New Equations of Motion of an Electron. I. A Classical Point Particle

Xiaowen Tong*

Abstract

A new formula for the energy density of electrostatic field is derived. Based on the conservation of energy and momentum, the classical equations of motion of an electron, which is considered as a point particle, are then obtained by establishing a delay coordinate system. The resulting equations are exact but not covariant. Finally we calculate the self-energy of a free electron in quantum electrodynamics using a new cut-off procedure.

1 The Energy of Electrostatic Field

In many books about classical electrodynamics[1] , the electrostatic energy density

$$u = \frac{1}{8\pi}\mathbf{E}^2 \tag{1}$$

is derived from

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{2}$$

(1) can not be correct since one will get an infinite result after integrating it while (2) is always finite. Obviously (1) contains the potential energy shared by charges as well as the field energy owed by charges themselves which remains constant in electrostatic. (2) is just the former. The latter shall not be counted in the deduction of (1). Hence we will give a new derivation about the electrostatic energy density.

We rewrite (2) as

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V_{\not{i}i} \tag{3}$$

*Corresponding author: yztsztxw@163.com

where $V_{\not i i}$ is the total potential at $\mathbf{r}_i$ produced by all the charges except the charge i. The second subscript of $V_{\not i i}$ means 'at $\mathbf{r}_i$'. In order to use integral afterwards we write q_i as

$$q_i = \rho_i \Delta \mathcal{V}_i = q_i \delta(\mathbf{r}_i) \Delta \mathcal{V}_i$$

where $\Delta \mathcal{V}_i$ is a small volume at point $\mathbf{r}_i$. Substitute this equation into (3) we obtain

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \delta(\mathbf{r}_i) \Delta \mathcal{V}_i V_{\not i i} \tag{4}$$

According to Gauss's law we have at $\mathbf{r}_i$

$$\nabla \cdot \mathbf{E} = 4\pi \rho_i = 4\pi q_i \delta(\mathbf{r}_i)$$

We use $\mathbf{E}_{ij}$ to denote the field strength at $\mathbf{r}_j$ produced by q_i. Then the electric field strength $\mathbf{E}$ at $\mathbf{r}_i$ can be split into two parts*

$$\mathbf{E} = \mathbf{E}_{ii} + \mathbf{E}_{\not i i}$$

Where the two parts satisfy respectively

$$\nabla \cdot \mathbf{E}_{ii} = 4\pi q_i \delta(\mathbf{r}_i) \tag{5}$$

and

$$\nabla \cdot \mathbf{E}_{\not i i} = \nabla \cdot \mathbf{E}_{1i} + \nabla \cdot \mathbf{E}_{2i} + \cdots + \nabla \cdot \mathbf{E}_{i-1,i} + \nabla \cdot \mathbf{E}_{i+1,i} + \nabla \cdot \mathbf{E}_{ni} = 0 \tag{6}$$

The reason for (6) is that except the charge q_i all the other charges are not at $\mathbf{r}_i$, i.e.

$$\nabla \cdot \mathbf{E}_{ki} = 0, \ for \ k \neq i$$

Thus we can retain the term $\nabla \cdot \mathbf{E}_{\not i i}$ or abandon it. Retaining it means we are calculating the action $V_{\not i i}$ of on $\mathbf{E}_{\not i i}$, which is exactly the self-action. Therefore we abandon it. Plugging (5) into (4) we get

$$W = \frac{1}{8\pi} \sum_{i=1}^{n} \nabla \cdot \mathbf{E}_{ii} V_{\not i i} \Delta \mathcal{V}_i \tag{7}$$

As mentioned previously, the divergence of $\mathbf{E}_{ij}$ vanishes everywhere except at $\mathbf{r}_i$. We can add the term below to every term of the sum on the rhs of (7)

$$\sum_{j=1, j \neq i}^{\infty} \nabla \cdot \mathbf{E}_{ij} V_{\not j j} \Delta \mathcal{V}_j$$

Consequently

*The meaning of the subscripts of $\mathbf{E}_{\not i i}$ is the same as that of $V_{\not i i}$. Similarly, $\mathbf{E}_{ii}$ denotes the field strength at $\mathbf{r}_i$ produced only by q_i.

$$\begin{aligned}
W &= \frac{1}{8\pi}\sum_{i=1}^{n}\left(\nabla\cdot\mathbf{E}_{ii}V_{\not{i}i}\Delta\mathcal{V}_i + \sum_{j=1,j\neq i}^{\infty}\nabla\cdot\mathbf{E}_{ij}V_{\not{i}j}\Delta\mathcal{V}_j\right)\\
&= \frac{1}{8\pi}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\nabla\cdot\mathbf{E}_{ij}V_{\not{i}j}\Delta\mathcal{V}_j\\
&= \frac{1}{8\pi}\sum_{i=1}^{n}\int_{\mathcal{V}}\nabla\cdot\mathbf{E}_i V_{\not{i}}\,d\mathcal{V}
\end{aligned}$$

