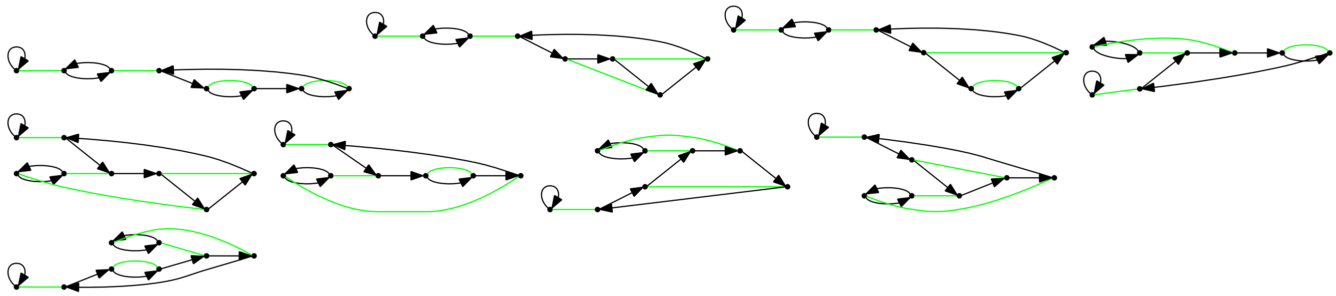
3 fermions 0 bubbles 2 tadpoles (9 graphs):



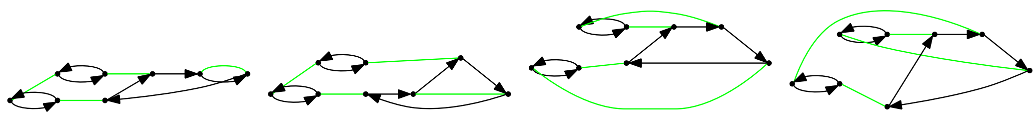
3 fermions 1 bubble 0 tadpoles (4 graphs):



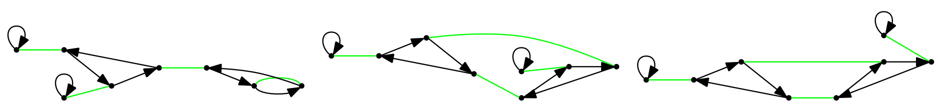
3 fermions 1 bubble 1 tadpole (9 graphs):



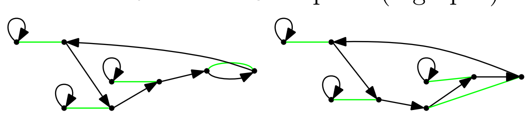
3 fermions 2 bubbles 0 tadpoles (4 graphs):



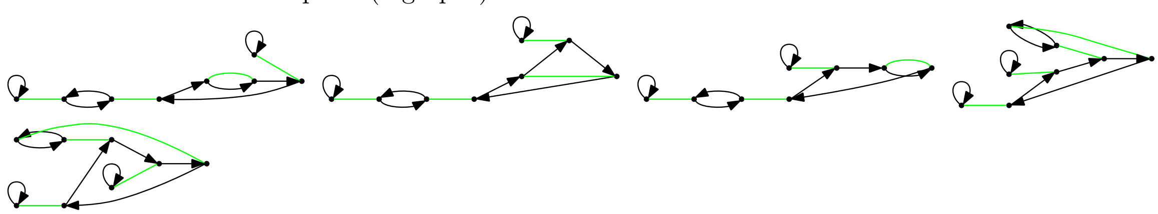
4 fermions 0 bubbles 2 tadpoles (3 graphs):



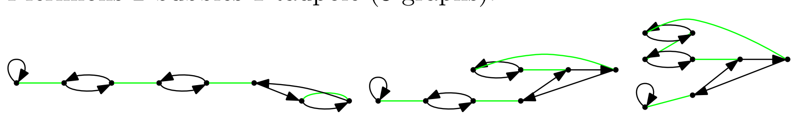
4 fermions 0 bubbles 3 tadpoles (2 graphs):



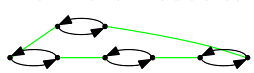
4 fermions 1 bubble 2 tadpoles (5 graphs):



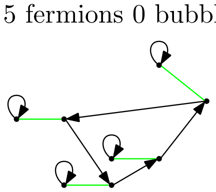
4 fermions 2 bubbles 1 tadpole (3 graphs):



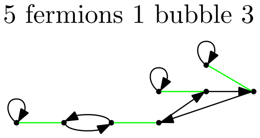
4 fermions 4 bubbles 0 tadpoles (1 graph):



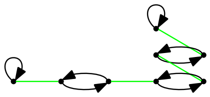
5 fermions 0 bubbles 4 tadpoles (1 graph):



5 fermions 1 bubble 3 tadpoles (1 graph):



5 fermions 3 bubbles 2 tadpoles (1 graph):



Total: 107 graphs on 8 vertices, 50 without tadpoles, 32 with neither bubbles nor tadpoles.

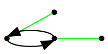
VI. EFFECTIVE INTERACTION (DIRECTED FERMIONS)

Cutting an interaction line in a diagram of Section V creates 1, 1, 4, 29, 272, 3237 … polarization diagrams with 0, 2, 4,… internal vertices.

The subset of graphs with 1 fermion (0, 1, 2, 9, 56, 485 …) is counted by the number of 1-rooted chord diagrams, which means, by marking one of the vertices of the of the 0, 1, 2, 5, 17, 79,… diagrams mentioned in Section II. Some of these rooted chord diagrams without bubbles have been plotted by Marie and Yeats [21].

A. 2 vertices

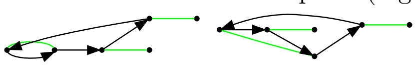
1 fermion 1 bubble 0 tadpoles (1 graph):



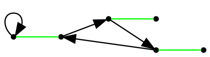
Total: 1 graphs on 2 vertices, 1 without tadpoles, 0 with neither bubbles nor tadpoles.

B. 4 vertices

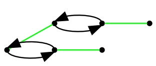
1 fermion 0 bubbles 0 tadpoles (2 graphs):



2 fermions 0 bubbles 1 tadpole (1 graph):



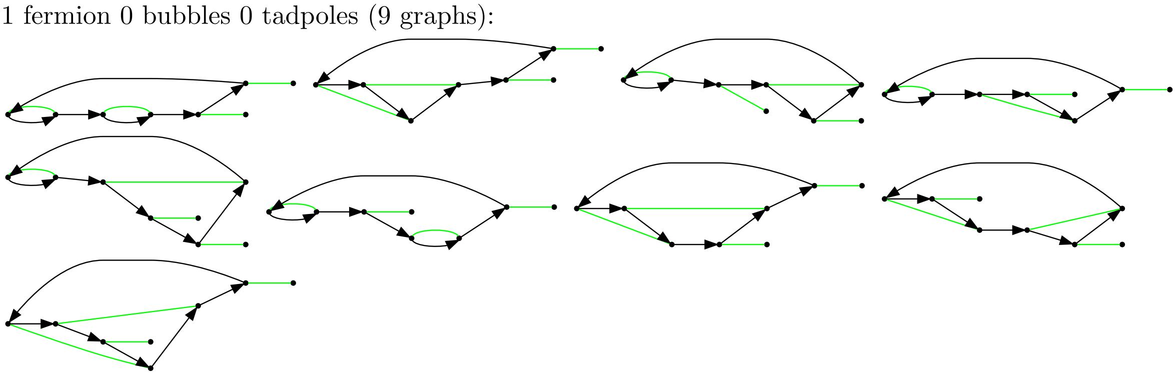
2 fermions 2 bubbles 0 tadpoles (1 graph):



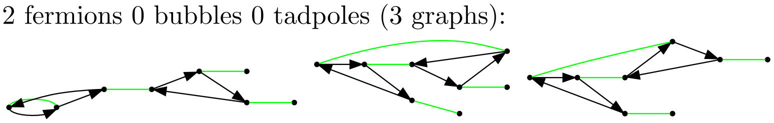
Total: 4 graphs on 4 vertices, 3 without tadpoles, 2 with neither bubbles nor tadpoles.

C. 6 vertices

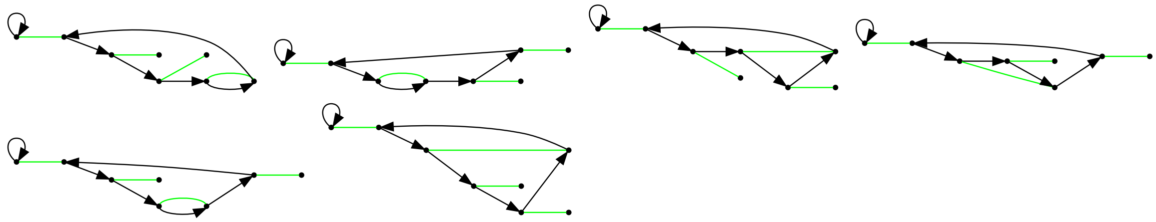
1 fermion 0 bubbles 0 tadpoles (9 graphs):



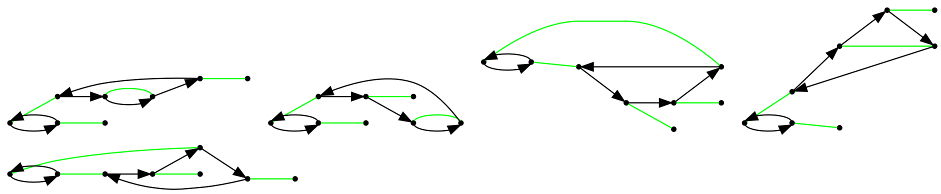
2 fermions 0 bubbles 0 tadpoles (3 graphs):



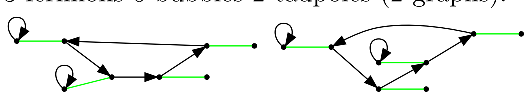
2 fermions 0 bubbles 1 tadpole (6 graphs):



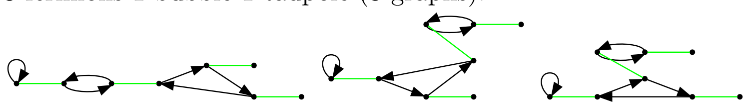
2 fermions 1 bubble 0 tadpoles (5 graphs):



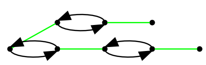
3 fermions 0 bubbles 2 tadpoles (2 graphs):



3 fermions 1 bubble 1 tadpole (3 graphs):



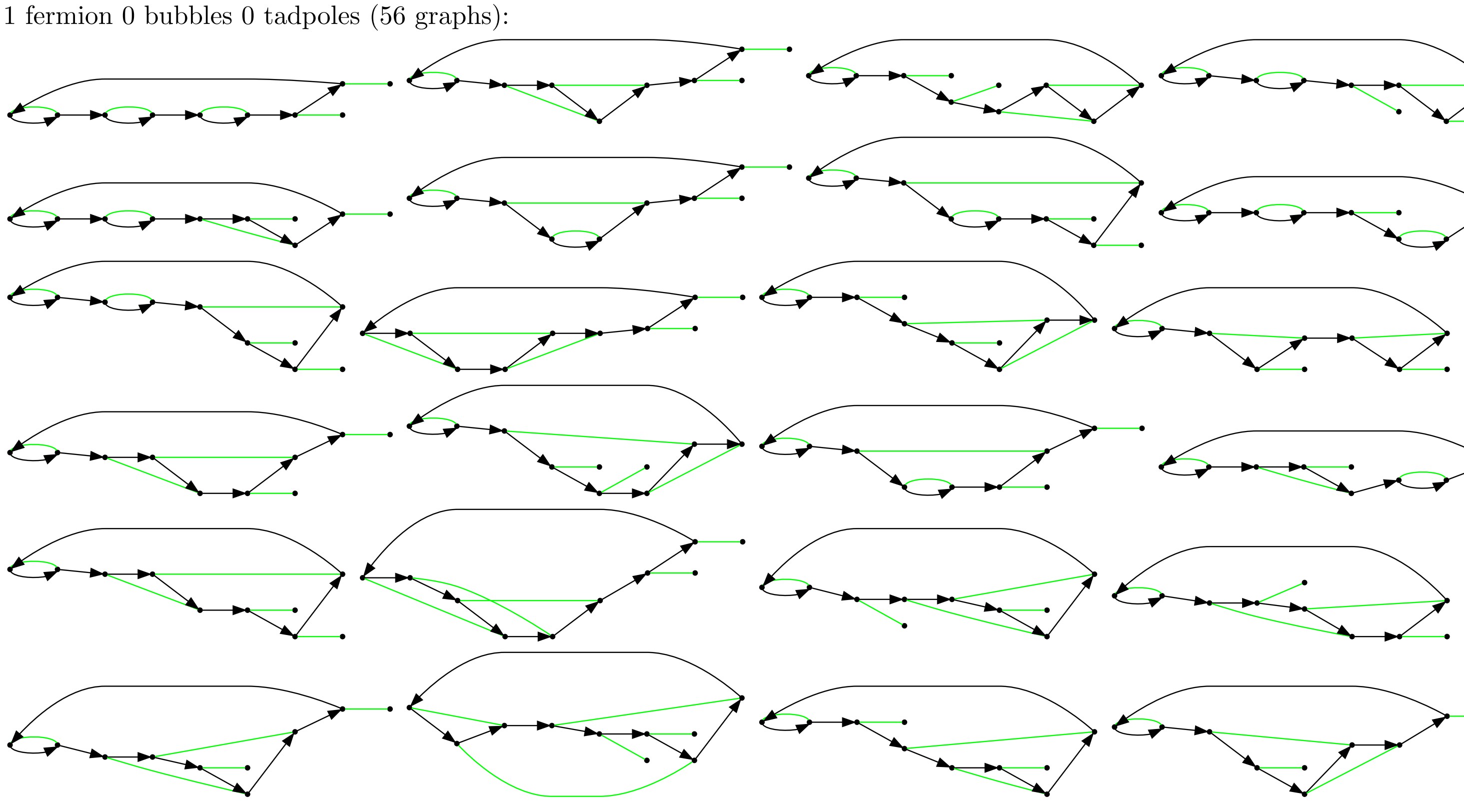
3 fermions 3 bubbles 0 tadpoles (1 graph):

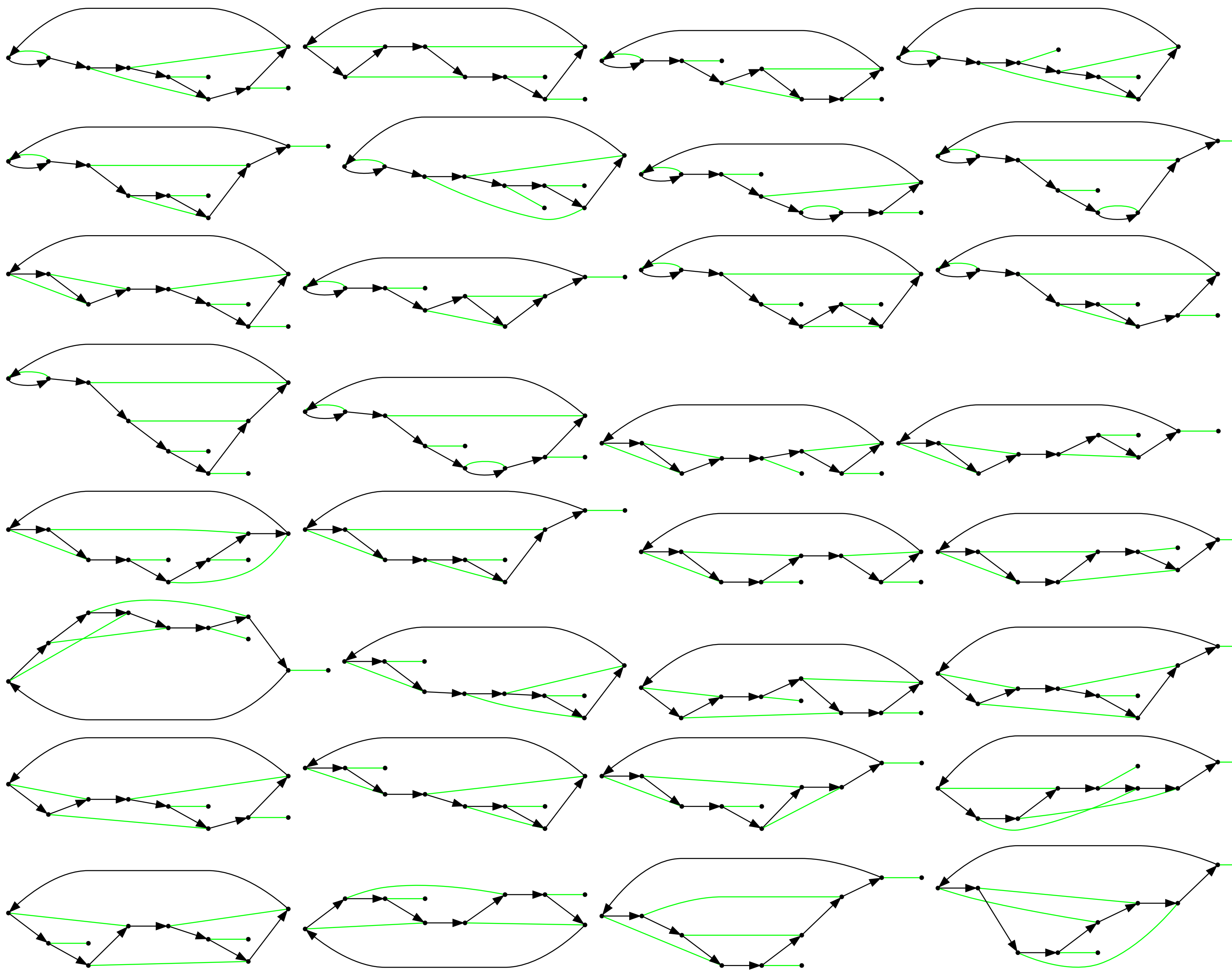


Total: 29 graphs on 6 vertices, 18 without tadpoles, 12 with neither bubbles nor tadpoles.

