The Friedmann-Lemaître-Robertson-Walker Metric in a de Sitter Universe

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Abstract

An elementary derivation of the Friedmann-Lemaître-Robertson-Walker metric is given for a pure (matter-free) de Sitter universe assuming a global time marker. The presentation shows that the cosmological constant is proportional to the Ricci scalar R, which appears in two variations of the metric typically given in textbooks. For a positive Ricci scalar, the universe is open and expanding at an exponential rate with time, while for a negative R the universe oscillates in size sinusoidally. For $R = 0$, the universe is Minkowskian, as expected. Because there is no matter in a de Sitter universe, there is no discussion of the Hubble relation or the evolution of the universe in terms of the Friedmann equations associated with mass-energy pressure and density.

Introduction

A de Sitter universe (not to be confused with a de Sitter *space*) refers to a possible state of the universe in the distant future, when matter has been thinned out to negligible density due to the expansion of the universe and black hole evaporation, leaving predominantly a rarified gas of stray high-entropy photons. Being devoid of matter, its primary characteristic is the assumed existence of a non-zero cosmological constant Λ in lieu of an energy-momentum tensor. Thus, the only difference between a completely empty universe and a de Sitter universe is the cosmological constant, which in many modern theories is thought to represent dark energy. Dark energy is currently believed to comprise roughly 70% of all the energy in the universe; if true, then one could rightly say we live in a 13.8 billion-year-old universe that is now nearly de Sitterian.

We also know that the universe is expanding and, if recent Type 1a supernovae data are to be believed, the rate of expansion is actually accelerating. How can an empty universe do *that*? It is generally believed that dark energy exerts a repulsive effect on matter (and on space itself), and as the universe expands it actually *creates* more space, which is then filled in with dark energy. As a result, the attractive force of gravity by all the matter in the universe is eventually exceeded by this repulsive force, which explains the acceleration.

We will not address these complicated issues here. Instead, we'll examine how the cosmological constant appears naturally as a constant, non-zero Ricci scalar R using ordinary Einsteinian gravity theory in de Sitter space. In particular, we will derive the Friedmann-Lemaître-Robertson-Walker spacetime from elementary principles, convincingly demonstrating that the Ricci scalar and the cosmological constant are one and the same.

It is assumed that the reader is familiar with the special and general theories of relativity along with some basic understanding of cosmology, but beyond that the material should be accessible to any undergraduate physics or astronomy student.

1. The Friedmann-Lemaître-Robertson-Walker Spacetime

In it basic form, the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model is based on a perfectly isotropic and homogeneous universe (that is, it looks the same in every direction and from one location to another) filled with an incoherent "dust" initially spread more or less uniformly throughout all space. Based on astronomical observations made in the 1920s, the universe is also known to be expanding in time. As a result, we might expect the matter to be subject to pressure and density variations as the universe undergoes expansion, but taken as a whole the makeup of the universe remains isotropic and homogenous in time.

In addition, the FLRW model assumes no particular center to the universe, so distant stars, galaxies and nebulae appear to rushing away from any given observer. As a result, the universe cannot be modeled with a Schwarzschild-type metric, which assumes a point acting as the center of a gravitating mass. Consequently, cosmological time itself cannot be treated as it is in the Schwarzschild metric, either. For simplicity, and in apparent violation of what might otherwise seem a completely covariant approach to time, we adopt a *global* concept of time, one in which imaginary clocks distributed throughout the universe tick away at the same rate. We also make the assumption that at some initial time (say, the start of the Big Bang) the clocks are all synchronized with one another and remain that way for all time.

In consideration of these simplfying assumptions, we expect the FLRW model to be expressible in a generalization of a *maximally-symmetric* spacetime, such as one set in ordinary spherical coordinates:

$$ds^2 = c^2 dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

However, since we know that space is expanding, we might expect the space part of the FLRW model to include a time-dependent expansion term $e^{g(ct)}$, as in

$$ds^2 = c^2 dt^2 - e^{g(ct)}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

Following the formalism of Adler et al., we can generalize this even further by including a purely space-dependent term $e^{f(r)}$, so that our FLRW metric finally appears as

$$ds^2 = c^2 dt^2 - e^{g(ct)} e^{f(r)}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{1.1}$$

We need not concern ourselves with terms involving the coordinates θ and ϕ, since perfect isotropy and homogeneity are assumed for the FLRW model.