In the last step we have thrown away the second index of $\mathbf{E}_{ij}$ and $V_{\not{i}j}$ since we are integrating them in a volume $\mathcal{V}$. $\mathcal{V}$ shall be big enough to include all the charges. After applying an integration by parts we obtain

$$W = \frac{1}{8\pi}\sum_{i=1}^{n}\left(-\int_{\mathcal{V}}\mathbf{E}_i\cdot\nabla V_{\not{i}}\,d\mathcal{V} + \oint_S V_{\not{i}}\mathbf{E}_i\cdot d\mathbf{S}\right) \tag{8}$$

If we choose the volume $\mathcal{V}$ to be the whole space, the second integral in the bracket of (8) would vanish. Due to

$$\nabla V_{\not{i}} = -\mathbf{E}_{\not{i}}$$

(8) becomes

$$W = \frac{1}{8\pi}\sum_{i=1}^{n}\int\mathbf{E}_i\cdot\mathbf{E}_{\not{i}}\,d\mathcal{V} \tag{9}$$

Consider a system of two point charges, we have

$$\mathbf{E}_{\not{1}} = \mathbf{E}_2,\ \mathbf{E}_{\not{2}} = \mathbf{E}_1$$

(9) gives

$$W = \frac{1}{4\pi}\int\mathbf{E}_1\cdot\mathbf{E}_2\,d\mathcal{V} \tag{10}$$

(3) being always finite guarantees the same holds for (9). The form of (9) implies that the energy of electrostatic fields origins from 'interaction' between charged particles. One can verify easily (10) for two point charges[2].

2 The Classical Equations of Motion of an Electron

The success of eliminating the self-action of a point charge in last section motivates us to go further. If we continue to apply the method to general electromagnetic phenomena in which electrons are usually in motion, however, we would still obtain only the energy of 'interaction'. As it is known, when an electron is accelerated its own field will 'twist', which implies the field has indeed changed. In addition, electromagnetic radiation always takes away some energy and momentum from the electron. Therefore reaction of the field exerting on the electron shall exist. It is expected then we would encounter some infinity in view of (1). But we shall keep in mind that only the change of the energy and the momentum of the field, which corresponds to the reaction, is meaningful to us. In other words, the motion of the electron interacting with the electromagnetic field may be learned by calculating the change of the latter.

We want to obtain an equation of motion with manifest Lorentz covariance. To this end, it is suitable to adopt the covariant form of electrodynamics*. The equations for the electron interacting with electromagnetic fields are

$$\frac{dp^{\mu}_{mech}}{dt} = -\frac{1}{c}\int F^{\mu\nu}_{tot} j_{\nu} d^3x \tag{11}$$

$$\partial_{\mu} F^{\mu\nu}_{tot} = -\frac{4\pi}{c} j^{\nu} \tag{12}$$

where p^{μ}_{mech} is the mechanical energy-momentum vector of the electron and $F^{\mu\nu}_{tot}$ is the total field strength tensor. According to the principle of superposition, the latter can be split into two terms, namely, those of external fields and that of the field of the electron itself:

$$F^{\mu\nu}_{tot} = F^{\mu\nu}_{sel} + F^{\mu\nu}_{ext} \tag{13}$$

There aren't any other charged particles except the electron in the space we consider. Therefore,

$$\partial_{\mu} F^{\mu\nu}_{ext} = 0$$

In view of (12) we obtain then a equation

$$\partial_{\mu} F^{\mu\nu}_{sel} = -\frac{4\pi}{c} j^{\nu} \tag{14}$$

Using the four-potential we write $F^{\mu\nu}_{sel}$ as

$$F^{\mu\nu}_{sel} = \frac{\partial A^{\nu}_{sel}}{\partial x_{\mu}} - \frac{\partial A^{\mu}_{sel}}{\partial x_{\nu}} \tag{15}$$

In Lorentz gauge (14) now becomes

$$\Box A^{\mu}_{sel} = -\frac{4\pi}{c} j^{\mu} \tag{16}$$

Let $\chi^{\mu}(\tau)$ be the world-line function of the electron which we regard it as a point charge. The current density is then

$$j^{\mu}(x) = ec\int_{-\infty}^{\infty} \delta(x - \chi(\tau)) v^{\mu}(\tau) d\tau$$

*We use $g_{\mu\nu} = diag\{-1, 1, 1, 1\}$ in this section.