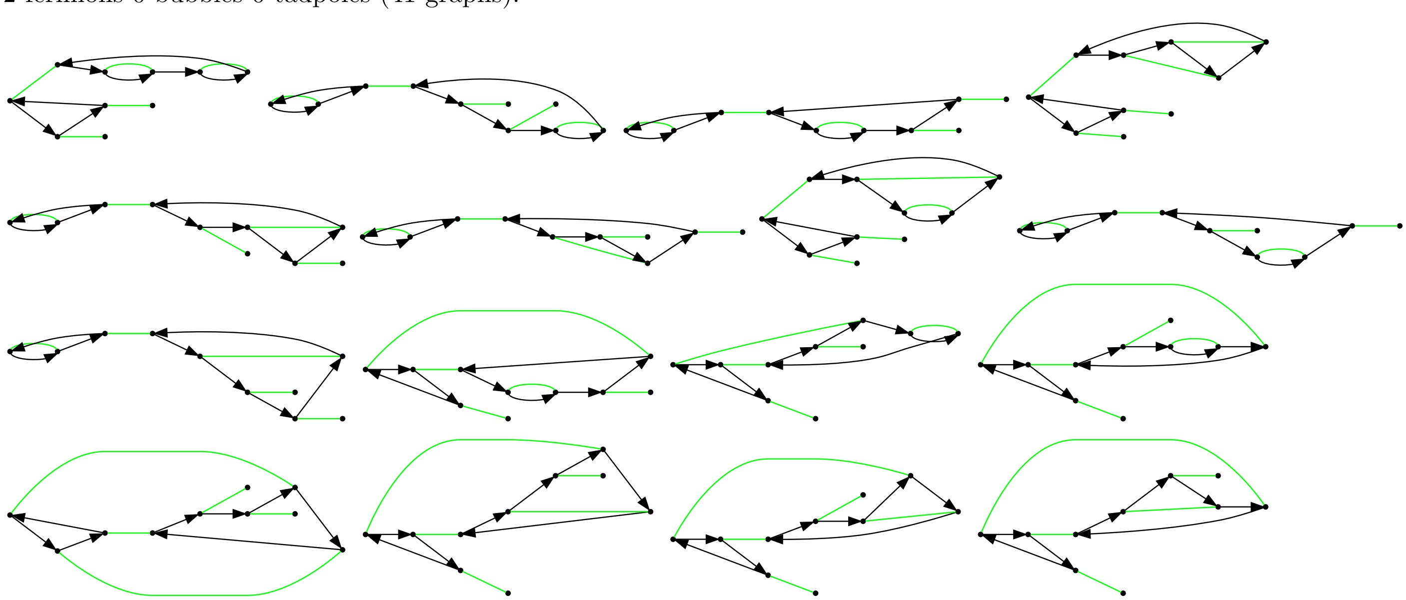
D. 8 vertices

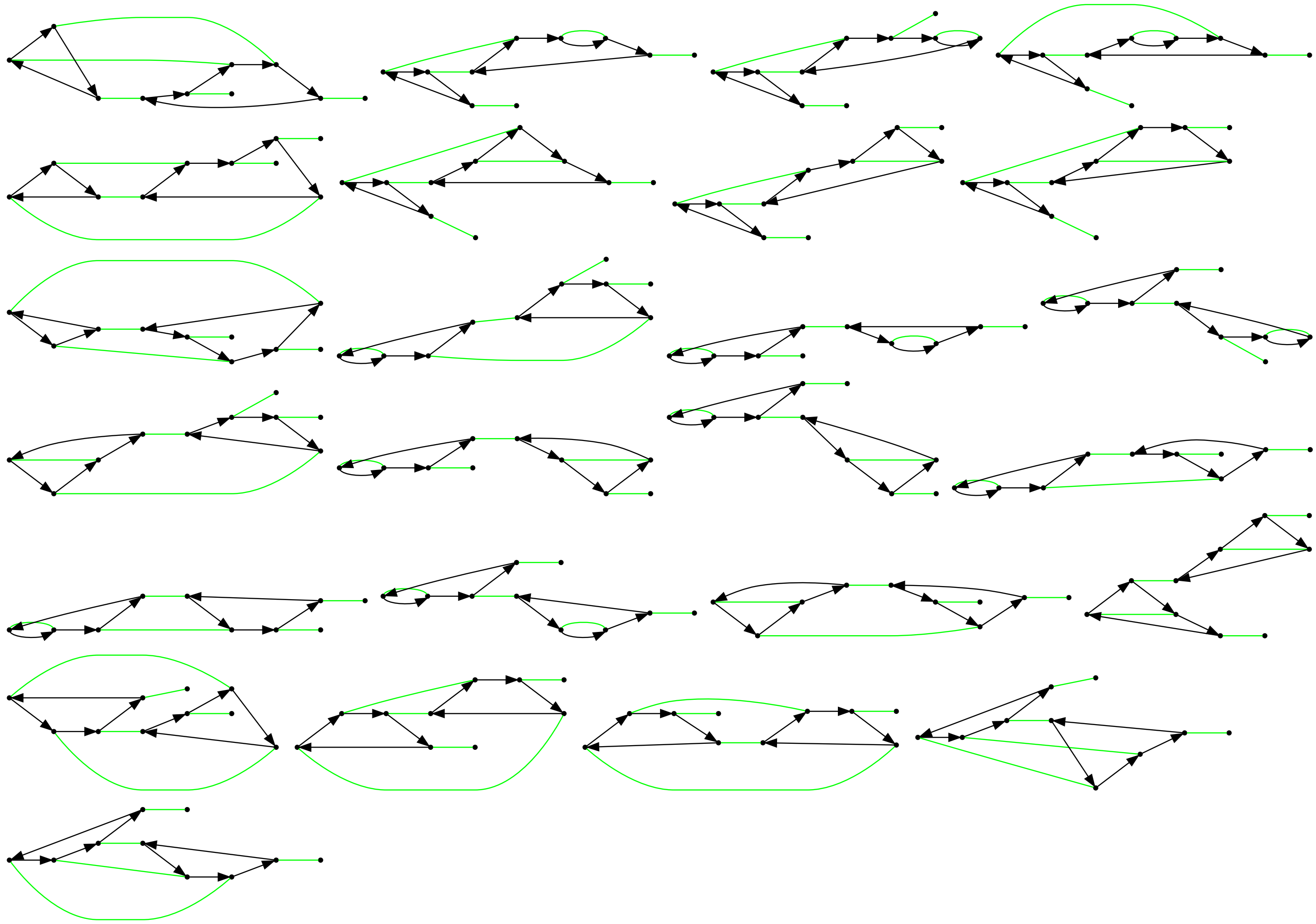
1 fermion 0 bubbles 0 tadpoles (56 graphs):



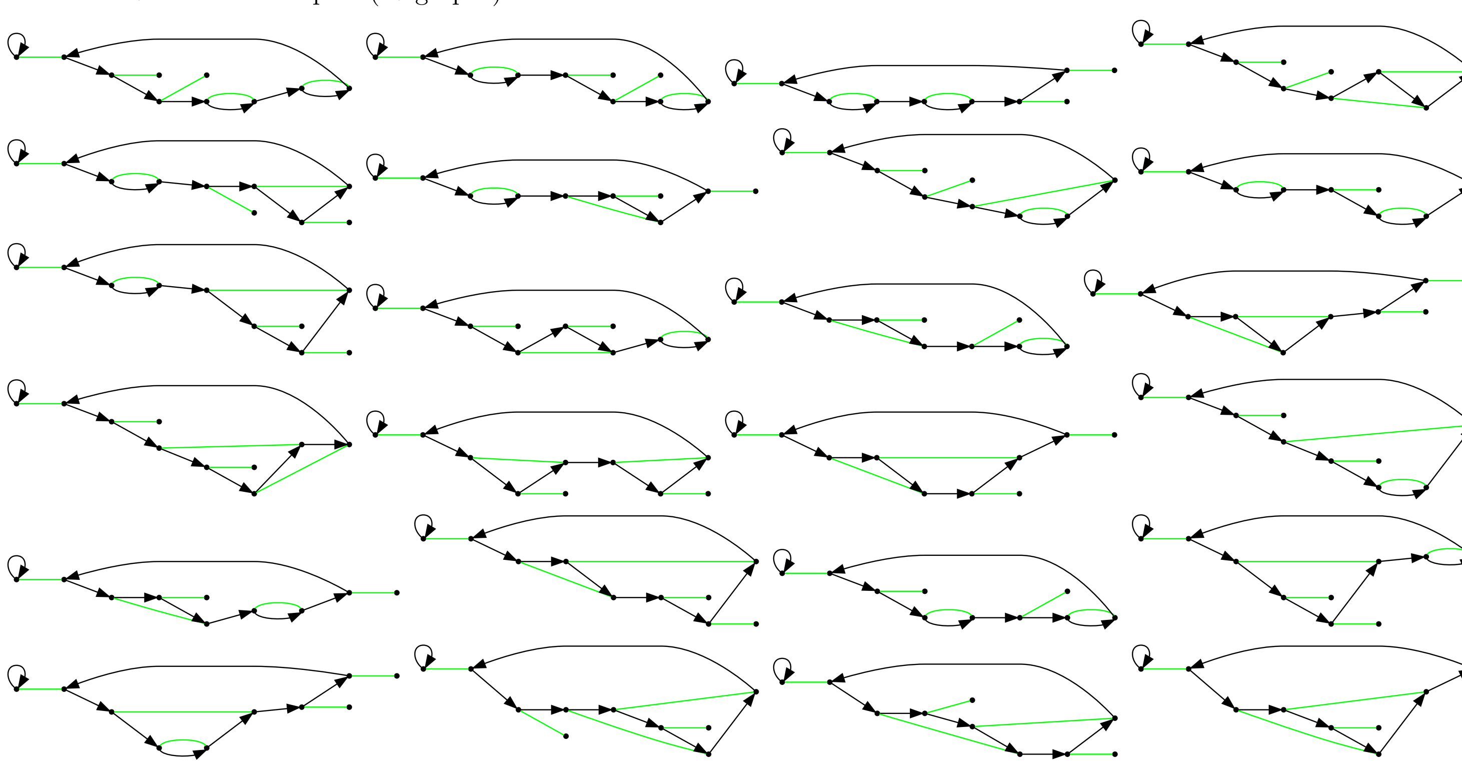


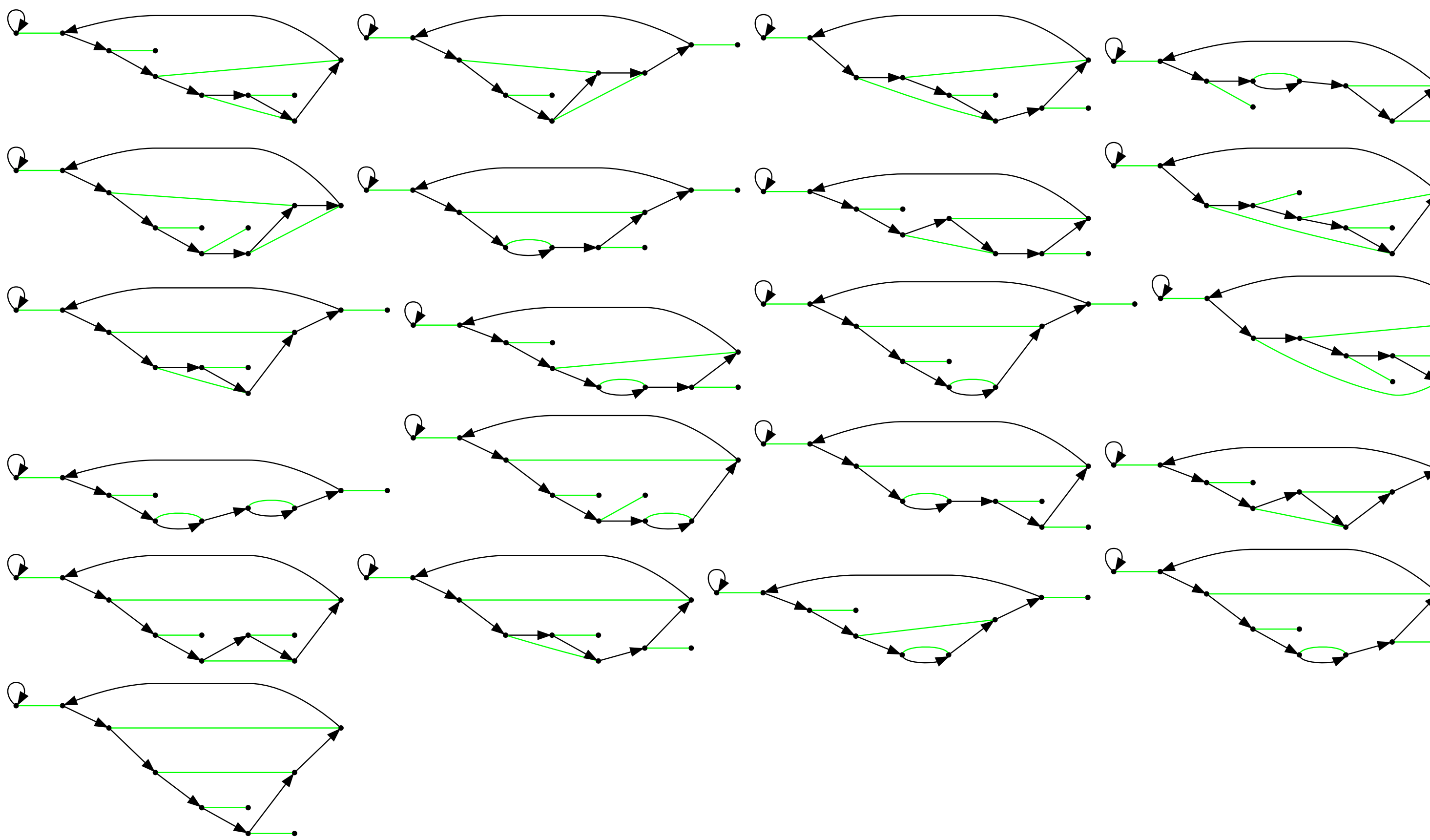
2 fermions 0 bubbles 0 tadpoles (41 graphs):



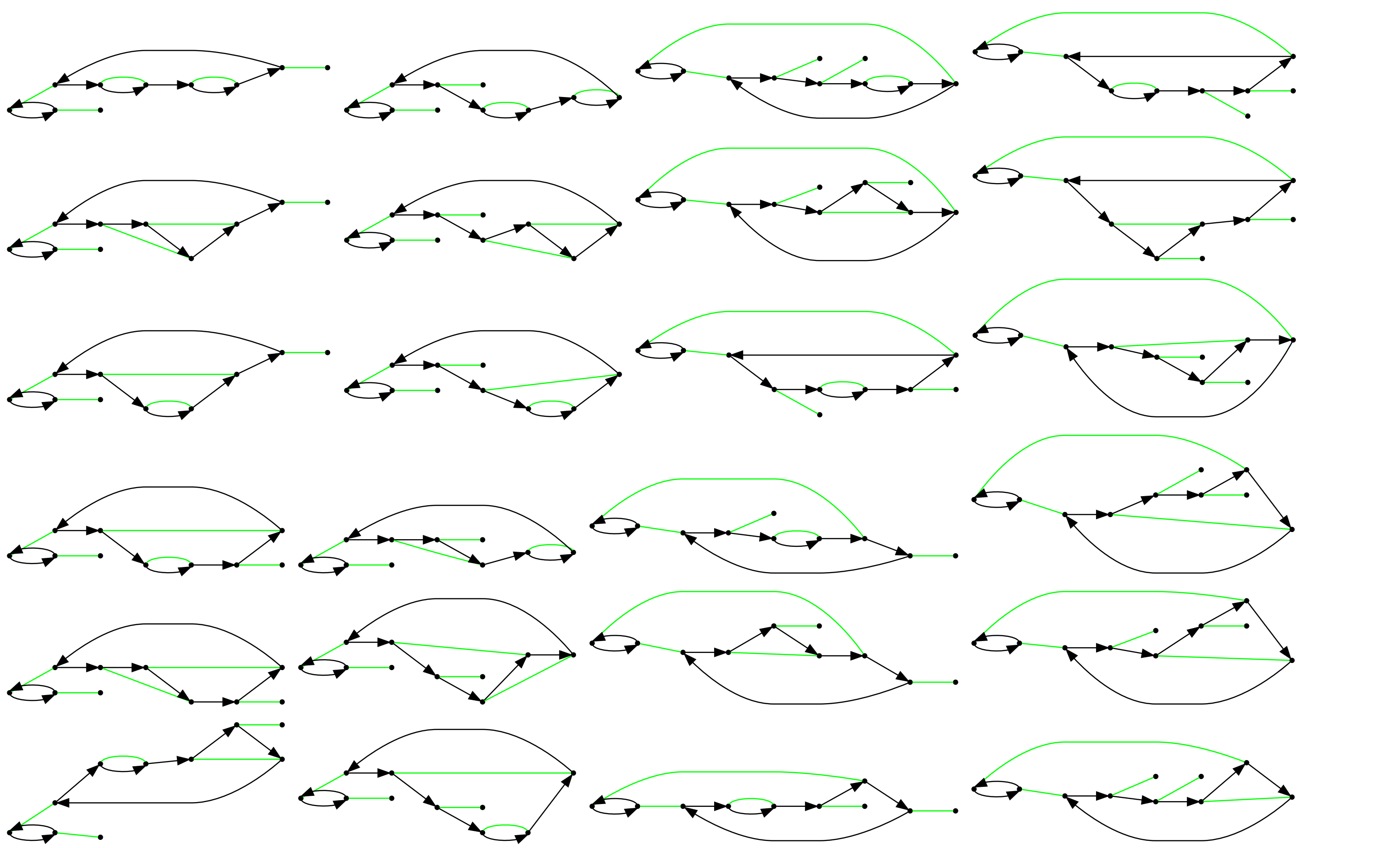


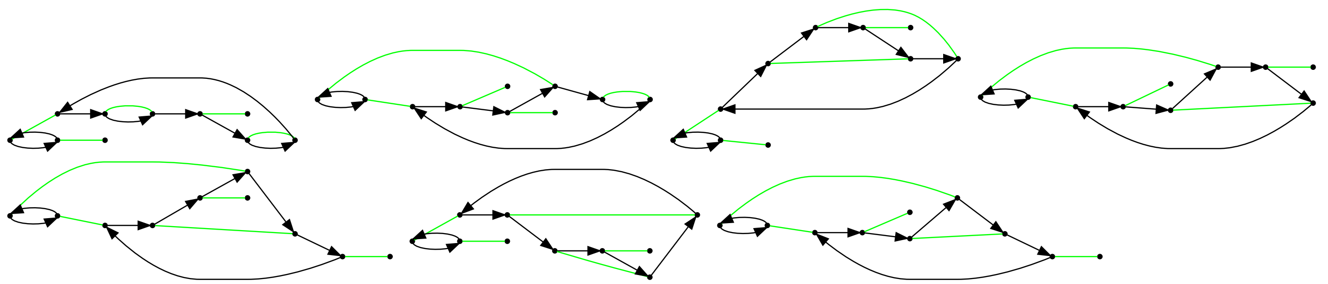
2 fermions 0 bubbles 1 tadpole (45 graphs):



