Proof that as the Standard Deviation of a Log Normal Distribution Approaches Zero the Distribution Becomes a Normal Distribution with a Mean of e^u Where u is the Mean of the Natural Logarithms of the Data Set Values.

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Abstract

While it is fairly easy to prove that the Log Normal distribution becomes a Benford distribution as the standard deviation approaches infinity (see appendix A), it is a bit more difficult to prove that as the standard deviation approaches zero that the distribution becomes a Normal distribution with a mean of e^u where u is the mean of the natural logarithms of the data set values.

Proof:

Proof that as the standard deviation of a Log Normal distribution approaches 0 the distribution becomes a Normal distribution with a mean of e^{u} where u is the mean of the natural logarithms of the data set values.

1) Log Normal probability density function: pdf(x) = $\frac{1}{x\sqrt{2\pi\sigma^2}} e^{\frac{-(Ln(x)-u)^2}{2\sigma^2}}$; u = mean(ln(x)), $\sigma =$ std_dev(ln(x))

2) Determine the mode of the Log Normal distribution i.e. $\frac{dy}{dx}$ = $\frac{1}{\sqrt{2\pi\sigma^2}}\frac{dy}{dx}(\frac{e^{-(\ln(x)-u)^2/2\sigma^2}}{x})$ = 0 ; solve for x

3) $\frac{dy}{dx} = e^{-(Ln(x)-u)^2/2\sigma^2}\ [\frac{-(Ln(x)+u)}{\sigma^2} - 1] = 0$

4) Solve x for $\frac{-Ln(x)+u}{\sigma^2} - 1 = 0$

5) $Ln(x) = u-\sigma^2$

6) $x = e^{(u-\sigma^2)}$

7) As $\sigma \to 0$; $x \to e^{u}$

8) $pdf(x) = \frac{1}{x\sqrt{2\pi\sigma^2}}\, e^{\frac{-(Ln(x)-u)^2}{2\sigma^2}}$

9) Taylor series of Ln(x) about e^{u} =

10) $Ln(e^{u}) + \frac{x- e^{u}}{e^{u}} - \frac{(x-e^{u})^2}{2e^{2u}} + \frac{(x-e^{u})^3}{3e^{3u}} - \frac{(x-e^{u})^4}{4e^{4u}} + \ldots\ldots =$

11) $\ln(e^{u}) + \sum_{k=1}^{\infty} \frac{-(-1)^{k}(x- e^{u})^{k}}{ke^{ku}}$

12) $Ln(x-e^{u}) \sim Ln(e^{u}) + \frac{x- e^{u}}{e^{u}}$ as $\sigma \to 0$

13) $Ln(x-e^{u}) \sim u + \frac{x- e^{u}}{e^{u}}$

14) $pdf(x) \sim \frac{1}{x\sqrt{2\pi\sigma^2}}\, e^{\frac{-(u+\frac{x-e^{u}}{e^{u}}-u)^2}{2\sigma^2}}$

15) $pdf(x) = \sim \frac{1}{e^{u}\sqrt{2\pi\sigma^2}}\, e^{\frac{-(\frac{x-e^{u}}{e^{u}})^2}{2\sigma^2}}$ as $\sigma \to 0$

16) $pdf(x) \sim \frac{1}{\sqrt{2\pi(\sigma e^{\upsilon})^2}}\, e^{\frac{-(x-e^{\upsilon})^2}{2(\sigma e^{\upsilon})^2}}$

17) u_x = mean(x); $\boldsymbol{\sigma}_x$ = std_dev(x)

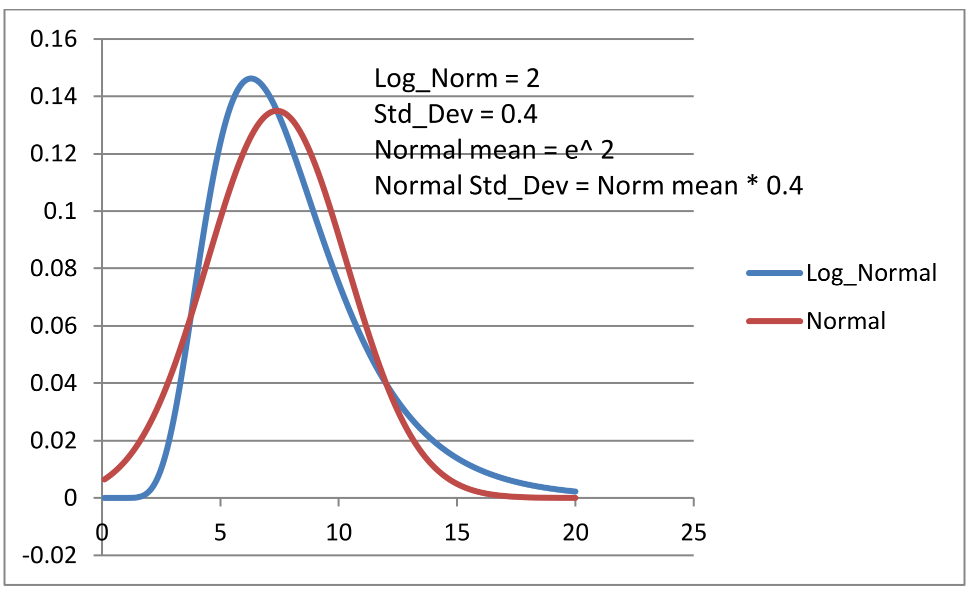
18) $u_x \sim e^{u}$; $\boldsymbol{\sigma}_x \sim u_x\,\sigma$

19) pdf(x) ~ $\frac{1}{\sqrt{2\pi(\sigma_x)^2}} e^{\frac{-(x-u_x)^2}{2(\sigma_x)^2}}$