In order to use some delay quantities to express the solution of (16), we introduce a vector u^μ which satisfies

$$u^\mu u_\mu = 1, \quad u^\mu v_\mu = 0$$

The potential to be solved at x^μ is produced by the electron when it was at χ^μ. If we let

$$R^\mu = x^\mu - \chi^\mu \tag{17}$$

The delay relation is then

$$R^\mu R_\mu = 0$$

We also introduce an invariant

$$\rho = -\frac{R^\mu v_\mu}{c}$$

(17) now can be written

$$R^\mu = \rho(u^\mu + \frac{v^\mu}{c}) \tag{18}$$

We find that the solution of (16) is[3]

$$A^\mu_{sel}(x) = \frac{e}{c}\frac{v^\mu}{\rho}$$

The field strength tensor can be calculated from (15). The result is[4]

$$F^{\mu\nu}_{sel} = \frac{e}{c\rho^2}(v^\mu u^\nu - v^\nu u^\mu) + \frac{e}{c^2\rho}[\frac{1}{c}(a^\mu v^\nu - a^\nu v^\mu) + u^\nu(a^\mu + \frac{a_u v^\mu}{c}) - u^\mu(a^\nu + \frac{a_u v^\nu}{c})] \tag{19}$$

where a_u is defined by

$$a_u = a^\mu u_\mu$$

We are interested in the symmetric tensor which is given by

$$\Theta^{\mu\nu}_{tot} = \frac{1}{16\pi} g^{\mu\nu}(F_{tot})_{\alpha\beta}(F_{tot})^{\alpha\beta} + \frac{1}{4\pi}(F_{tot})^{\mu\alpha}(F_{tot})_\alpha{}^\nu$$

Substituting (13) into it yields

$$\Theta^{\mu\nu}_{tot} = \Theta^{\mu\nu}_{sel} + \Theta^{\mu\nu}_{ext} + \Theta^{\mu\nu}_{crs}$$

where

$$\Theta^{\mu\nu}_{crs} = \frac{1}{8\pi} g^{\mu\nu}(F_{sel})_{\alpha\beta}(F_{ext})^{\alpha\beta} + \frac{1}{4\pi}[(F_{ext})^{\mu\alpha}(F_{sel})_\alpha{}^\nu + (F_{ext})^{\nu\alpha}(F_{sel})_\alpha{}^\mu] \tag{20}$$

We call $\Theta^{\mu\nu}_{crs}$ the 'cross' term. It is composed of the field variables of the electron and the external field variables while $\Theta^{\mu\nu}_{sel}$ contains only the former and $\Theta^{\mu\nu}_{ext}$ the latter. From (19) it can be shown that

$$\begin{aligned}\Theta^{\mu\nu}_{sel} =& \underbrace{\frac{e^2}{4\pi\rho^4}(u^\mu u^\nu - \frac{v^\mu v^\nu}{c^2} - \frac{1}{2}g^{\mu\nu})}_{\Theta^{\mu\nu}_{v}} + \underbrace{\frac{e^2}{2\pi c^2\rho^3}(a_u\frac{R^\mu R^\nu}{\rho^2} - a_u\frac{v^{(\mu}R^{\nu)}}{c\rho} - \frac{a^{(\mu}R^{\nu)}}{\rho})}_{\Theta^{\mu\nu}_{va}} \\ &+ \underbrace{\frac{e^2}{4\pi c^4\rho^2}({a_u}^2 - a_\lambda a^\lambda)\frac{R^\mu R^\nu}{\rho^2}}_{\Theta^{\mu\nu}_{a}} \\ =&\Theta^{\mu\nu}_{v} + \Theta^{\mu\nu}_{va} + \Theta^{\mu\nu}_{a}\end{aligned} \tag{21}$$

The divergence of the total symmetric tensor is

$$\partial_\mu \Theta^{\mu\nu}_{tot} = -\frac{1}{c}F^{\mu\nu}_{tot}j_\nu$$

We integrate the two sides of the above equation in a four-dimensional volume V of which we choose the time coordinate varies from 0 to t_0. Due to (11) it follows that

$$\int_V \mathrm{d}^4x\partial_\mu\Theta^{\mu\nu}_{tot} = \int_0^{t_0} c\mathrm{d}t\frac{\mathrm{d}\mathbf{p}^\nu_{mech}}{\mathrm{d}t} \tag{22}$$