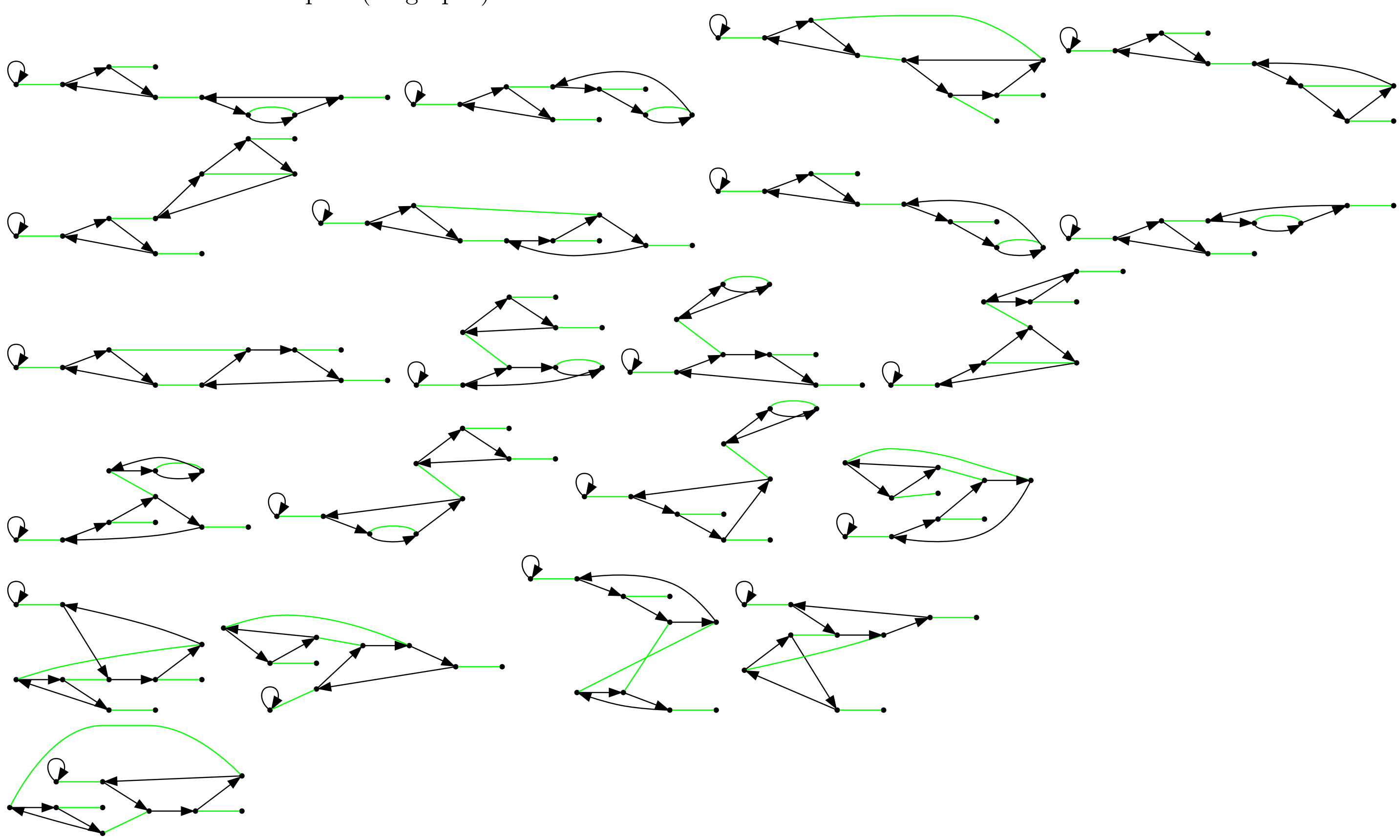


2 fermions 1 bubble 0 tadpoles (31 graphs):

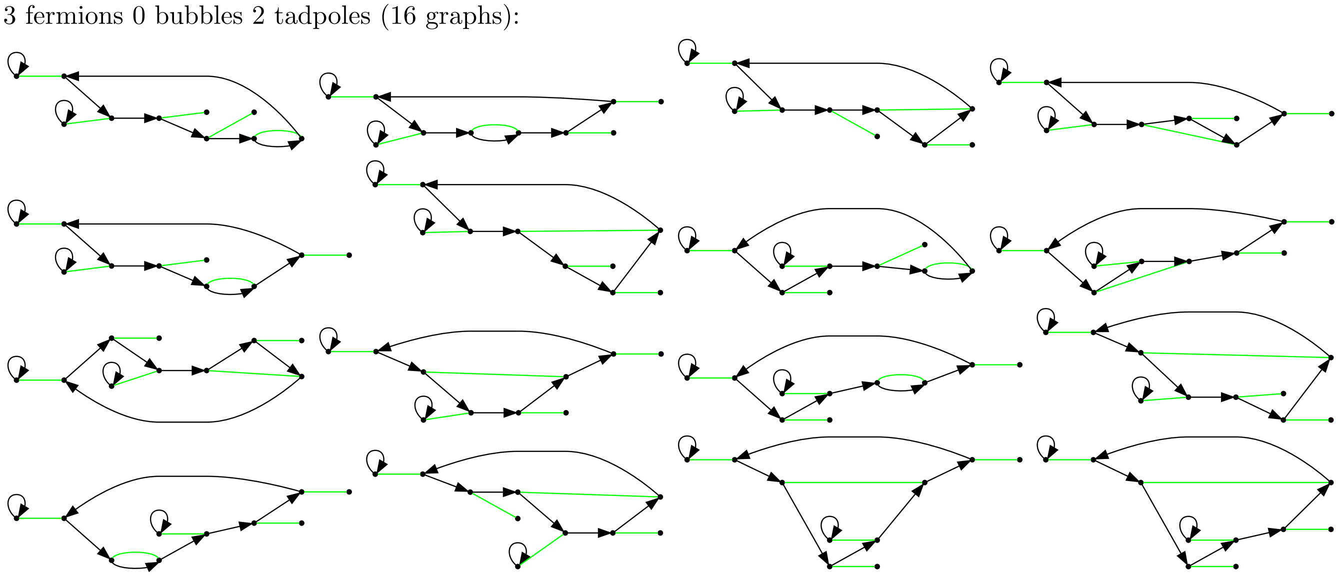




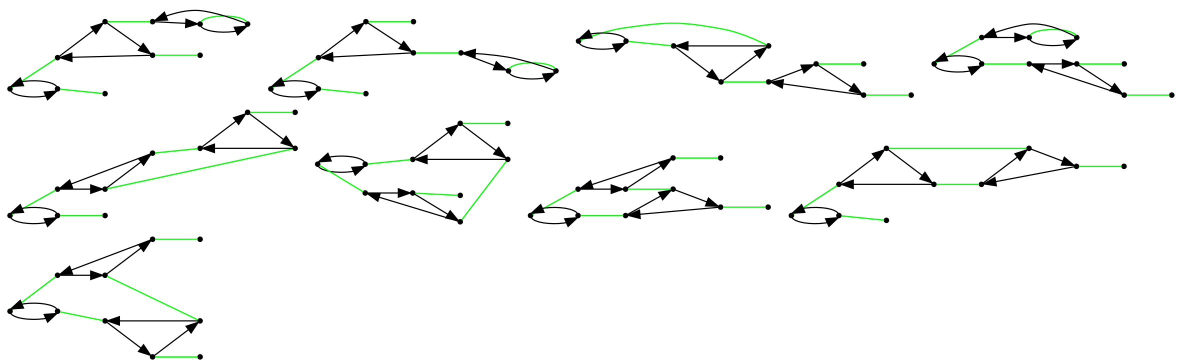
3 fermions 0 bubbles 1 tadpole (21 graphs):



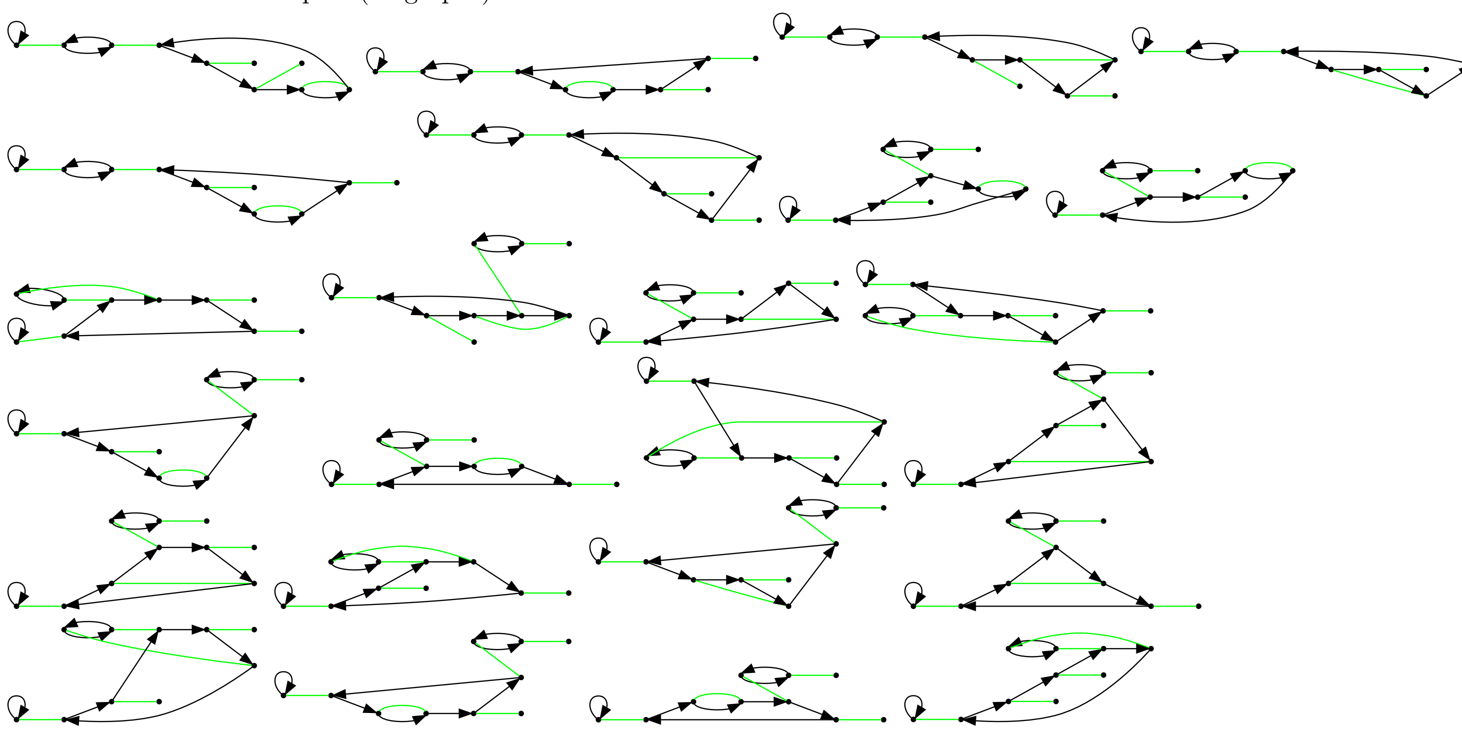
3 fermions 0 bubbles 2 tadpoles (16 graphs):



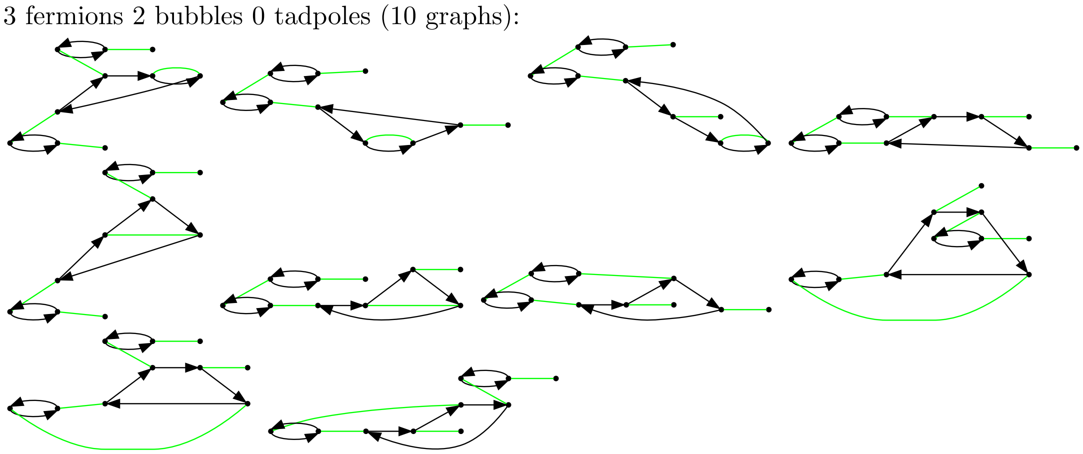
3 fermions 1 bubble 0 tadpoles (9 graphs):



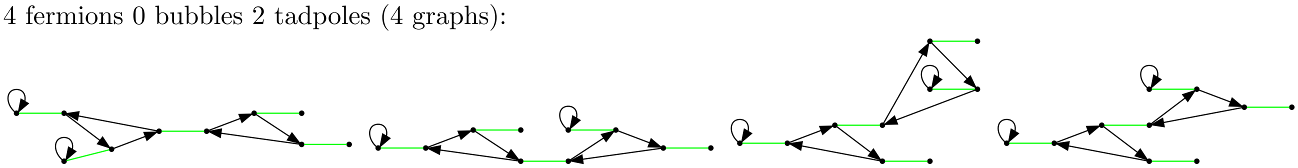
3 fermions 1 bubble 1 tadpole (24 graphs):



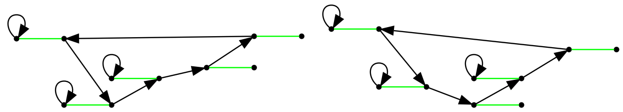
3 fermions 2 bubbles 0 tadpoles (10 graphs):



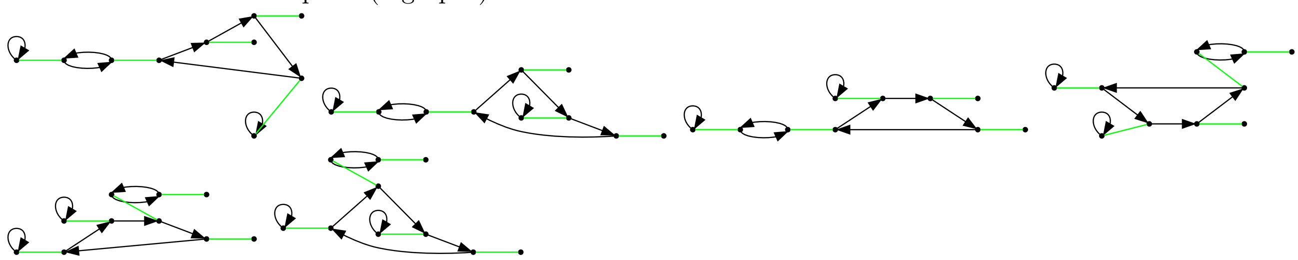
4 fermions 0 bubbles 2 tadpoles (4 graphs):



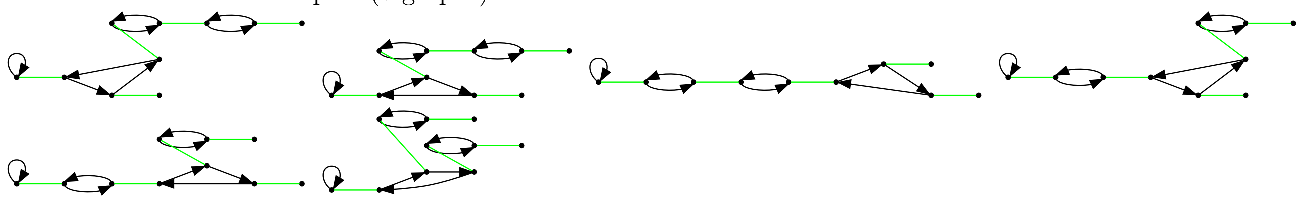
4 fermions 0 bubbles 3 tadpoles (2 graphs):



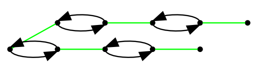
4 fermions 1 bubble 2 tadpoles (6 graphs):



4 fermions 2 bubbles 1 tadpole (6 graphs):



4 fermions 4 bubbles 0 tadpoles (1 graph):



Total: 272 graphs on 8 vertices, 148 without tadpoles, 97 with neither bubbles nor tadpoles.

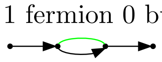
VII. GREENS FUNCTION (DIRECTED FERMIONS)

Cutting a fermion line in a diagram of Section V creates 1, 2, 10, 74, 706, 8162... Feynman diagrams of the Green's function with 0, 2, 4,... vertices [22–25][1, A000698].

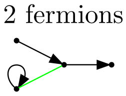
1, 1, 3, 15, 105,... of them have one fermion [1, A001147].