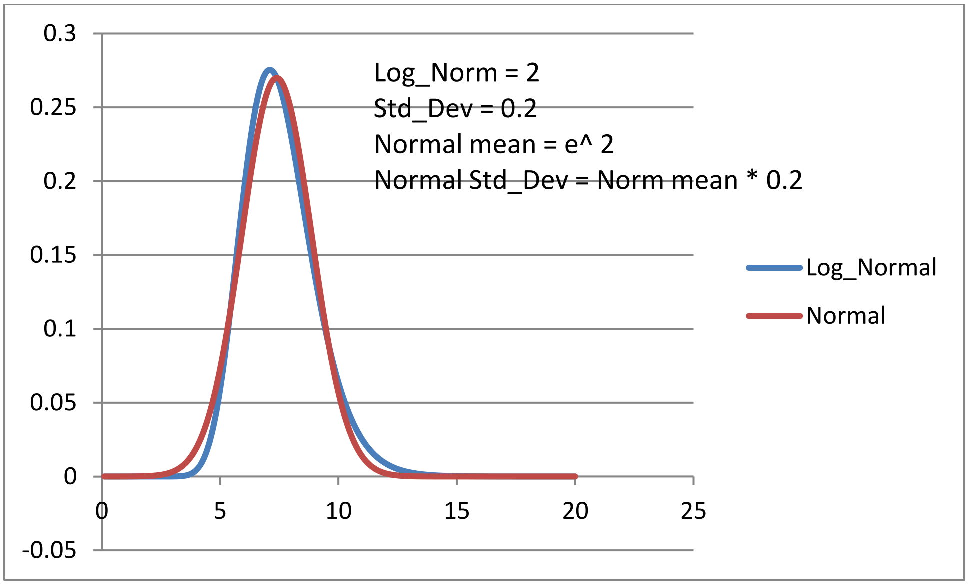
20) Which is a Normal Distribution with a mean of e^u and a standard deviation of σ e^u

The following graphs are plots of the Log Normal distribution with given values Of mean (u) and standard deviations of σ v. the Normal distribution with a mean of e^u and a standard deviation of σ e^u.

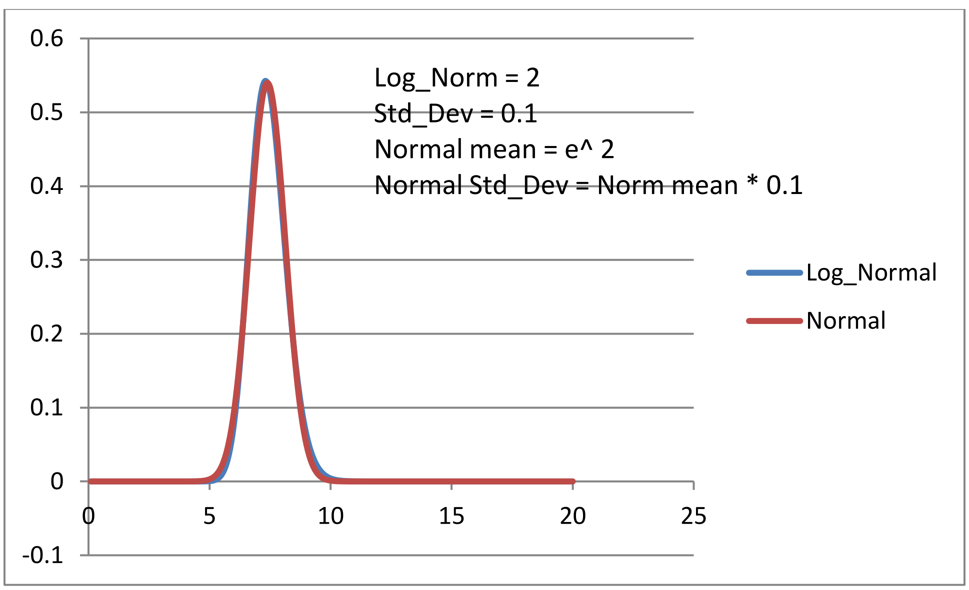
fig#1



fig#2



fig#3



The plots were derived from a Microsoft Excel spreadsheet. They strongly indicate empirically that as the standard deviation does approach zero the Log Normal distribution with a given mean and standard deviation does converge to a Normal distribution with the mean equal to e raised to Log Normal mean and the standard deviation equal to the Normal times the Log Normal standard deviation.

Appendix A

Proof that as the standard deviation of a Log Normal distribution approaches infinity the distribution becomes a Benford distribution i.e. the probability density function approaches k/x

1) The Benford probability density function = 1/xln(10).
2) The Log Normal probability density function = $\frac{1}{x\sqrt{2\pi\sigma^2}}$ $e^{-(Ln(x)-u)^2/2\sigma^2}$
3) For x=1: 1/xln(10) = 1/ln(10); $\frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(Ln(x)-u)^2/2\sigma^2}$ = $\frac{1}{\sqrt{2\pi\sigma^2}}$ $e^{-(u)^2/2\sigma^2}$
4) Normalize by multiplying the Log Normal distribution by $\frac{\sqrt{2\pi\sigma^2}}{\ln(10)} e^{(u)^2/2\sigma^2}$
5) The difference between the two distributions is : $\frac{1}{xln(10)}$ - $\frac{1}{xln(10)}$ ($e^{-(Ln(x)-u)^2/2\sigma^2}$) =
6) $\frac{1}{xln(10)}$ (1-$e^{-(Ln(x)-u)^2/2\sigma^2}$)
7) For any given value of x the value 1-$e^{-(Ln(x)-u)^2/2\sigma^2}$ approaches 0; since $e^{\frac{k(constant)}{\sigma^2}}$ approaches 1 as σ approaches ∞.