According to Gauss's theorem, the lhs of the above equation can be rewritten as

$$\int_V d^4x\partial_\mu\Theta^{\mu\nu}_{tot} = \int_\Sigma dS\varepsilon(n)n_\mu\Theta^{\mu\nu}_{tot}$$

where n_μ is the tangent vector of the three-dimensional area element dS. $\varepsilon(n)$ is $+1$ when dS is space-like and -1 when dS is time-like. Suppose that the relation between the coordinate time t and the proper time τ of the electron is

$$t = t(\tau)$$

We set

$$t(0) = 0, \ \ t(\tau_0) = t_0$$

Differentiate both sides of (22) with respect to τ_0 we obtain

$$\frac{d}{d\tau_0}\int_\Sigma dS\varepsilon(n)n_\mu\frac{\Theta^{\mu\nu}_{tot}}{c} = \frac{dp^v_{mech}}{d\tau_0} \tag{23}$$

We choose that the volume V contains only the electron being studied. Hence,

$$\partial_\mu\Theta^{\mu\nu}_{ext} = 0$$

Thus we do not need to consider the external term in (22). Consequnetly, (23) becomes

$$\frac{d}{d\tau_0}\int_\Sigma dS\varepsilon(n)n_\mu\frac{1}{c}(\Theta^{\mu\nu}_{sel}+\Theta^{\mu\nu}_{crs})=\frac{dp^v_{mech}}{d\tau_0} \tag{24}$$

We assume that the electron was moving uniformly with velocity v_0 until it begins to accelerate at $t=0$ under the action of the external field, as shown below*. The shape of volume V can be chosen arbitrarily without affecting our result of calculation as long as the electron is located in V. We choose it to be a rectangle so that the calculation would be simple. As exhibited in the figure, the furthest points at which the electromagnetic signals arrive at t_0 are labeled by $P^{'}$ and $P^{'}$. With the exception of the segment $PP^{'}$, we denote with $\Sigma^{'}$ the remainder of the boundary Σ. The electromagnetic fields on $\Sigma^{'}$ are all produced by the uniform motion of the electron, of which we use $\Theta^{\mu\nu}_{v_0}$ to represent the symmetric tensor. When evaluating the integral on the lhs of (24), we don't

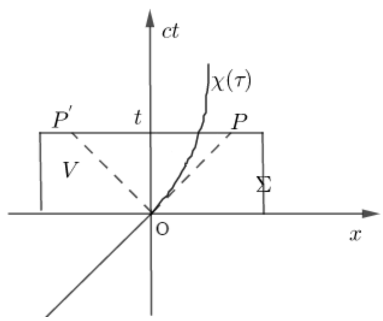

Figure 1: the world-line of the electron in two dimensional space-time.

need to calculate directly that involving $\Theta^{\mu\nu}_{v_0}$. If the electron continues to move uniformly after $t=0$ with the absence of external fields, we have from (23)

$$\frac{d}{d\tau_0}\int_\Sigma dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu\nu}_{v_0}=\frac{d}{d\tau_0}(\int_{P'}^{P} dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu\nu}_{v_0}+\int_{\Sigma'} dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu\nu}_{v_0})=0 \tag{25}$$

or

$$\frac{d}{d\tau_0}\int_{\Sigma'} dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu\nu}_{v_0}=-\frac{d}{d\tau_0}\int_{P'}^{P} dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu\nu}_{v_0}$$

According to (21) and the above equation, the first term on the lhs of (24) is

*For the sake of brevity, we drew the diagram in only two dimensions.

$$
\begin{aligned}
&\frac{d}{d\tau_0}\int_{\Sigma} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{sel} \\
&= \frac{d}{d\tau_0}[\int_{P'}^{P} dS\varepsilon(n)n_\mu \frac{1}{c}(\Theta^{\mu\nu}_a + \Theta^{\mu\nu}_{va} + \Theta^{\mu\nu}_v) + \int_{\Sigma'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{v_0}] \\
&= \frac{d}{d\tau_0}\int_{P'}^{P} dS\varepsilon(n)n_\mu \frac{1}{c}(\Theta^{\mu\nu}_a + \Theta^{\mu\nu}_{va} + \Theta^{\mu\nu}_v - \Theta^{\mu\nu}_{v_0})
\end{aligned}
$$

(24) becomes then

$$
\frac{d}{d\tau_0}\int_{\Sigma} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs} + \frac{d}{d\tau_0}\int_{P'}^{P} dS\varepsilon(n)n_\mu \frac{1}{c}(\Theta^{\mu\nu}_a + \Theta^{\mu\nu}_{va} + \Theta^{\mu\nu}_v - \Theta^{\mu\nu}_{v_0}) = \frac{dp^v_{mech}}{d\tau_0} \tag{26}
$$