A. 2 vertices

1 fermion 0 bubbles 0 tadpoles (1 graph):



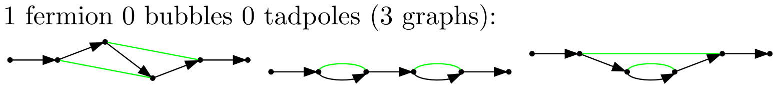
2 fermions 0 bubbles 1 tadpole (1 graph):



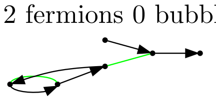
Total: 2 graphs on 2 vertices, 1 without tadpoles, 1 with neither bubbles nor tadpoles.

B. 4 vertices

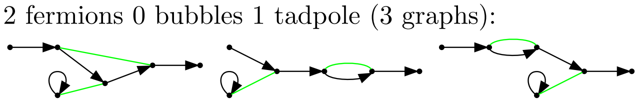
1 fermion 0 bubbles 0 tadpoles (3 graphs):



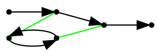
2 fermions 0 bubbles 0 tadpoles (1 graph):



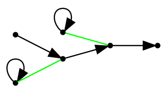
2 fermions 0 bubbles 1 tadpole (3 graphs):



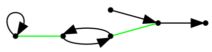
2 fermions 1 bubble 0 tadpoles (1 graph):



3 fermions 0 bubbles 2 tadpoles (1 graph):



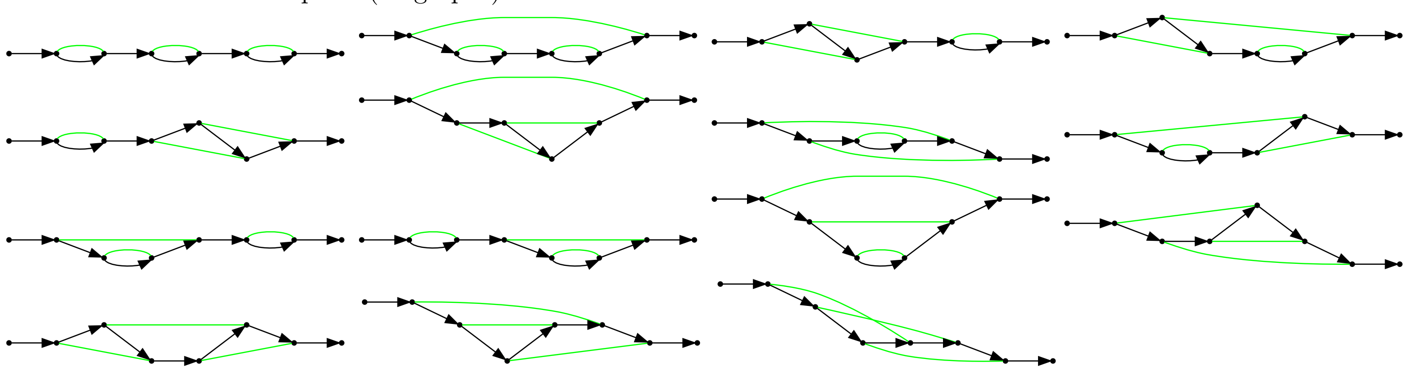
3 fermions 1 bubble 1 tadpole (1 graph):



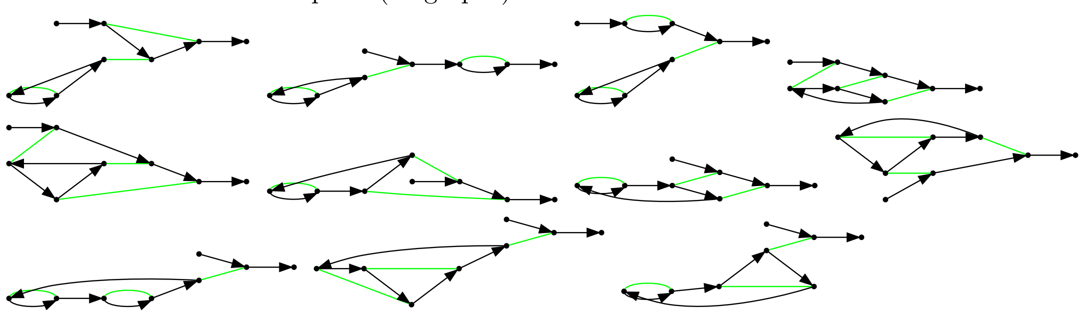
Total: 10 graphs on 4 vertices, 5 without tadpoles, 4 with neither bubbles nor tadpoles.

C. 6 vertices

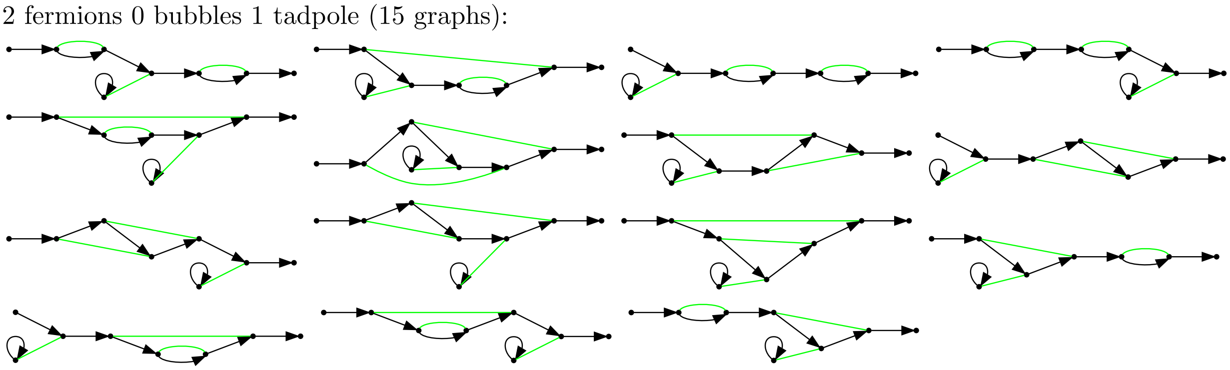
1 fermion 0 bubbles 0 tadpoles (15 graphs):



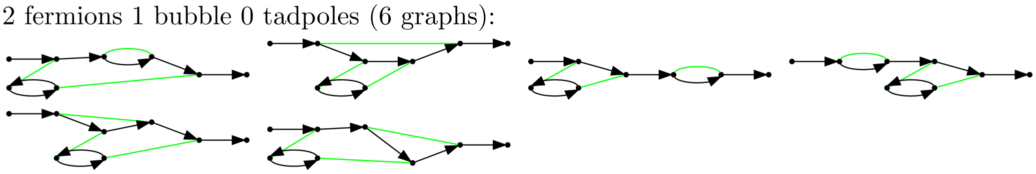
2 fermions 0 bubbles 0 tadpoles (11 graphs):



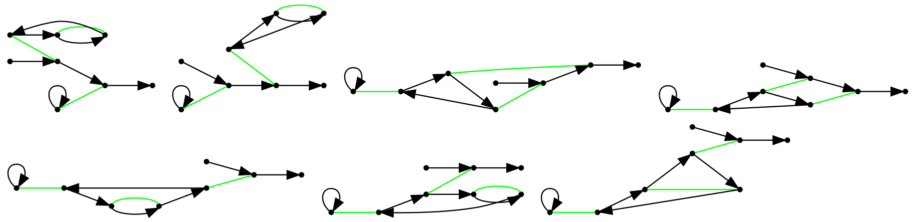
2 fermions 0 bubbles 1 tadpole (15 graphs):



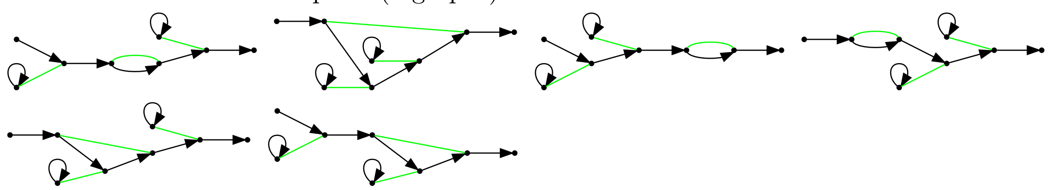
2 fermions 1 bubble 0 tadpoles (6 graphs):



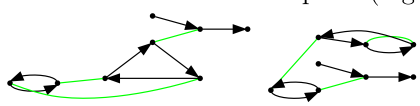
3 fermions 0 bubbles 1 tadpole (7 graphs):



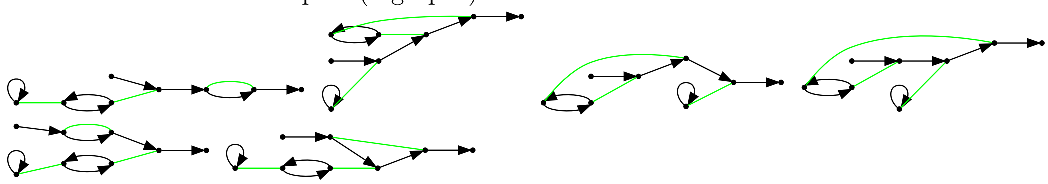
3 fermions 0 bubbles 2 tadpoles (6 graphs):



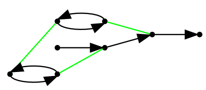
3 fermions 1 bubble 0 tadpoles (2 graphs):



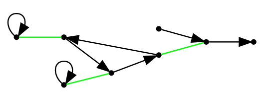
3 fermions 1 bubble 1 tadpole (6 graphs):



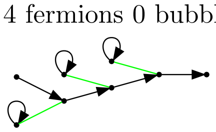
3 fermions 2 bubbles 0 tadpoles (1 graph):



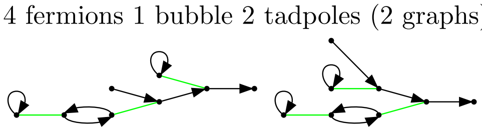
4 fermions 0 bubbles 2 tadpoles (1 graph):



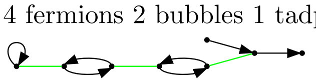
4 fermions 0 bubbles 3 tadpoles (1 graph):



4 fermions 1 bubble 2 tadpoles (2 graphs):



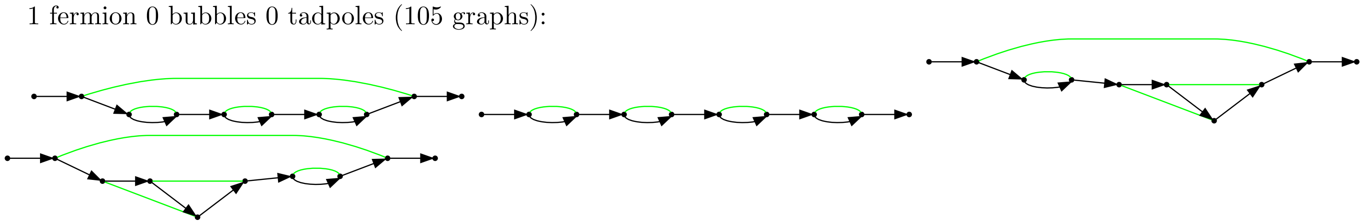
4 fermions 2 bubbles 1 tadpole (1 graph):

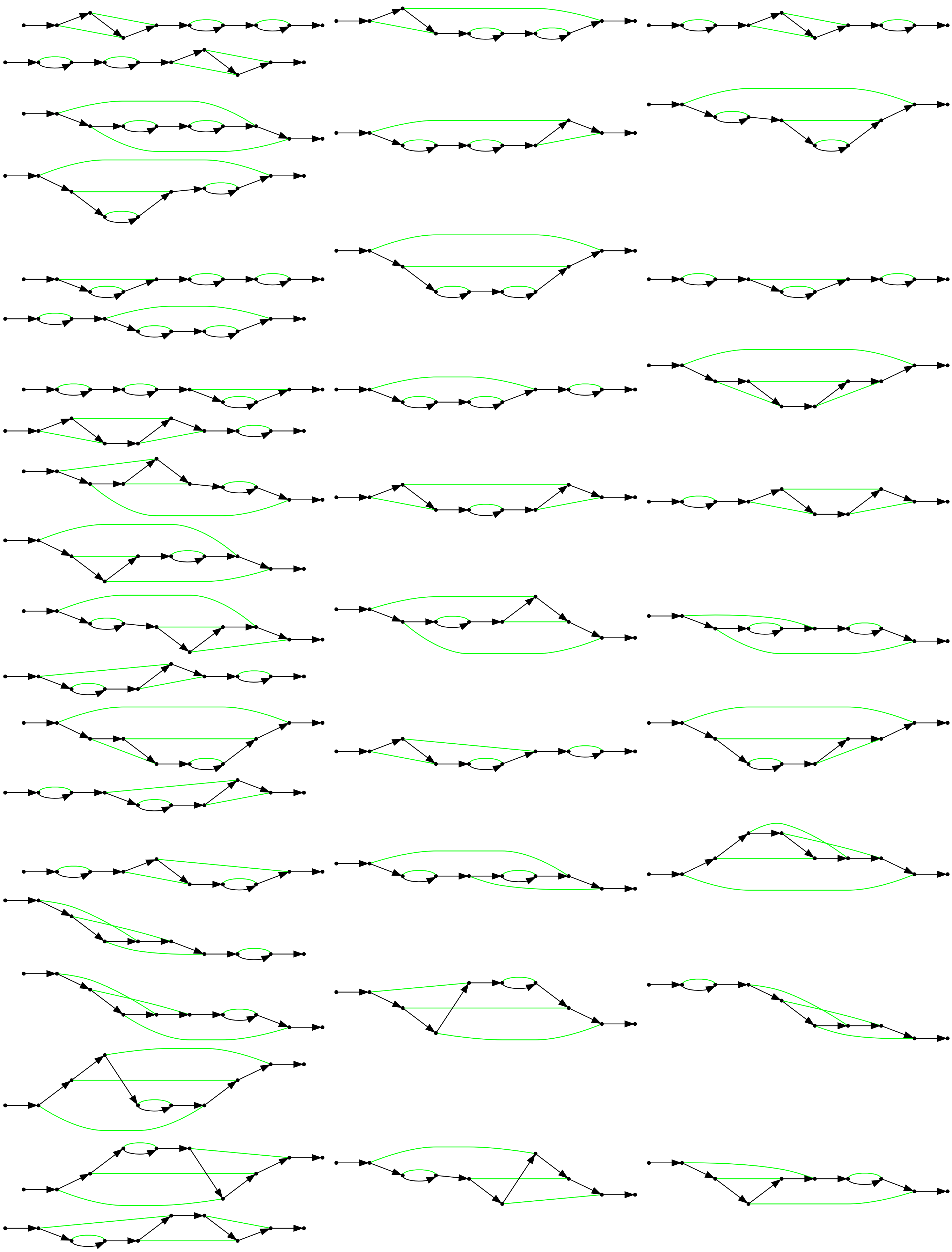


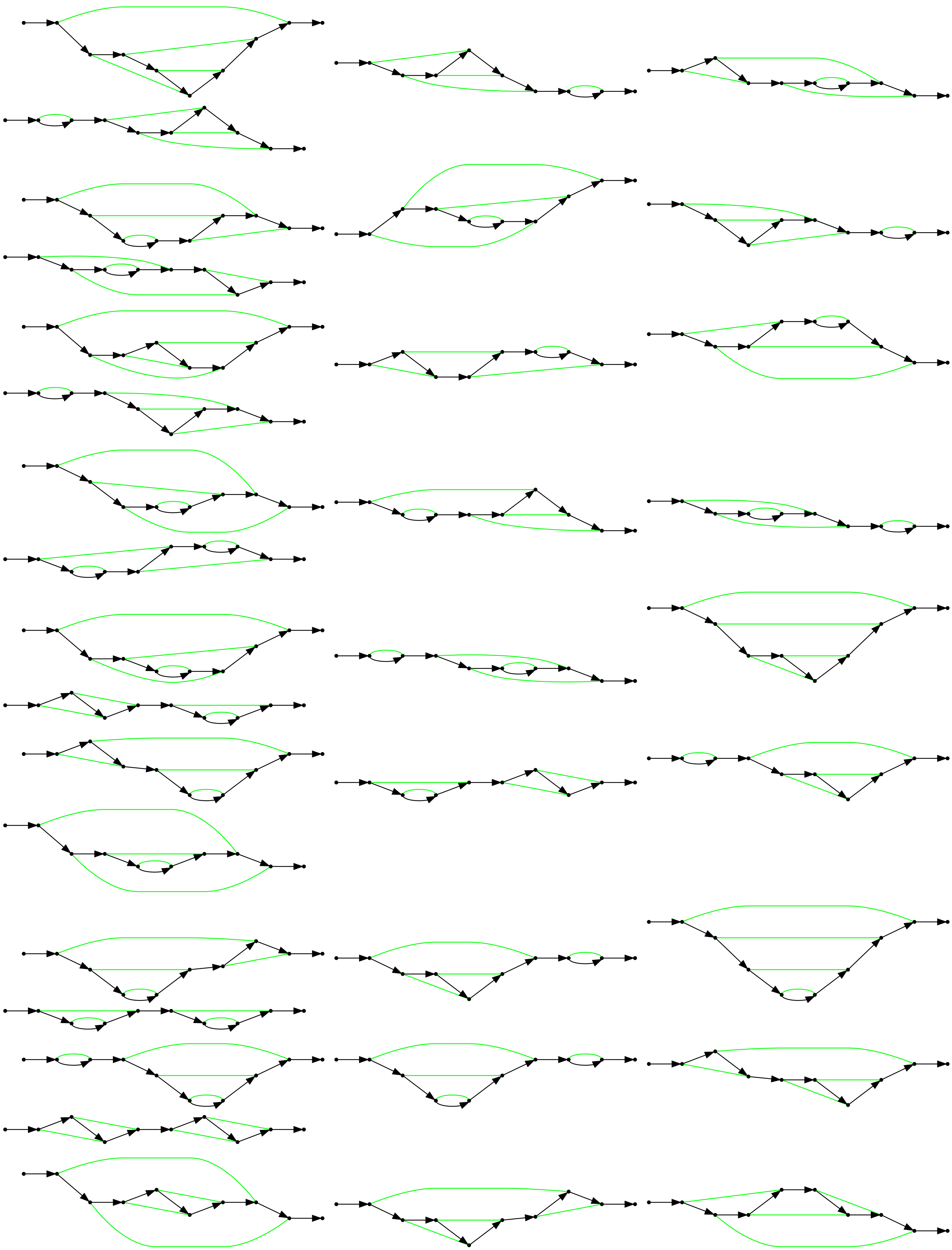
Total: 74 graphs on 6 vertices, 35 without tadpoles, 26 with neither bubbles nor tadpoles.