2. Derivation of the FLRW Metric

In a pure de Sitter spacetime, the ten Einstein gravitational field equations are simple:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{2.1}$$

where Λ is the cosmological constant. Assuming normalization of the metric tensor, the $g_{\mu\nu}$ are dimensionless, while the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are of dimension length^{-2}. Consequently, the cosmological constant must also be of dimension length^{-2}. Contraction of (2.1) with respect to $g^{\mu\nu}$ shows that $\Lambda = R/4$, so the expression reduces to

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = 0 \tag{2.2}$$

These field equations were solved long ago. For the Schwarzschild-like metric

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

the solution is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{Rr^2}{12} \tag{2.3}$$

where $2m$ is the usual geometric mass term. But in a de Sitter space there is no matter, so the metric should look like

$$ds^2 = \left(1 + \frac{Rr^2}{12}\right) c^2 dt^2 - \frac{dr^2}{1 + Rr^2/12} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The r^2 term in this metric has interesting consequences, and in fact appears as an acceleration term for a vanishingly-small test particle with respect to the origin. This is intriguing, because this is exactly the kind of property we would expect for Λ (which is proportional to R). However, in the FLRW model there is no origin either, so this solution must be rejected. Nevertheless, we might expect similar behavior from a suitable FLRW metric, even with its assumption of a global time marker.

We therefore proceed to solve the FLRW metric (1.1) by deriving solutions to (2.2) directly using the lower- and upper-case metric tensors

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^g e^f & 0 & 0 \\ 0 & 0 & -r^2 e^g e^f & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta e^g e^f \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-g} e^{-f} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} e^{-g} e^{-f} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} e^{-g} e^{-f} \end{bmatrix}$$

Using these quantities, the Ricci tensor $R_{\mu\nu}$ and scalar R are easily calculated (see Adler et al. for a summary of the required Christoffel symbols):

$$R_{00} = \frac{3}{2}\,g'' + \frac{3}{4}\,(g')^2 \tag{2.4}$$

$$R_{11} = f'' + \frac{1}{r}\,f' - e^g e^f \left(\frac{1}{2}\,g'' + \frac{3}{4}\,(g')^2\right) \tag{2.5}$$

$$R_{22} = r^2\left[\frac{1}{2}\,f'' + \frac{1}{4}\,(f')^2 + \frac{3}{2r}\,f' - e^g e^f \left(\frac{1}{2}\,g'' + \frac{3}{4}\,(g')^2\right)\right] \tag{2.6}$$

$$R_{33} = \sin^2\theta R_{22} \tag{2.7}$$

$$R = 3\left[g'' + (g')^2\right] - 2\,e^{-(g+f)}\left[f'' + \frac{1}{4}\,(f')^2 + \frac{2}{r}\,f'\right] \tag{2.8}$$

where the primes represent differentiation with respect to their arguments:

$$g' = \frac{1}{c}\,\frac{dg}{dt},\;\; g'' = \frac{1}{c^2}\,\frac{d^2g}{dt^2} \;\text{ and }\; f' = \frac{df}{dr},\;\; f'' = \frac{d^2f}{dr^2}$$

Using (2.4), (2.5) and (2.6), the associated field equations (2.2) then reduce to

$$\frac{3}{2}\,g'' + \frac{3}{4}\,(g')^2 - \frac{1}{4}\,R = 0 \tag{2.9}$$

$$f'' + \frac{1}{r}\,f' - e^g e^f \left(\frac{1}{2}\,g'' + \frac{3}{4}\,(g')^2\right) + \frac{1}{4}\,e^g e^f\,R = 0 \tag{2.10}$$

$$r^2\left[\frac{1}{2}\,f'' + \frac{1}{4}\,(f')^2 + \frac{3}{2r}\,f' - e^g e^f \left(\frac{1}{2}\,g'' + \frac{3}{4}\,(g')^2\right)\right] + \frac{1}{4}\,r^2 e^g e^f R = 0 \tag{2.11}$$

Despite their apparent complexity, these differential equations are quite easy to solve for e^g and e^f separately. The solution to (2.9) is simply

$$e^g = \cosh^2(\beta c t) \tag{2.12}$$

where

$$\beta = \sqrt{\frac{R}{12}} \tag{2.13}$$

and has the dimension length^{-1}. By combining (2.4) and (2.5), we have

$$f'' + \frac{1}{r}\,f' = -e^{g+f}\,g'' \tag{2.14}$$

Similarly, combining (2.4) and (2.6) gives

$$\frac{1}{2}\,f'' + \frac{1}{4}\,\left(f'\right)^2 + \frac{3}{2\,r}\,f' = -e^{g+f}\,g'' \tag{2.15}$$

Equating (2.14) and (2.15), we have

$$f'' - \frac{1}{2}\,\left(f'\right)^2 - \frac{1}{r}\,f' = 0 \tag{2.16}$$

The solution to this differential equation is simply

$$e^f = \frac{1}{(1 + \frac{1}{4}\,\beta^2\,r^2)^2} \tag{2.17}$$

It is easily shown that (2.12) and (2.17) satisfy all the other differential equations. The FLRW metric for a pure de Sitter spacetime is therefore

$$ds^2 = c^2 dt^2 - \cosh^2(\beta c t)\,\frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2}{(1 + \frac{1}{4}\,\beta^2\,r^2)^2} \tag{2.18}$$