To get an equation of motion of the electron, what we need to do next is to calculate the integrals on the lhs of the above equation. The calculation has been exhibited in Appendix A. The result is(See (A.10))

$$
\frac{\mathrm{d}\mathbf{p}_{mech}}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2}{3}\frac{e^2\gamma\beta}{ca_0}\right) = \gamma\left(e\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) - \frac{2}{3}\frac{e^2}{c^5}a^\lambda a_\lambda \gamma \mathbf{v} \tag{27a}
$$

$$
\frac{\mathrm{d}E_{mech}}{c\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\frac{e^2\gamma\ \left(1+\frac{1}{3}\beta^2\right)}{ca_0}\right) = \gamma\frac{e\mathbf{v}\cdot\mathbf{E}}{c} - \frac{2}{3}\frac{e^2}{c^4}a^\lambda a_\lambda \gamma \tag{27b}
$$

where a_0 is the radius of cut-off. Note that this equation is exact without any approximation.

3 The Self-Action of a Free Electron in Quantum Electrodynamics

We immediately recognize the terms in the brackets on the LHS of (27) if we write down the energy and the momentum according to (A.5) and (A.6) of the field of a point charge moving uniformly

$$
\mathbf{p}_{field} = \frac{2}{3}\frac{e^2\gamma\beta}{ca_0}, \quad E_{field} = \frac{1}{2}\frac{e^2\gamma(1+\frac{1}{3}\beta^2)}{a_0} \tag{28}
$$

For comparison, the electrostatic energy of a point charge is $e^2/2a_0$. Therefore, the terms in the brackets on the LHS of (27), which do not form a 4-vector, correspond to the electromagnetic mass of an electron. The LHS of (22) is generally a 4-vector unless the symmetric tensor $\Theta^{\mu\nu}_{tot}$ has singularities, which is our case in virtue of (21). An interesting aspect of our result (27) is that the radiation reaction

$$
\frac{2}{3}\frac{e^2}{c^5}a_\lambda a^\lambda v^\mu
$$

exactly constitutes a 4-vector, although $\Theta_a^{\mu\nu}$ has a quadratic singularity ρ^{-2}. This is due to the quadratic zero of the volume element of integral (See (A.4)). But this zero can not cancel the divergent in $\Theta_v^{\mu\nu}$ and $\Theta_{va}^{\mu\nu}$. It has been a problem up to now[5] that the electromagnetic mass being not covariant. It is very interesting that if the electron moves with velocity c, the energy and the momentum of the field would form a 4-vector in view of (28).

To solve the problem many attempts have be made[2]. One of them, for example, was brought forward by Born and Infeld. They modified the Maxwell equations in the region of small distance away from an electron to reduce the infinity of the fields to a finite value. But we are not in the situation yet in which we have to propose hypotheses. Because it has been proved in quantum electrodynamics that the effect of the self-action of a free electron can be included in the observed mass[6]. However, it seems that the meaning of this procedure-renormalization, hasn't been clarified thoroughly, especially the link between it and the classical theory.

The self-energy of a free electron can be calculated by using the perturbation theory. What the change of the wave-function the self-action leads to is just a factor $e^{-i\Delta ET}$. The first-order of the change is given by[7]($\bar{u}u = 1, g^{\mu\nu} = \{1, -1, -1, -1\}$)

$$\lim_{T\to\infty} -i\Delta ET = \frac{1}{N}e^2 \int\int d^4x d^4y \bar{\psi}(x)(-i\gamma_\mu)iS_F(x-y)(-i\gamma_\nu)iD_F^{\mu\nu}(x-y)\psi(y) \tag{29}$$

where N is a normalization factor. It equals to $\delta^3(0)$ if continuum normalization is applied to $\psi(x)$. After integrating over one of the coordinate variables, we get

$$\Delta E = e^2 i \int d^4x \bar{\psi}(x)\gamma_\mu S_F(x-y)\gamma_\nu D_F^{\mu\nu}(x-y)\psi(y)$$

It reads in momentum space

$$\Delta E = -4\pi e^2 i \frac{m}{E}\int \frac{d^4k}{(2\pi)^4}\bar{u}\gamma^\mu \frac{1}{\not{p} - \not{k} - m}\gamma_\mu u \frac{1}{k^2} \tag{30}$$

This integral divergences. One can regulate it using Pauli-Villars' procedure or Feynman's method. Here we will use a simpler regularization. We first transform the Minkowski space of the integral to Euclidean space by a Wick rotation[9]. Then we impose a cut-off on it at a large momentum.