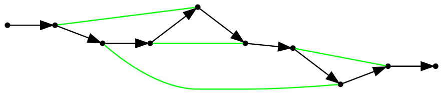
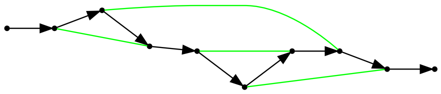
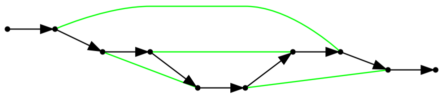
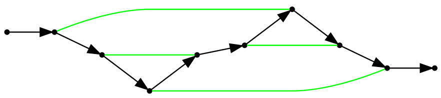
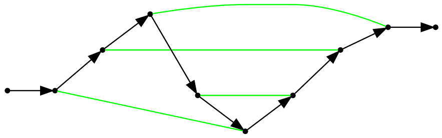
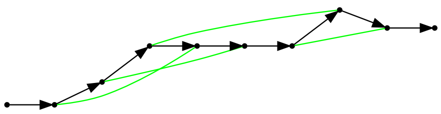
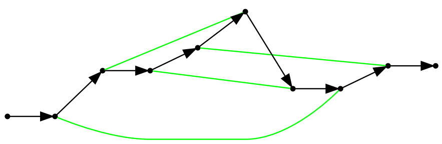
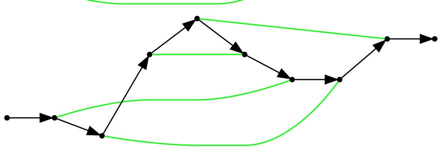
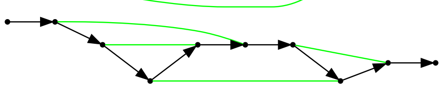
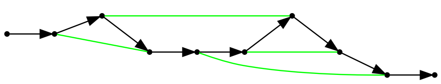
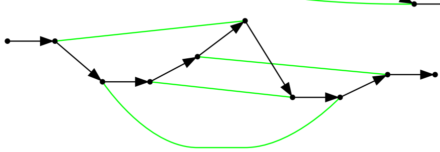
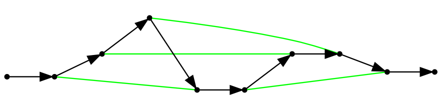
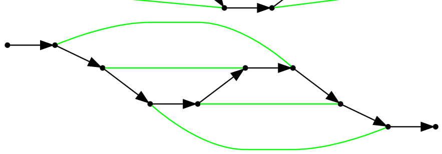
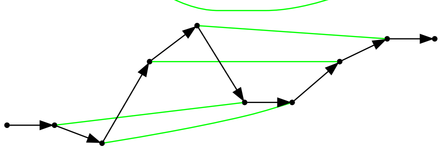
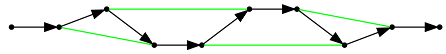
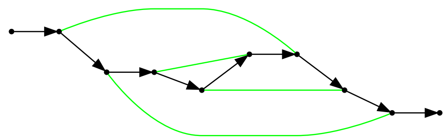
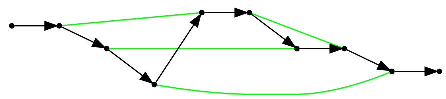
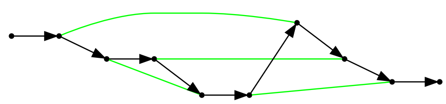
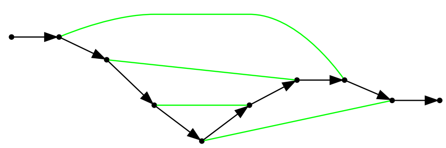
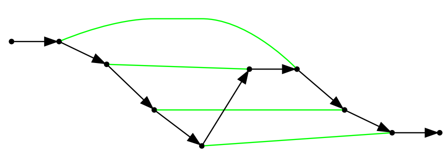
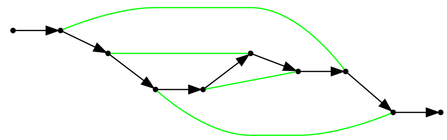
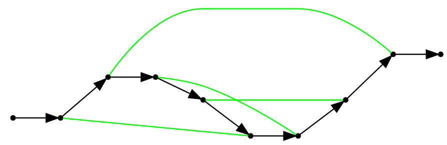
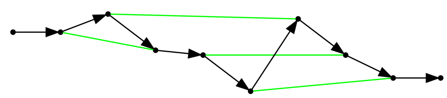
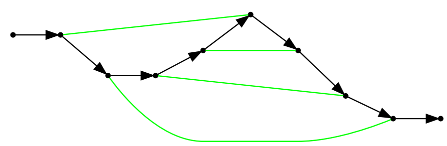
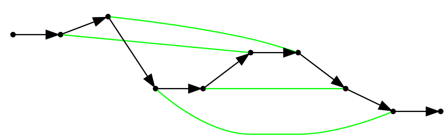
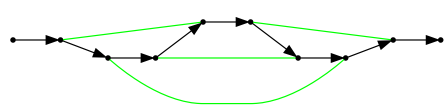
D. 8 vertices

1 fermion 0 bubbles 0 tadpoles (105 graphs):

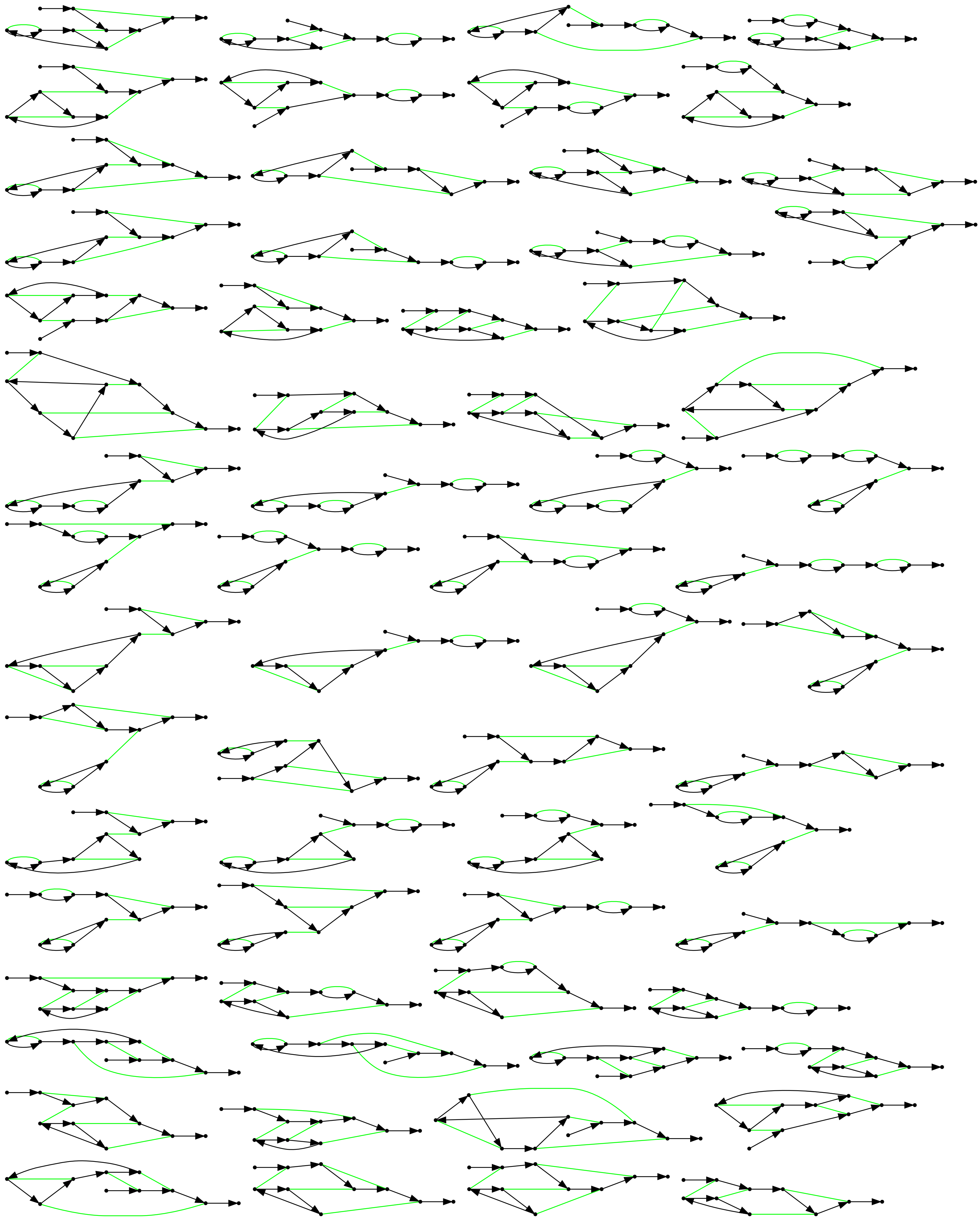


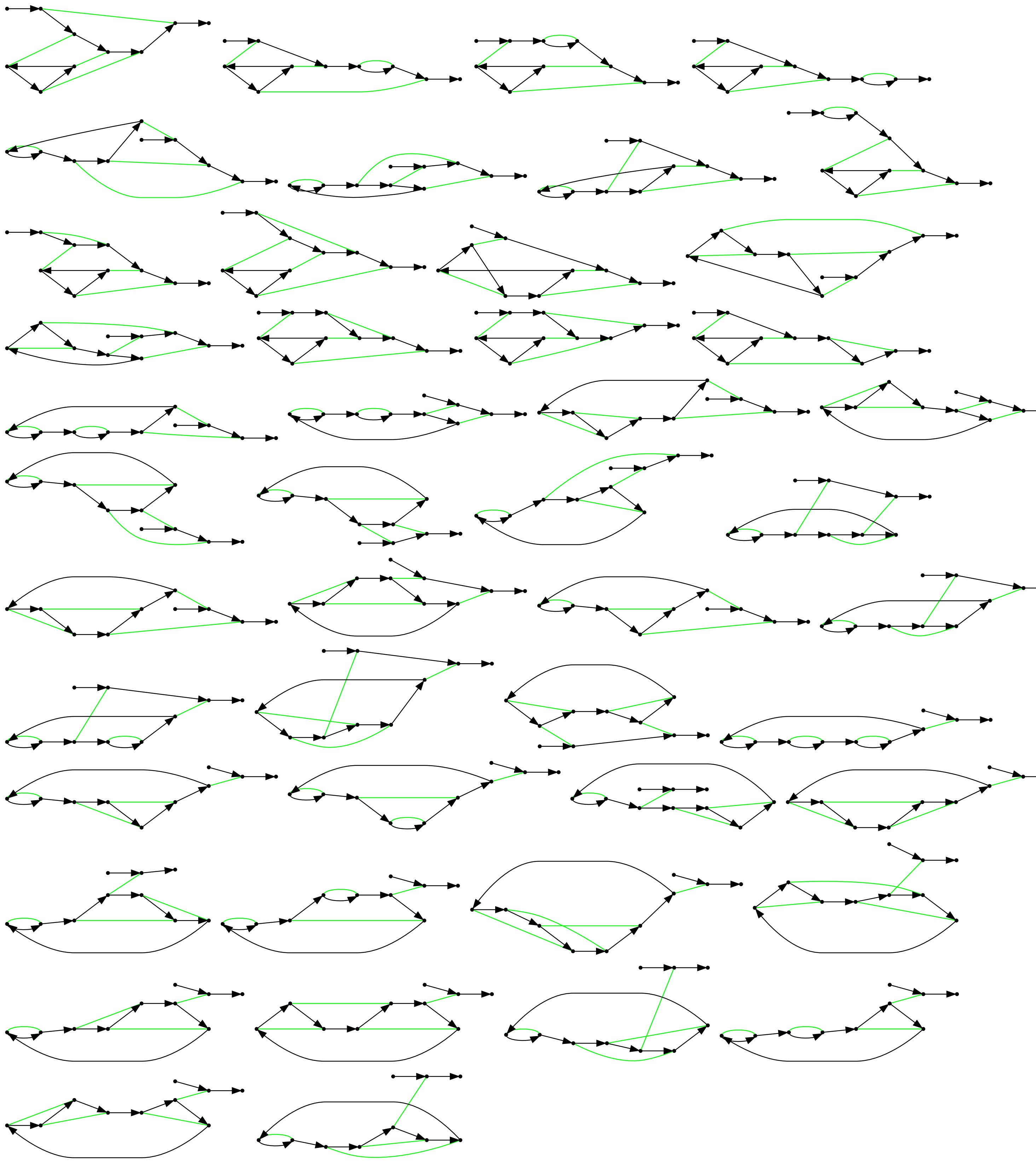




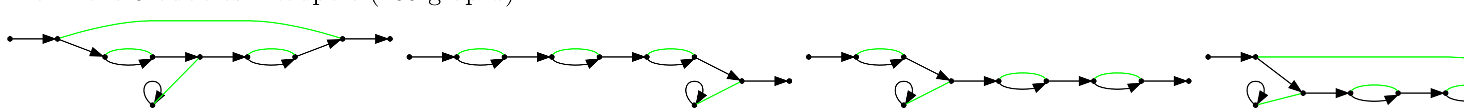



























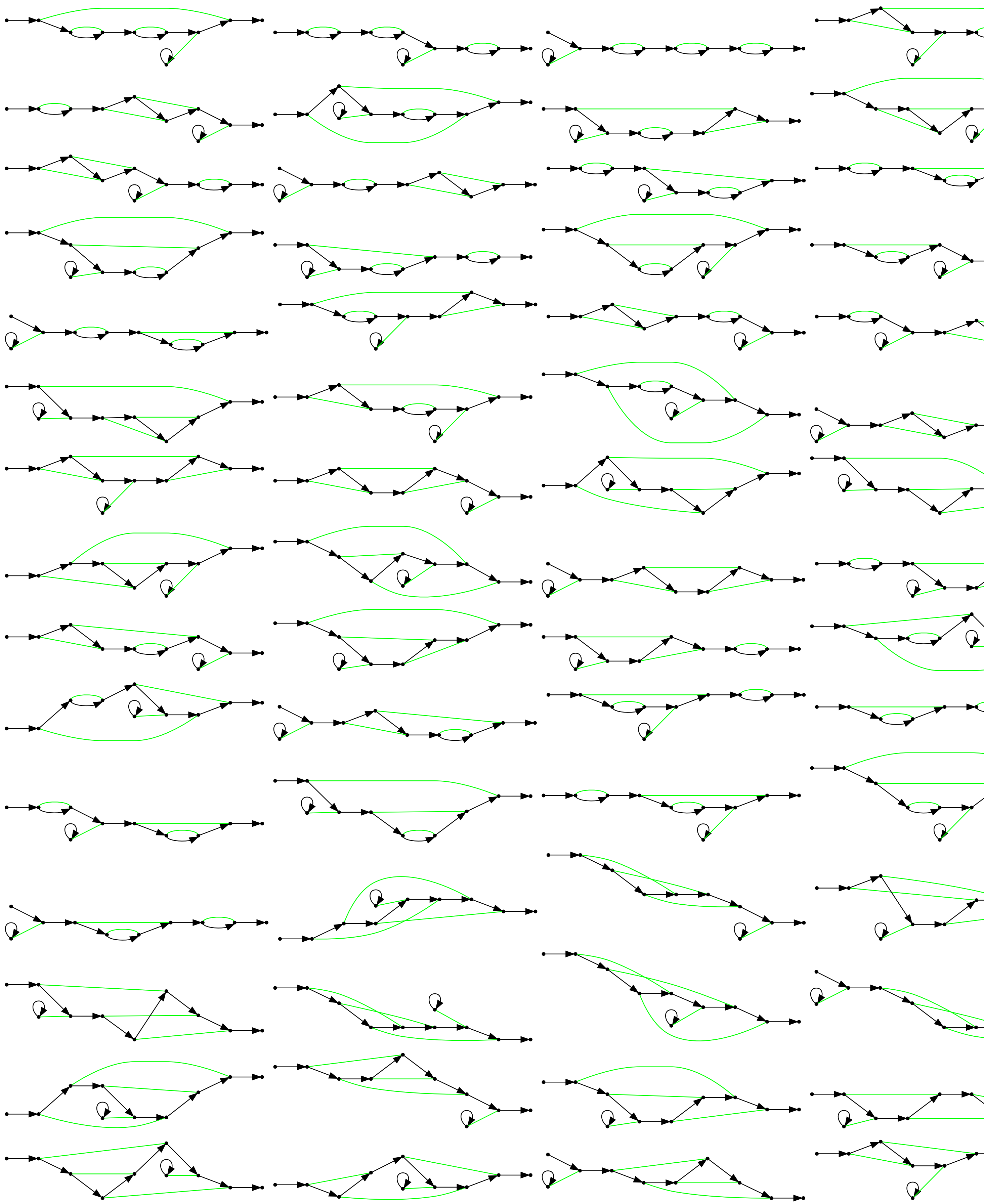
2 fermions 0 bubbles 0 tadpoles (110 graphs):