An equivalent (and perhaps more familiar) expression for the FLRW metric can be obtained using the coordinate transformation

$$u = \frac{r}{1 + \frac{1}{4}\beta^2 r^2}$$

which results in

$$ds^2 = c^2 dt^2 - \cosh^2(\beta c t)\left[\frac{du^2}{1-\beta^2 u^2} + u^2 d\theta^2 + u^2 \sin^2\theta\, d\phi^2\right] \tag{2.19}$$

3. Comments and Conclusions

The FLRW expansion term

$$\cosh^2(\beta c t) = \frac{1}{4}\left(e^{\beta c t} + e^{-\beta c t}\right)^2$$

obviously blows up exponentially without limit provided that $\beta = \sqrt{R/12}$ is real, which requires that $R = 4\Lambda$ be a positive number. When $R = 0$, the FLRW metric degenerates into ordinary Minkowski spacetime, as expected. However, when $R < 0$ then β is imaginary and the expansion term becomes sinusoidal, since $\cosh(ix) = \cos(x)$. We would then expect the FLRW metric to describe a universe that is oscillating in physical size with time. These conclusions are compatible with the predictions of the conventional FLRW model, where the universe contains mass-energy (non-de Sitterian spacetime).

The notion of a global time coordinate for the entire universe is usually tied to that of *co-moving coordinates*, which itself is related a conjecture that the German mathematical physicist Hermann Weyl proposed in 1923. The *Weyl conjecture* (or hypothesis) refers to the notion of matter expanding along worldlines that never intersect and in general remain parallel to their nearby neighbors. An observer moving along with the expansion would see nothing different than any other observer for all time, and for all intents and purposes the observer could consider herself to be at rest. In fact, all other observers in the universe would also consider themselves to be at rest, and as a consequence of this line of thinking all 3-space movement can be considered to be time-independent, so that the spacial time derivative $d\vec{x}/dt$ for every observer can be set to zero. This is of great benefit when considering the motion of matter fields in cosmology, since it is obviously unrealistic to think that the precise motions of even small aggregates of cosmological matter could be derived from any theory.

The de Sitter equation (2.2) is traceless, unlike the usual Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.1}$$

in which the Ricci scalar R is generally not a constant. This points to a problem with (3.1), given the fact that the energy-momentum tensor $T_{\mu\nu}$ for the electromagnetic field is itself traceless and therefore seemingly incompatible with the Einstein equations, which are derivable from the Einstein-Hilbert action

$$S = \int \sqrt{-g}\, R\, d^4x$$

In an early attempt to unify the gravitational and electromagnetic interactions (the only forces of Nature known at the time), in 1918 Weyl proposed a theory in which the gravitational action is given instead by

$$S = \int \sqrt{-g}\, R^2\, d^4x$$

Weyl's scalar R was based on an ingenious generalization of Riemannian geometry, but it was shown to fail (by Einstein himself) upon certain physical considerations we need not go into here. However, it can be shown that the field equations for Weyl's action in traditional Riemannian geometry give

$$R\left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right) + D_\mu D_\nu R - g_{\mu\nu} R^{\alpha\beta} D_\alpha D_\beta R = 0$$

where D_μ is the covariant differentiation operator. If the scalar R is a constant, this becomes the traceless expression

$$R\left(R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R\right) = 0 \tag{3.2}$$

We then have the option of choosing $R = 0$ (for which we have no theory) or $R =$ a non-zero constant. In the latter case we can divide R out of (3.2), which results in the de Sitter equation (2.2). However, in this approach to Weyl's theory we have no need of requiring an empty universe. Indeed, the field equations

$$R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R = 0 \tag{3.3}$$

have the same solution (2.3) given earlier, but now there is no reason to require that $2m = 0$ as before. Furthermore, note that in (2.3) we can make R as small as possible (but not zero), and all the traditional predictions of Einstein's gravity theory remain intact, including the perihelion advance of the planet Mercury, the deflection of starlight and the gravitational redshift. Finally, (3.3) is of second order, eliminating a problem that typical fourth-order approaches are prone to exhibit.

References

1. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, 2nd Edition, 1975. See Chapter 12 for a detailed summary of the Christoffel symbols and associated Ricci expressions needed in Section 2.

2. G. 't Hooft, *Introduction to General Relativity*, 2002. This is a short paper available for download at

   http://srv2.fis.puc.cl/ mbanados/Cursos/RelatividadGeneral/tHooft-Notes.pdf

   't Hooft uses a different approach in the derivations of (2.18) and (2.19) based on a maximally-symmetric 3-space. His discussion is not limited to de Sitter spacetime and includes a review of the usual Friedmann equations dealing with Hubble's velocity-distance relation and related issues.

3. H. Weyl, *Space-Time-Matter*, Dover Publications, 1952. Weyl was one of the first proponents of Einstein's 1915 gravity theory and was instrumental in both promoting the theory and developing many basic applicable ideas in differential geometry. Weyl's failed 1918 theory is still seen as containing the germs of a future fruitful unified theory of gravitation and electromagnetism.