To illustrate our method, let us calculate first

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + i\varepsilon - L}$$

we make a substitution: $k_0 \to ik_0$, so that

$$k^2 = -(k_0^2 + \mathbf{k}^2) = -k_E^2 \tag{31}$$

k_E is obviously a Lorentz invariant. Therefore,

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon - L} &= \frac{-i}{(2\pi)^4} \int d^4k_E \frac{1}{k_E^2 + L} \\
&= \frac{-i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dk_E \frac{k_E^3}{k_E^2 + L} \\
&= \frac{-i}{8\pi^2} (\frac{\lambda^2}{2} - \frac{L}{2} \log \frac{\lambda^2 + L}{L})
\end{aligned}$$

Here in the last step we have cut off the integral at λ. Replacing the integration variable k with $k - p$ gives

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - 2p \cdot k - \Delta} = \frac{-i}{8\pi^2} (\frac{\lambda^2}{2} - \frac{\Delta + p^2}{2} \log \frac{\lambda^2 + \Delta + p^2}{\Delta + p^2})$$

Differentiating both sides of the above equation with respect to Δ and p^α respectively yields

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - 2p \cdot k - \Delta)^2} = \frac{i}{8\pi^2} \left[\frac{1}{2} \log \frac{\lambda^2 + \Delta + p^2}{\Delta + p^2} - \frac{\lambda^2}{2(\lambda^2 + \Delta + p^2)} \right]$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\alpha}{(k^2 - 2p \cdot k - \Delta)^2} = \frac{i}{16\pi^2} \left[p_\alpha \log \frac{\lambda^2 + \Delta + p^2}{\Delta + p^2} - \frac{\lambda^2 p_\alpha}{\lambda^2 + \Delta + p^2} \right]$$

With the help of these integrals and Feynman parameters, we are now able to calculate (30):

$$\Delta E = \frac{m}{E} \frac{e^2}{2\pi} \cdot 3m \log \frac{\lambda}{m}$$

Or

$$\Delta m = \frac{e^2}{2\pi} \cdot 3m \log \frac{\lambda}{m}$$

This additional mass would lead to additional momentum and energy which are of course relativistically covariant*. The self-action of a free electron in quantum electrodynamics differs greatly with that in classical theory. The difference can be classified into three aspects: spin, vacuum fluctuations and the electrons in negative energy states in vacuum[8].

What if when the electron is accelerated? (27) are far from being the appropriate equations of motion. The actual situation is definitely not that the electron can be treated as a point particle. But we have done our best without the aid of hypothesises. Continuing to seek an answer in the classical regime of electromagnetic dynamics is hopeless and unwise.

*Note that the way we prove the covariance is not the same as that we used in previous discussion about classical electrodynamics. What the perturbation theory of quantum mechanics concerns is the change of the energies of free eigenstates when their momentums are fixed.

Appendix A

We first establish a coordinate system for the four-dimensional space-time. It is convenient to employ the delay quantities as the coordinates. From (17) and (18), we obtain

$$x^\mu = \chi^\mu(\tau) + \rho(u^\mu + \frac{v^\mu}{c}) \tag{A.1}$$

Consider the case where the electron is moving along the x axis of our reference frame. We set

$$\chi^\mu(\tau) = (ct(\tau), x(\tau), 0, 0) \tag{A.2}$$

$$v^\mu = (\gamma c, \gamma v, 0, 0) \tag{A.3}$$

Note that the speed v is a unknown function of τ and so is γ. ρ and u^μ is R and $(0, \hat{R})$ respectively in the frame where the electron is instantaneously rest. $\hat{R}$ is the unit delay vector. More precisely,

$$(u^\mu)_{Rest} = (0, \cos\theta, \sin\theta\cos\varphi, \sin\theta sin\varphi)$$

We can infer then u^μ is in our reference system

$$u^\mu = (\gamma\beta\cos\theta, \gamma\cos\theta, \sin\theta\cos\varphi, \sin\theta sin\varphi)$$

According to (A.1), (A.2), (A.3)and the above equation, every point x^μ in the space-time now can be determined by using ρ, τ, θ and φ. Thus the coordinate system has been established as we want.