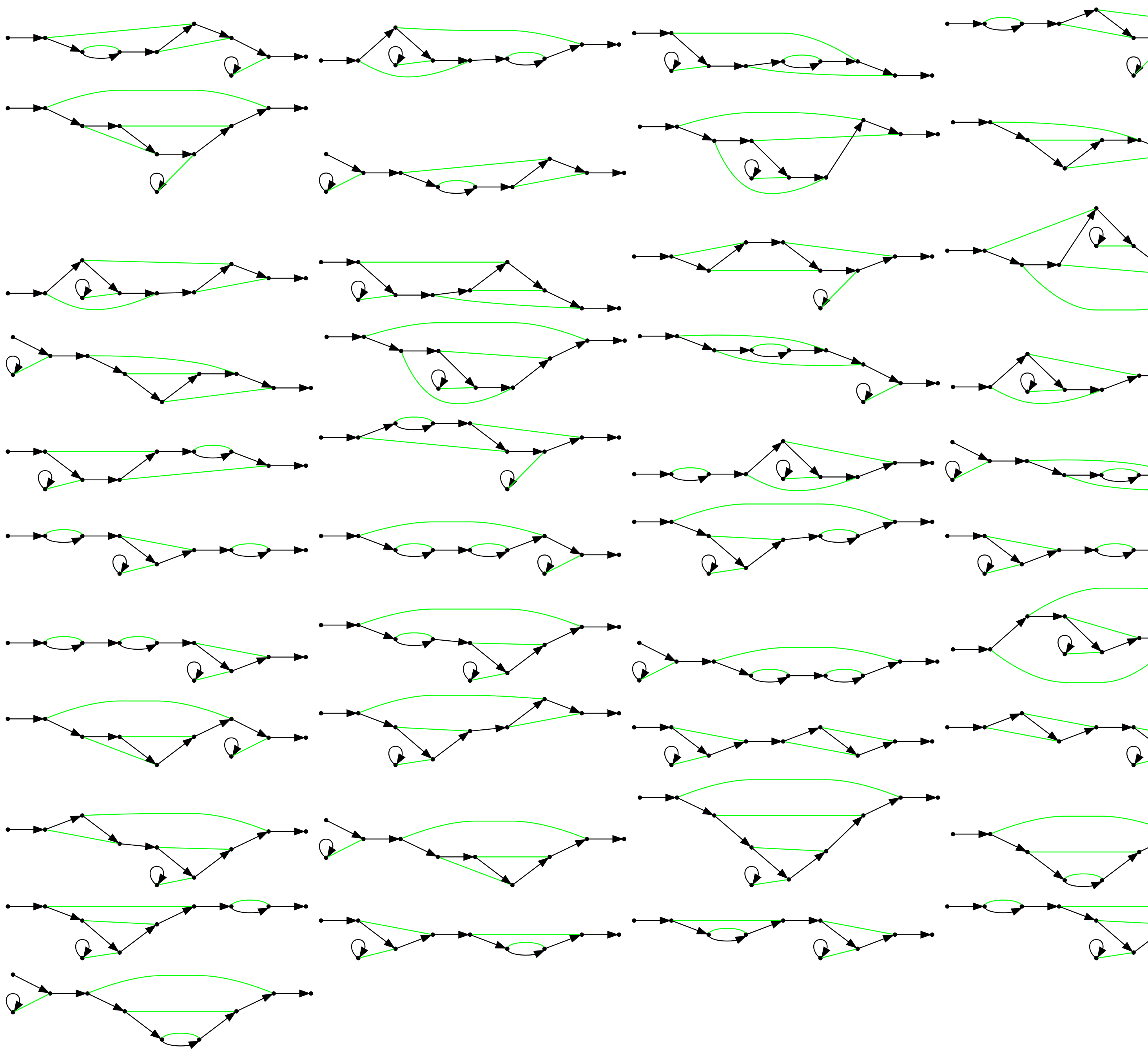




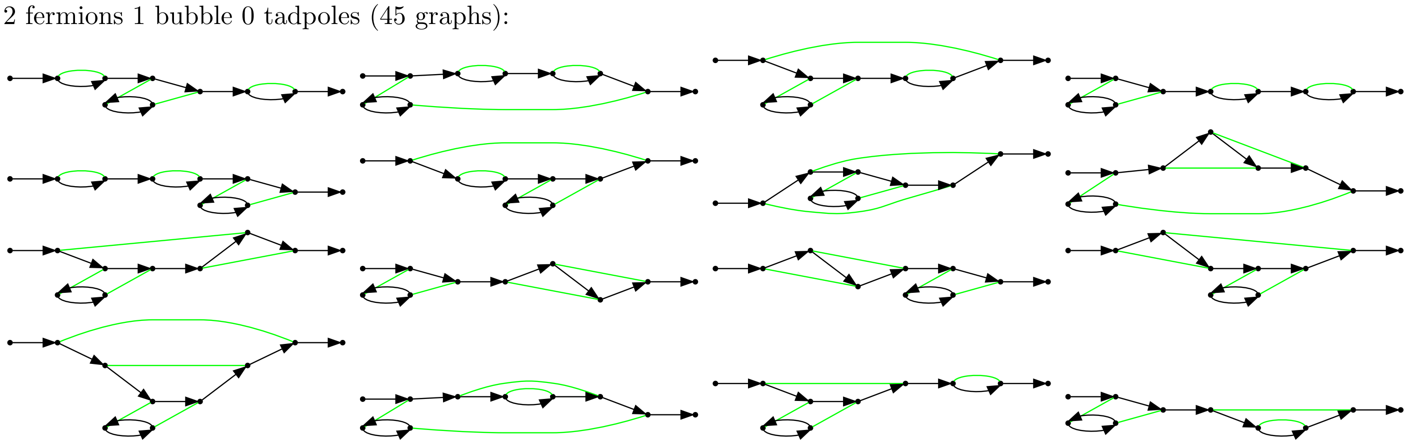
2 fermions 0 bubbles 1 tadpole (105 graphs):

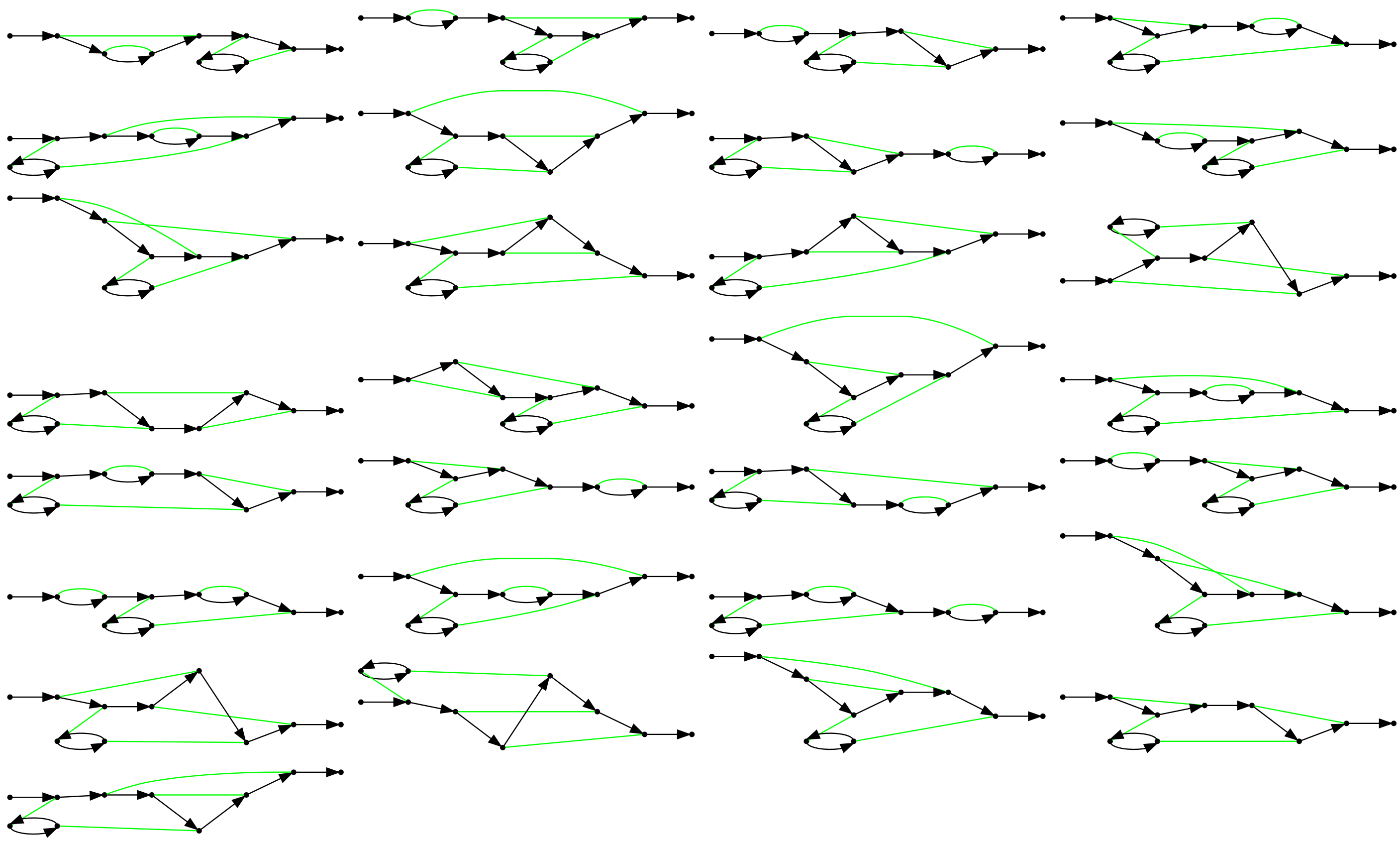




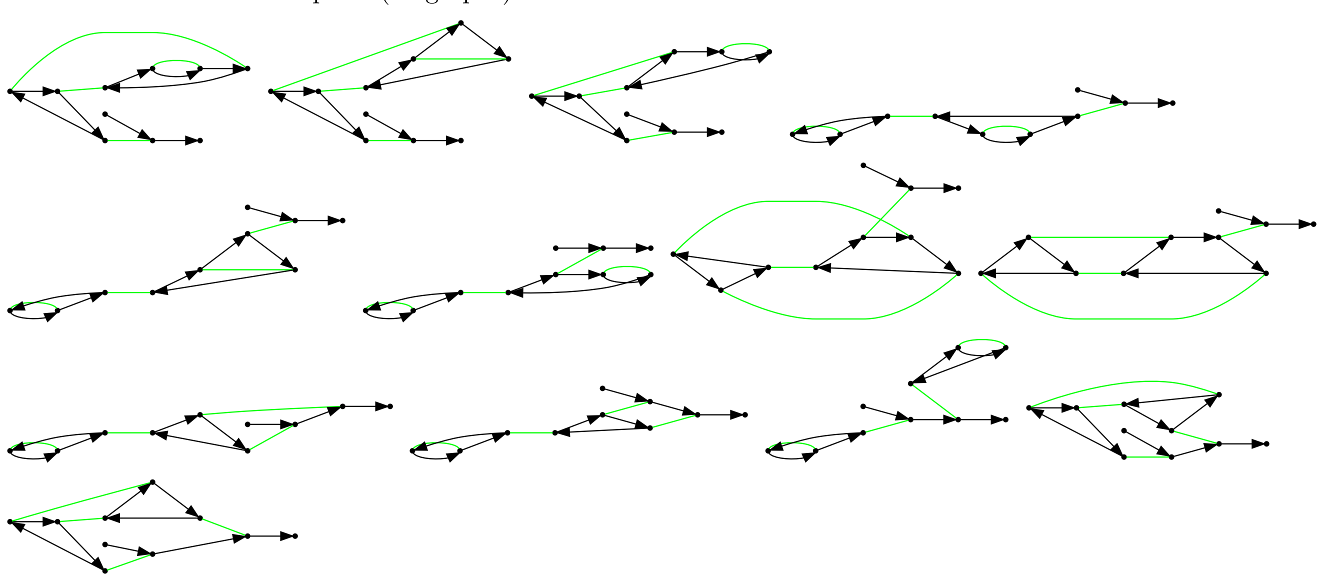


2 fermions 1 bubble 0 tadpoles (45 graphs):

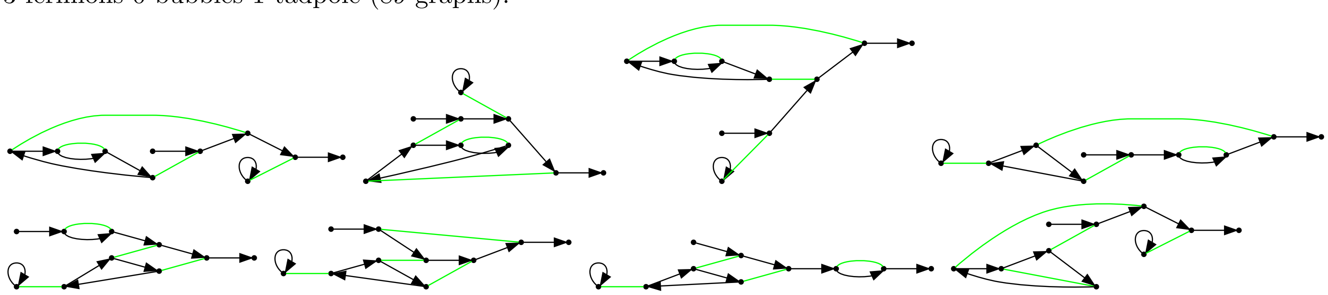


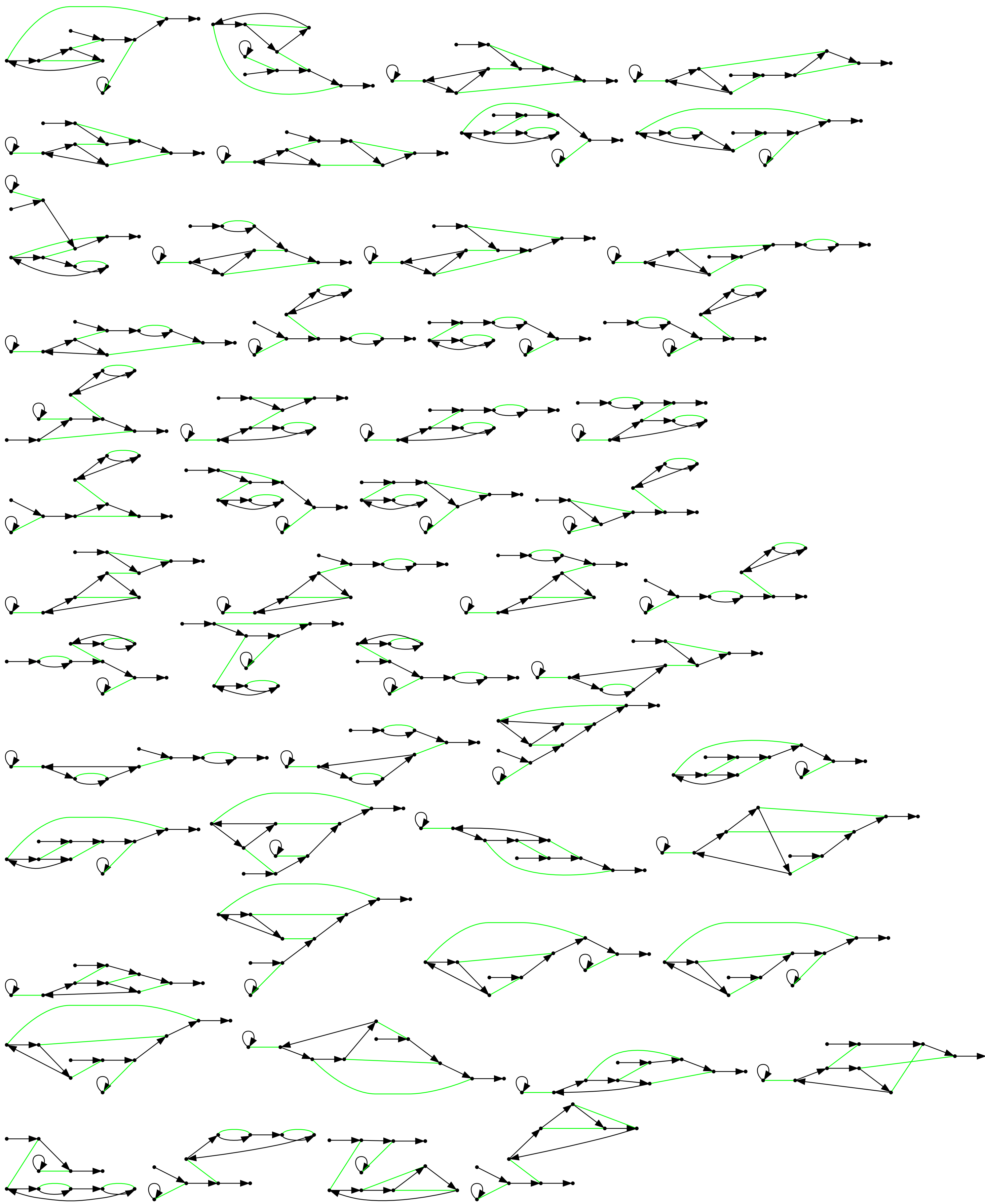


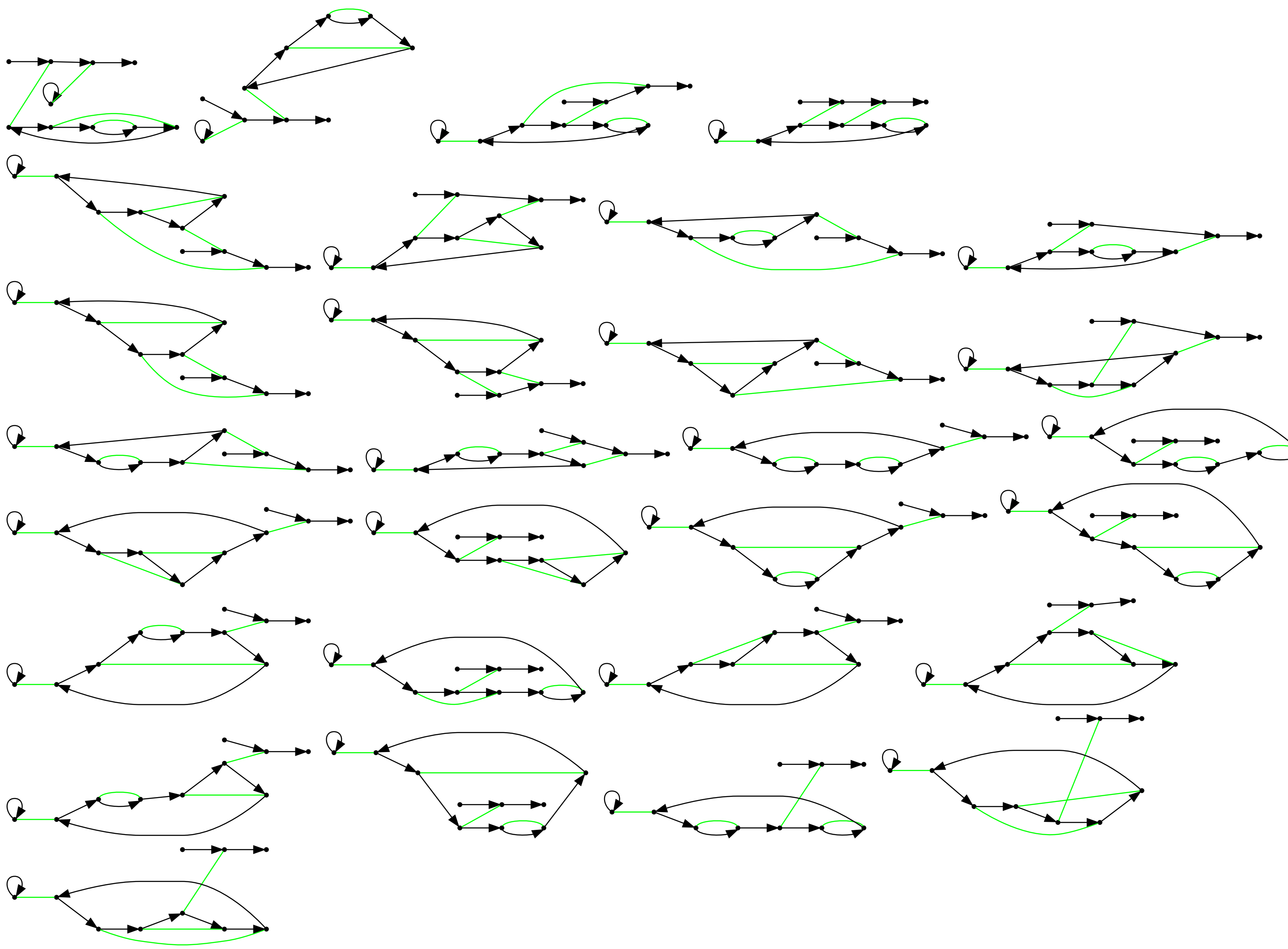
3 fermions 0 bubbles 0 tadpoles (13 graphs):



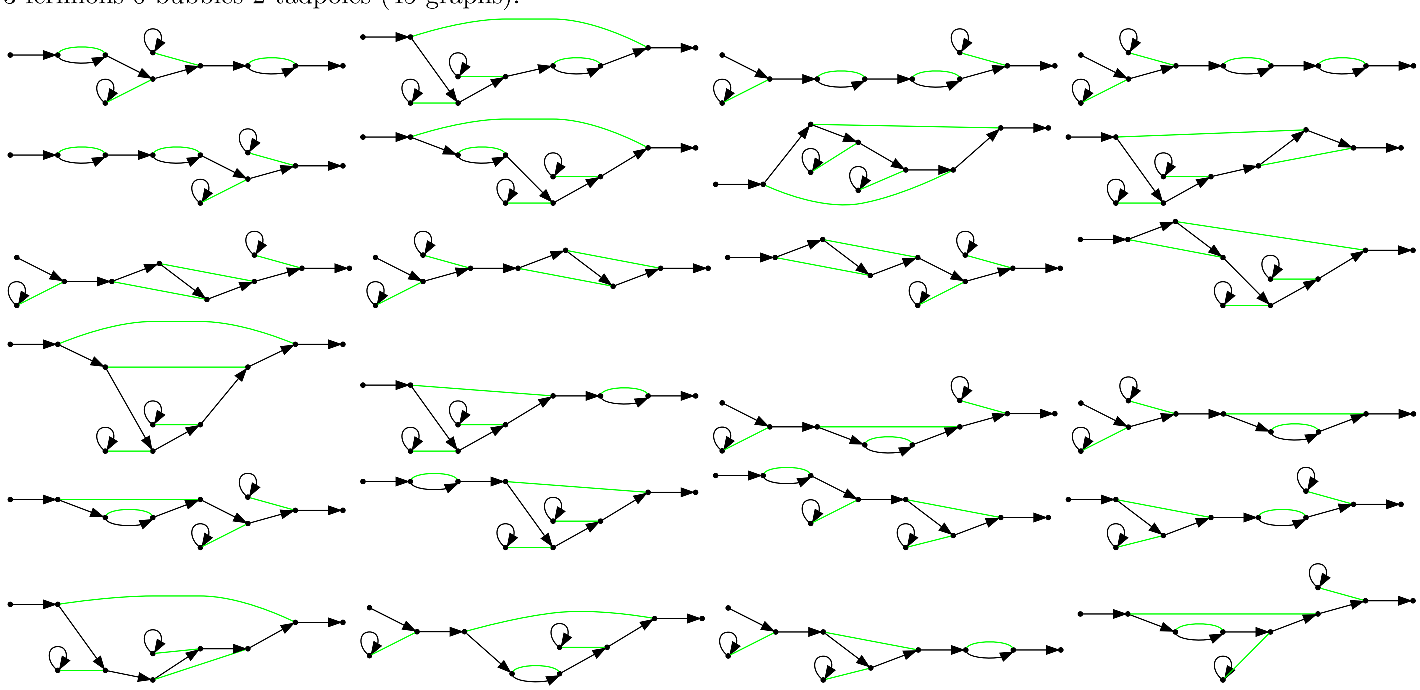
3 fermions 0 bubbles 1 tadpole (89 graphs):

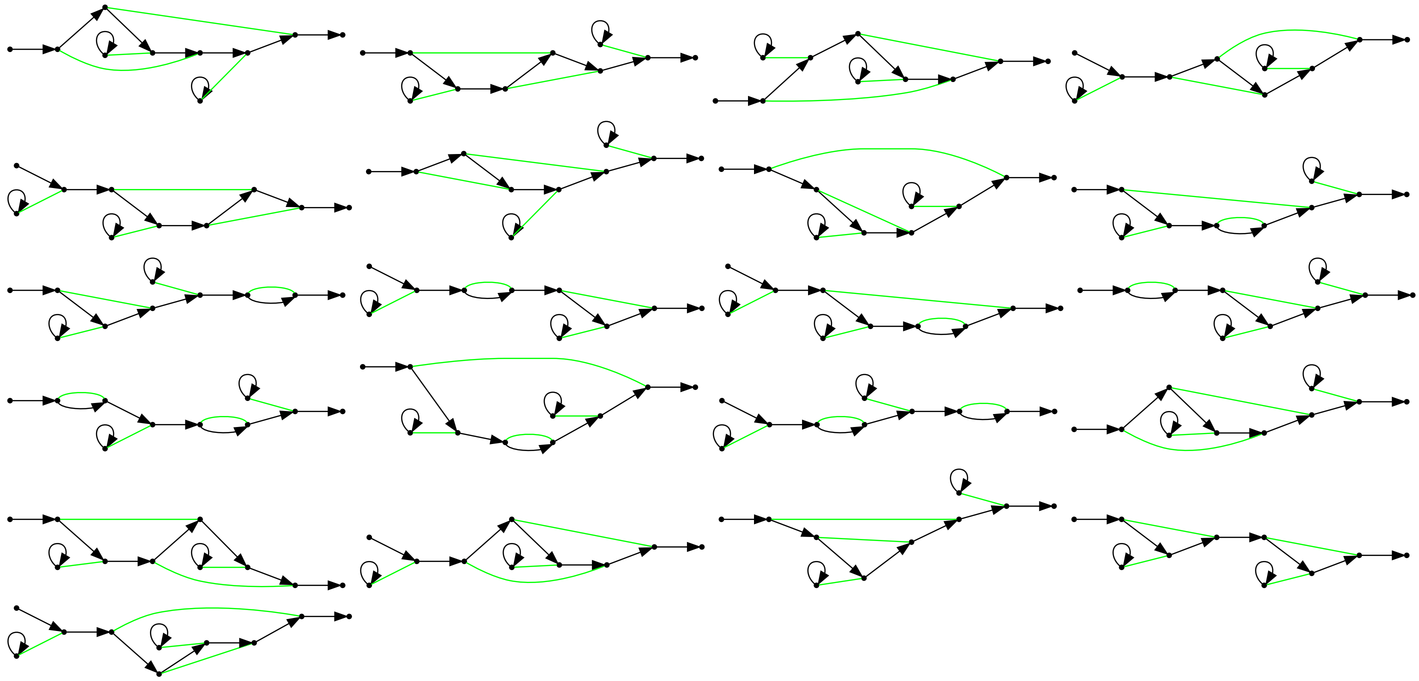




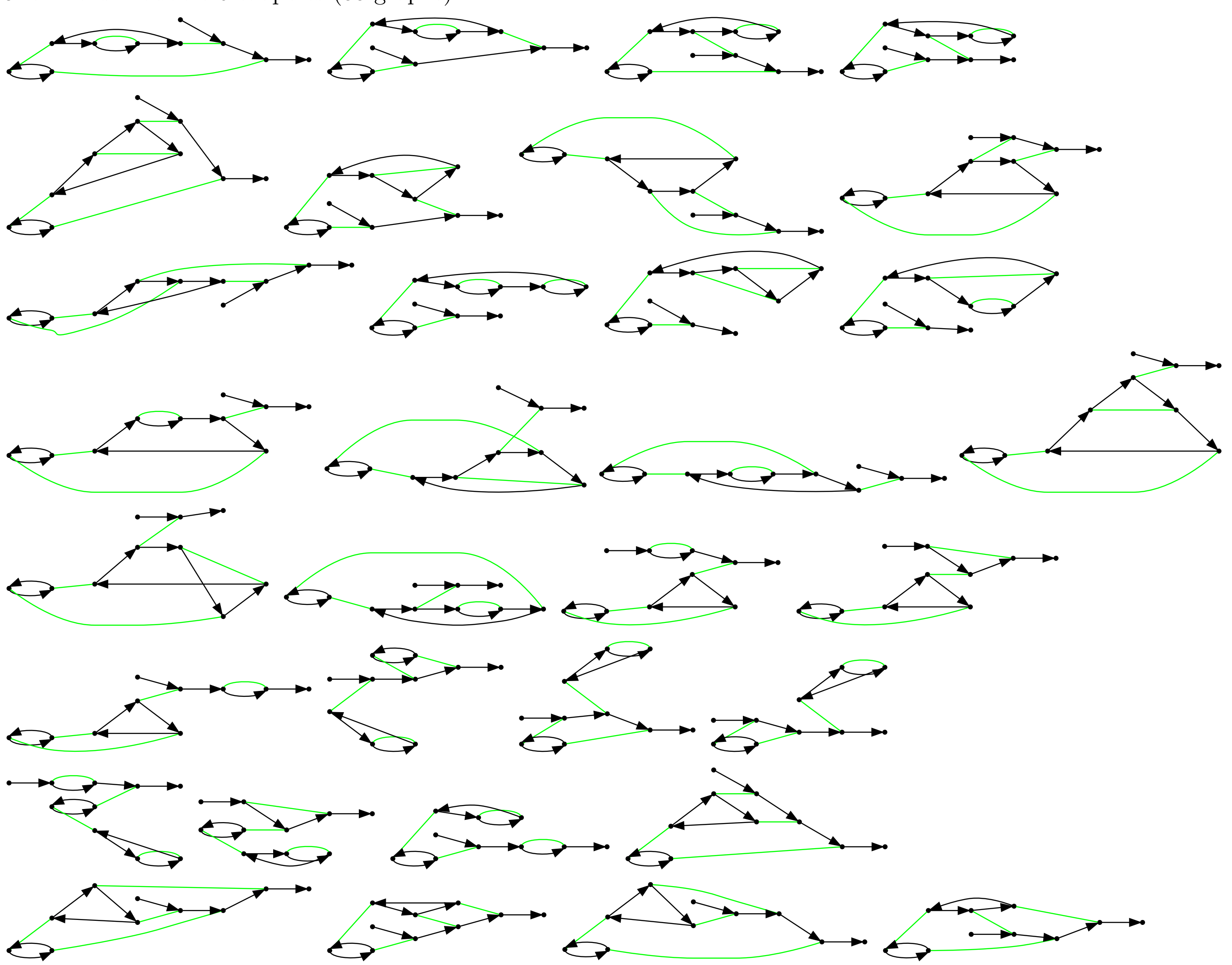


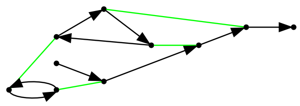
3 fermions 0 bubbles 2 tadpoles (45 graphs):



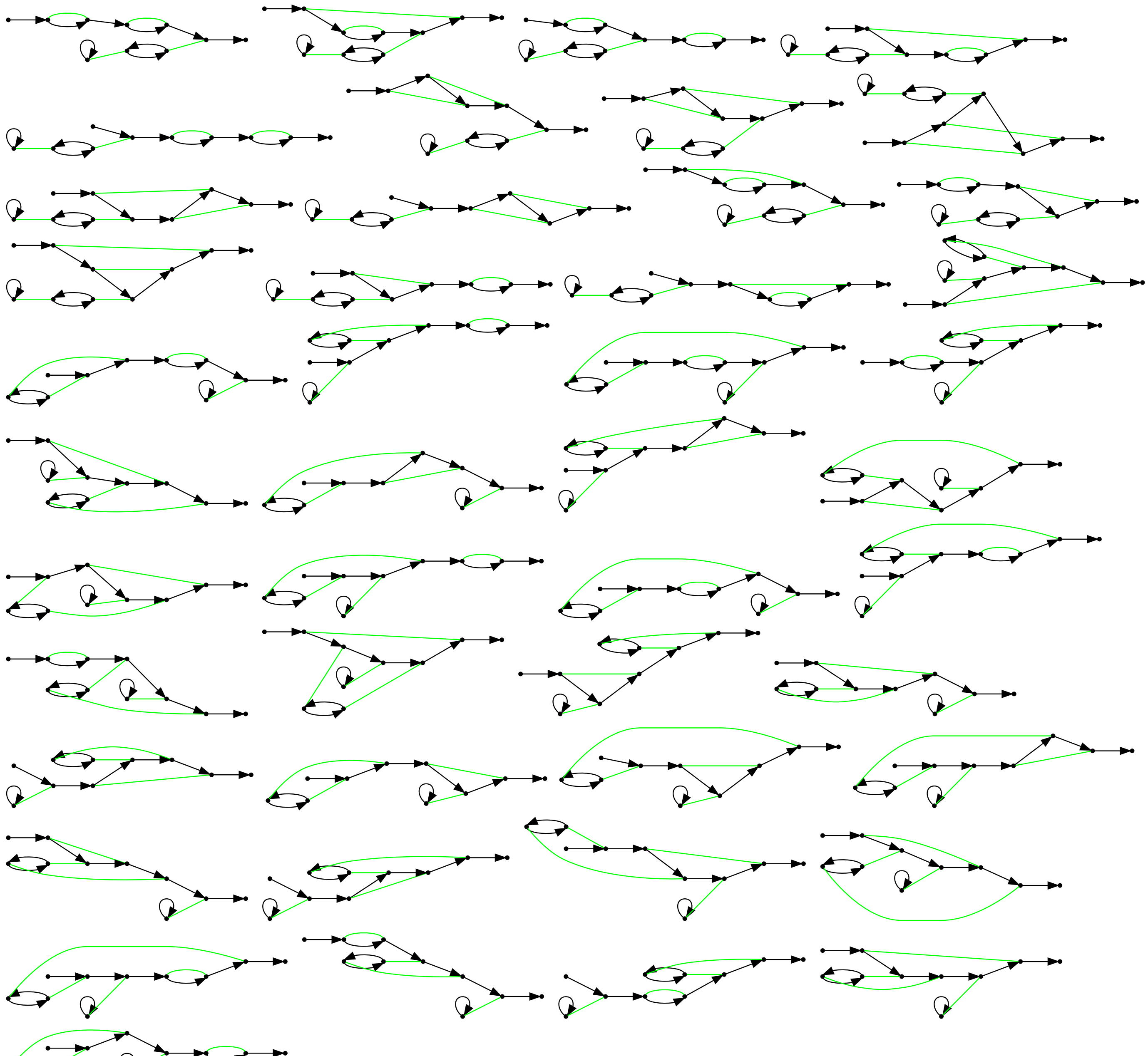


3 fermions 1 bubble 0 tadpoles (33 graphs):

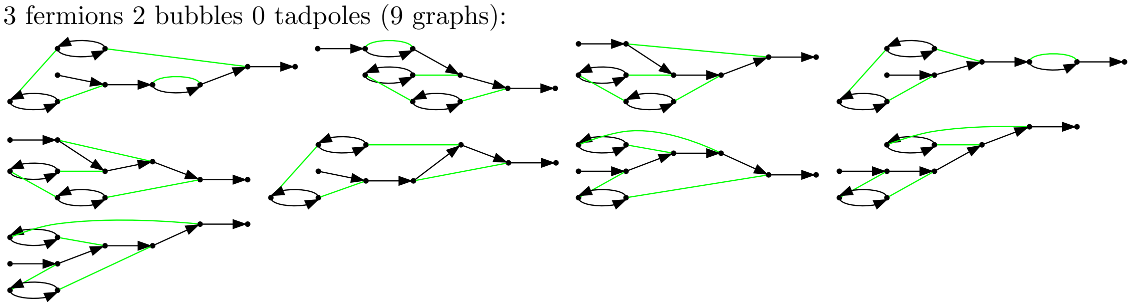




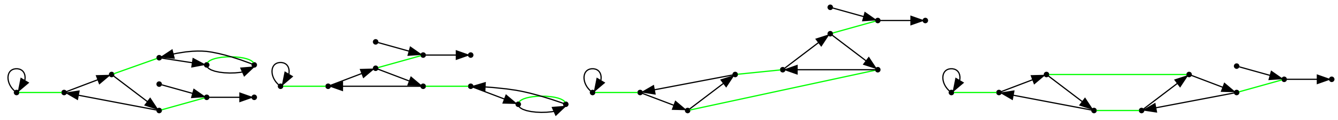
3 fermions 1 bubble 1 tadpole (45 graphs):