The segment $PP^{'}$ in Figure 1 is actually a three-dimensional sphere in our space-time. To label the points of this part, one needs only 3 coordinates. In fact, the $0th$ component of (A.1) is

$$ct = ct(\tau) + \rho\gamma\beta\cos\theta + \rho\gamma$$

where t is fixed and equal to t_0 on $PP^{'}$. Therefore we can eliminate from the above equation a coordinate such as ρ which would be given by

$$\rho = \frac{ct_0 - ct(\tau)}{\gamma(1+\beta\cos\theta)}$$

The points of $PP^{'}$ are now labeled by τ, θ and φ. After some cumbersome calculation we find the Jacob's determinant from the space components of (A.1)

$$\frac{\partial(x,y,z)}{\partial(\tau,\theta,\varphi)} = -\frac{c\rho^2\sin\theta}{\gamma(1+\beta\cos\theta)} \tag{A.4}$$

where we have retained the appearance of ρ. With the above equation we are able to compute the integrals on the lhs of (26). The normal vector of the surface represented by $PP^{'}$ is

$$n_\mu = (-1, 0, 0, 0)$$

From (21), we get

$$p_a^1 = \int_{P'}^{P} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta_a^{\mu 0} = \frac{1}{4\pi c}\int_0^{2\pi} d\varphi \int_0^{\tau_0} d\tau \int_{-1}^{1} d\cos\theta \cdot (-1)\cdot \frac{e^2}{c^4}({a_u}^2 - a_\lambda a^\lambda)$$

$$\cdot [-\gamma(1+\beta\cos\theta)]\cdot\gamma(\beta\cos\theta+1)\cdot\frac{c}{\gamma(1+\beta\cos\theta)}$$

$$= -\int_0^{\tau_0} d\tau \frac{2}{3}\frac{e^2}{c^5} a_\lambda a^\lambda \gamma v$$

$$p_a^0 = -\int_0^{\tau_0} d\tau \frac{2}{3}\frac{e^2}{c^4}\gamma a_\lambda a^\lambda$$

where we have used:

$$a_\lambda a^\lambda - a_u^2 = a_\lambda a^\lambda (1-\cos^2\theta)$$

The remaining components of p_a^μ are equal to 0. The velocity term is

$$p_v^0 = \int_{P'}^{P} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta_v^{\mu 0} = -\int_0^{\tau_0} d\tau \frac{1}{2}\frac{e^2\gamma^3}{(ct_0 - ct(\tau))^2}(1+\frac{1}{3}\beta^2)$$

$$p_v^1 = -\int_0^{\tau_0} d\tau \frac{2}{3}\frac{e^2\gamma^3 v}{c(ct_0 - ct(\tau))^2}$$

The term $\Theta_{va}^{\mu\nu}$ gives

$$p_{va}^0 = \int_{P'}^{P} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta_{va}^{\mu 0} = -\int_0^{\tau_0} d\tau \frac{e^2\gamma(2a^0 - \frac{2}{3}\beta a^1)}{c^2\,(ct_0 - ct(\tau))}$$

$$p_{va}^1 = -\int_0^{\tau_0} d\tau \frac{2}{3}\frac{e^2\gamma(\beta a^0 + a^1)}{c^2\,(ct_0 - ct(\tau))}$$

We may expect the term p_{va}^μ can be canceled out. This is true. We apply an integral by parts to p_v^0 and p_v^1:

$$p_v^0 = -\frac{1}{2}\frac{e^2\gamma^2(1+\frac{1}{3}\beta^2)}{c\,(ct_0 - ct(\tau))}\bigg|_0^{\tau_0} + \int_0^{\tau_0} d\tau \frac{e^2\gamma(2a^0 - \frac{2}{3}\beta a^1)}{c^2\,(ct_0 - ct(\tau))}$$

$$p_v^1 = -\frac{2}{3}\frac{e^2\gamma^2\beta}{c\,(ct_0 - ct(\tau))}\bigg|_0^{\tau_0} + \int_0^{\tau_0} d\tau \frac{2}{3}\frac{e^2\gamma(\beta a^0 + a^1)}{c^2\,(ct_0 - ct(\tau))}$$

The result is what we expected.

The term corresponding to $\Theta_{v_0}^{\mu\nu}$ is

$$p_{v_0}^0 = -\int_0^{\tilde{\tau}_0} d\tau \frac{1}{2}\frac{e^2\gamma_0^3}{\left(ct_0 - c\tilde{t}(\tau)\right)^2}(1+\frac{1}{3}{\beta_0}^2) = -\frac{1}{2}\frac{e^2\gamma_0^2(1+\frac{1}{3}{\beta_0}^2)}{c\,(ct_0 - c\gamma_0\tau)}\bigg|_0^{\tilde{\tau}_0} \tag{A.5}$$

$$p_{v_0}^1 = -\frac{2}{3}\frac{e^2\gamma_0^2\beta_0}{c\,(ct_0 - c\gamma_0\tau)}\bigg|_0^{\tilde{\tau}_0} \tag{A.6}$$

where $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$ and

$$\tilde{t}(\tau) = \gamma_0 \tau \ , \quad t_0 = \gamma_0 \tilde{\tau_0} \tag{A.7}$$