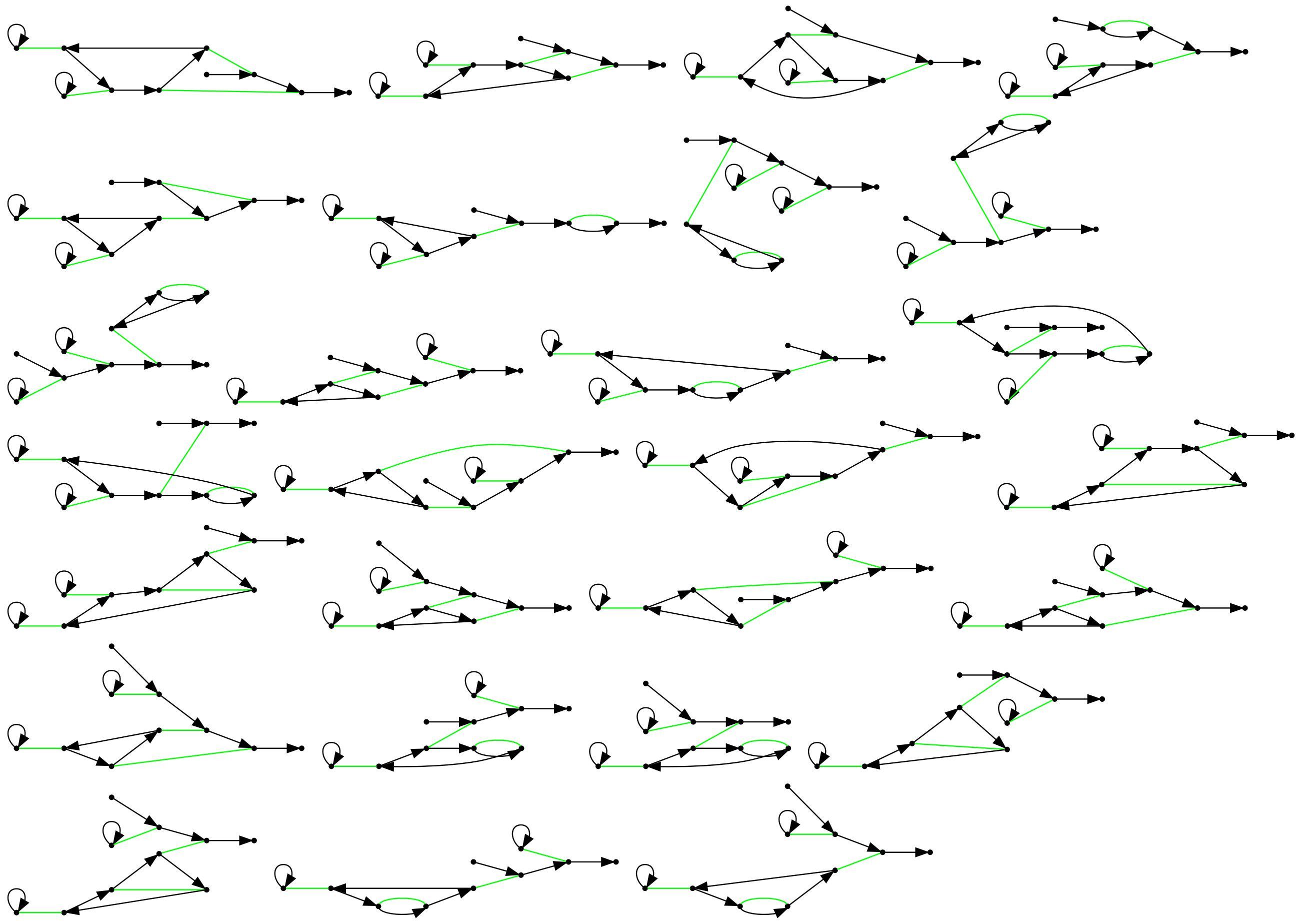
3 fermions 2 bubbles 0 tadpoles (9 graphs):



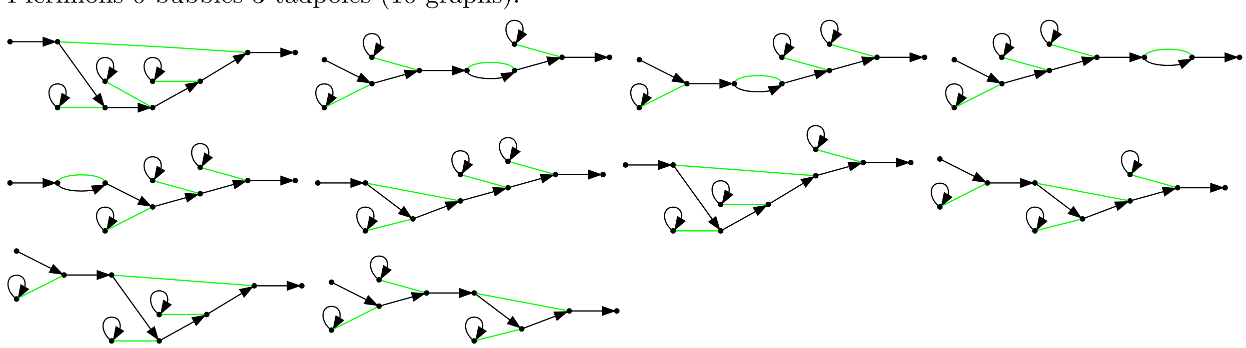
4 fermions 0 bubbles 1 tadpole (4 graphs):



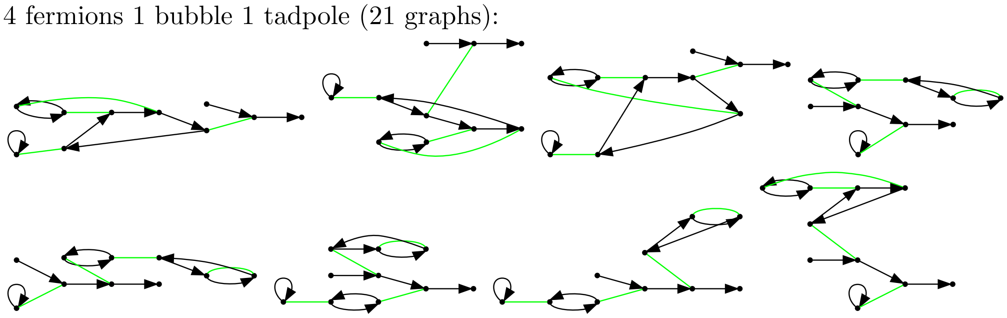
4 fermions 0 bubbles 2 tadpoles (27 graphs):

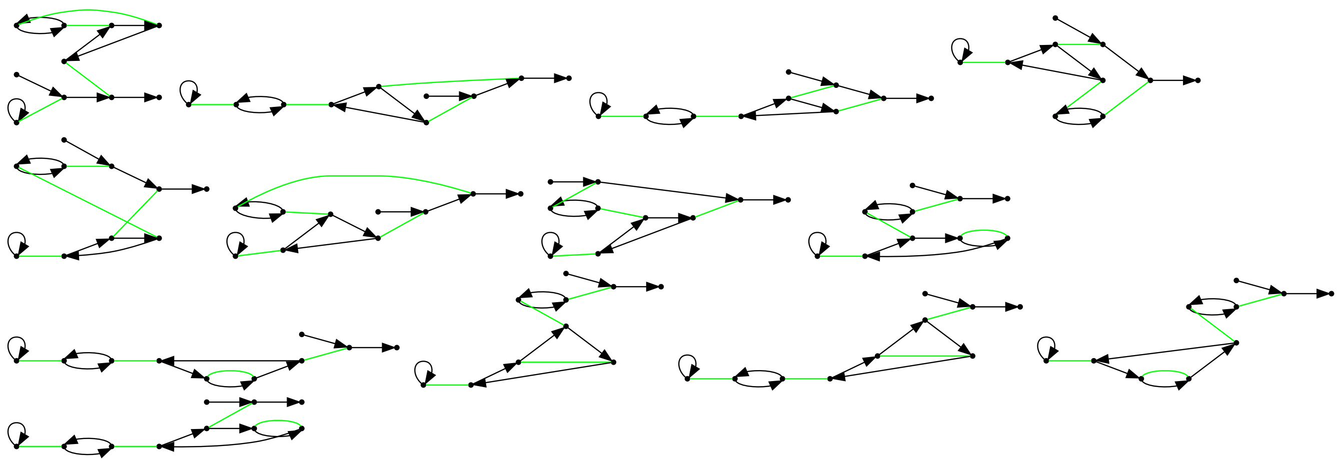


4 fermions 0 bubbles 3 tadpoles (10 graphs):

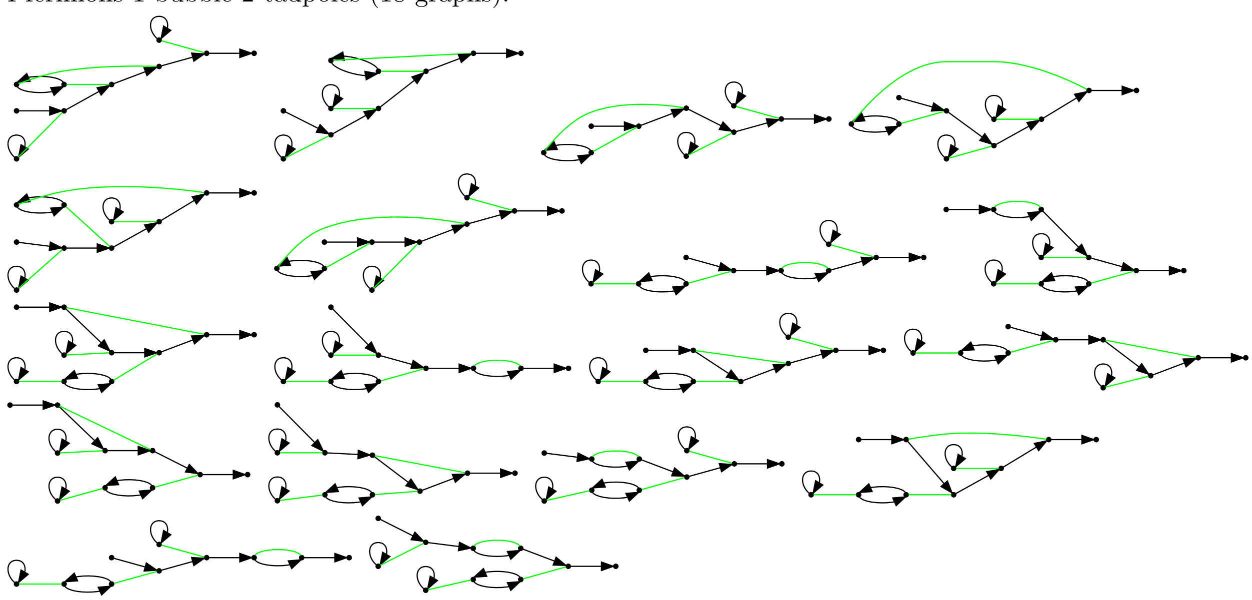


4 fermions 1 bubble 1 tadpole (21 graphs):

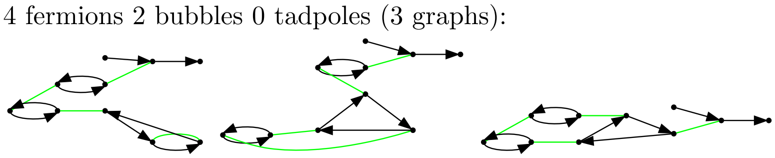




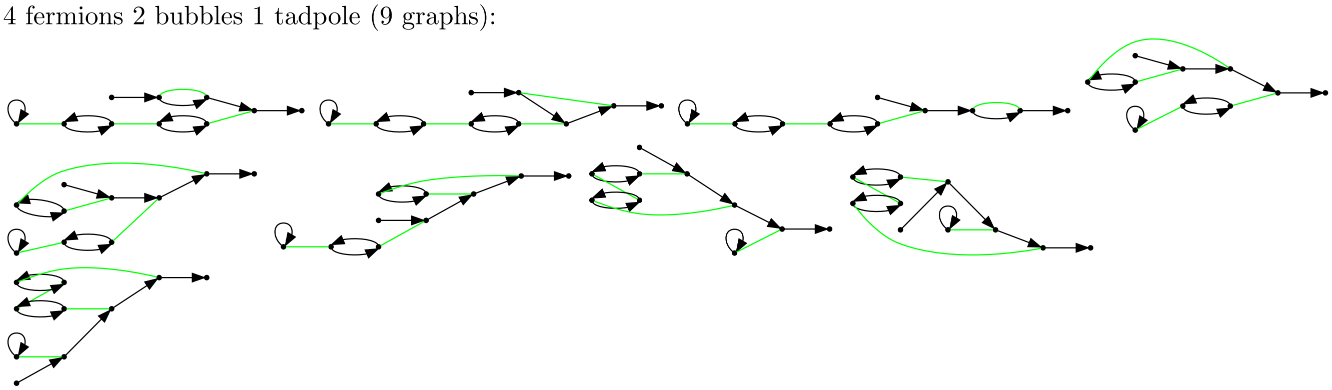
4 fermions 1 bubble 2 tadpoles (18 graphs):



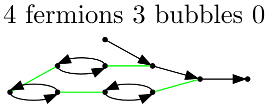
4 fermions 2 bubbles 0 tadpoles (3 graphs):



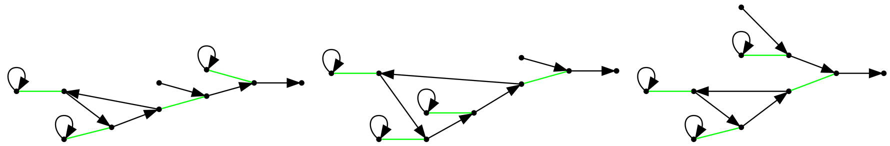
4 fermions 2 bubbles 1 tadpole (9 graphs):



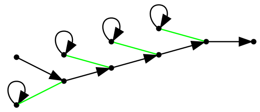
4 fermions 3 bubbles 0 tadpoles (1 graph):



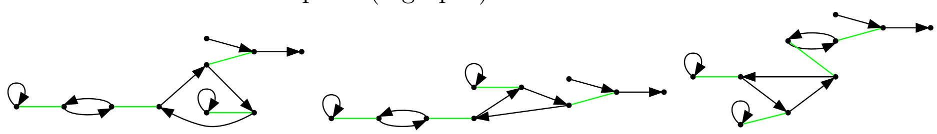
5 fermions 0 bubbles 3 tadpoles (3 graphs):



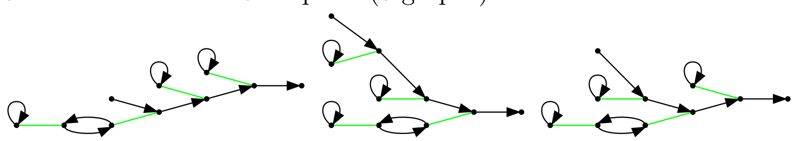
5 fermions 0 bubbles 4 tadpoles (1 graph):



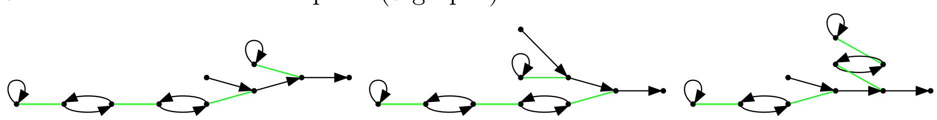
5 fermions 1 bubble 2 tadpoles (3 graphs):



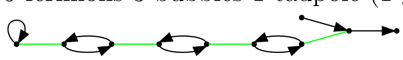
5 fermions 1 bubble 3 tadpoles (3 graphs):



5 fermions 2 bubbles 2 tadpoles (3 graphs):



5 fermions 3 bubbles 1 tadpole (1 graph):



Total: 706 graphs on 8 vertices, 319 without tadpoles, 228 with neither bubbles nor tadpoles.

[1] O. E. I. S. Foundation Inc., (2018), http://oeis.org/.
[2] M. Meringer, J. Graph Theory **30**, 137 (1999).
[3] G. Brinkmann, J. Goedgebeur, and N. V. Cleemput, Int. J. Chem. Model. **5**, 67 (2013).
[4] J.-P. Börnsen and A. E. M. van de Ven, arXiv:1807.04817 (2018), arXiv:1807.04817.
[5] G. Brinkmann, N. V. Cleemput, and T. Pisanski, Theor. Comput. Sci. **502**, 16 (2013).
[6] R. de Mello Koch and S. Ramgoolam, Phys. Rev. D **85**, 026007 (2012).
[7] E. M. Palmer, R. C. Read, and R. W. Robinson, SIAM J. Disc. Math. **16**, 65 (2002).
[8] N. V. Alexeev, J. E. Andersen, R. C. Penner, and P. G. Zograf, Adv. Math. **2389**, 1056 (2016).
[9] H. Kleinert, A. Pelster, B. Kastening, and M. Bachmann, Phys. Rev. E **62**, 1537 (2000).
[10] A. Stoimenow, Disc. Math. **218**, 209 (2000).
[11] W. Y.-C. Chen and D. C. Torney, Disc. Appl. Math. **145**, 349 (2005).
[12] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).
[13] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).
[14] N. C. Wormald, J. Lond. Math. Soc. **2**, 1 (1979).
[15] G.-B. Chae, E. M. Palmer, and R. W. Robinson, Discr. Math. **307**, 2979 (207).
[16] T. Kinoshita and M. Nio, Phys. Rev. Lett. **82**, 3240 (1999).
[17] T. Kinoshita and W. B. Lindquist, Phys. Rev. D **42**, 636 (1990).
[18] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Phys. Rev. Lett. **99**, 110406 (2007).
[19] A. B. d'Azevedo, A. Mednykh, and R. Nedela, Discr. Math. **310**, 1184 (2010).
[20] A. Pelster and K. Glaum, Phys. Stat. Sol. (b) **237**, 72 (2003).
[21] N. Marie and K. Yeats, arXiv:1210.5457 (2012), arXiv:1210.5457.
[22] P. Cvitanović, B. Lautrup, and R. B. Pearson, Phys. Rev. D **18**, 1939 (1978).
[23] R. J. Mathar, Int. J. Quant. Chem. **107**, 1975 (2007).
[24] A. Prunotto, W. M. Alberico, and P. Czerski, Open Physics **16**, 149 (2018).
[25] J. Touchard, Can. J. Math. **4**, 2 (1952), $S_6(1) = 109960$ is erroneous and should read 110410.