is the relation between the coordinate t and the proper time τ for uniform motion of the electron. When substituting the bound τ_0(or $\tilde{\tau_0}$) of the integrals we would get infinities. These infinities will not bother us, as we will see in section 3. Hence we retain the zeros in the denominators but in the limit form. For instance,

$$ct_0 - ct(\tau_0) = \lim_{a \to 0} ct_0 - ct(\tau_0 - \frac{a}{c}) = \lim_{a \to 0} \gamma(\tau_0) \frac{a}{c} \equiv \gamma(\tau_0) \frac{a_{\to 0}}{c}$$

where we have used

$$t(\tau_0 - \frac{a}{c}) \approx t(\tau_0) - \gamma(\tau_0) \frac{a}{c}$$

a is an invariant and has the dimension of length. The treating for the substituting of $\tilde{\tau_0}$ is analogous. Summing up p_v^μ, p_{va}^μ and $p_{v_0}^\mu$ together gives

$$\frac{d}{d\tau_0} \int_{P'}^{P} dS \varepsilon(n) n_\mu \frac{1}{c} (\Theta_{va}^{\mu 0} + \Theta_v^{\mu 0} - \Theta_{v_0}^{\mu 0}) = \frac{d}{d\tau_0} \left(\frac{1}{2} \frac{e^2 \gamma (1 + \frac{1}{3}\beta^2)}{c a_{\to 0}} \right)$$

$$\frac{d}{d\tau_0} \int_{P'}^{P} dS \varepsilon(n) n_\mu \frac{1}{c} (\Theta_{va}^{\mu 1} + \Theta_v^{\mu 1} - \Theta_{v_0}^{\mu 1}) = \frac{d}{d\tau_0} \left(\frac{2}{3} \frac{e^2 \gamma \beta}{c a_{\to 0}} \right)$$

When calculating the integral of the term $\Theta_{crs}^{\mu\nu}$ we regard the external fields as constants. This can be justified by two reasons. One is that we can make the volume V very small so that the external fields becomes homogeneous*. The second is the electron is subjected only to the field at the location where it is. Thus we can just choose the external field as a uniform electric field E_0 without loss of generality. Analogous to our previous discussion, it is not necessary to calculate directly the part on Σ'. Instead, it can be evaluated in the following way: Suppose that a constant force $\mathbf{f} = -eE_0$ begins to act on the electron at $t = 0$ so that it keeps moving uniformly. Thus (24) becomes

$$\frac{d}{d\tau_0} \int_\Sigma dS \varepsilon(n) n_\mu \frac{1}{c} (\Theta_{v_0}^{\mu\nu} + \Theta_{crs(v_0)}^{\mu\nu}) + f^\nu = \frac{d}{d\tau_0} \int_\Sigma dS \varepsilon(n) n_\mu \frac{1}{c} \Theta_{crs(v_0)}^{\mu\nu} + f^\nu = \frac{dp_{mech}^v}{d\tau_0} = 0$$

in virtue of (25). Here $\Theta_{crs(v_0)}^{\mu\nu}$ is the $\Theta_{crs}^{\mu\nu}$ when the electron is moving uniformly with velocity v_0 and f^ν is the covariant form of $\mathbf{f}$:

$$f^\nu = \gamma(\frac{\mathbf{f} \cdot \mathbf{v}}{c}, \ \mathbf{f})$$

Hence,

*We can't treat the fields of the electron in this way because they have a source-the electron, i.e. a singularity.

$$\int_{\Sigma'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs(v_0)} = -f^\nu\tilde{\tau}_0 - \int_P^{P'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs(v_0)}$$

Now we have

$$\begin{aligned}\int_{\Sigma} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs} &= \int_{\Sigma'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs(v_0)} + \int_P^{P'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs} \\ &= -f^\nu\tilde{\tau}_0 - \int_P^{P'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs(v_0)} + \int_P^{P'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs}\end{aligned} \tag{A.8}$$

Let the electric field be along the x axis, we have then

$$(F_{ext})^{01} = -E_0, \ \ (F_{ext})^{10} = E_0$$

All the other componemts of $(F_{ext})^{\mu\nu}$ are equal to 0. We only need to consider $\Theta^{0\nu}_{crs}$ due to n_μ. Let us calculate first its 0 component. From (20) it can be shown that

$$\Theta^{00}_{crs} = \frac{E_0}{4\pi}(F_{sel})_{10}$$

According to (19),

$$\begin{aligned}(F_{sel})_{10} &= \frac{e}{c\rho^2}(v_1u_0 - v_0u_1) + \frac{e}{c^2\rho}[\frac{1}{c}(a_1v_0 - a_0v_1) + u_0(a_1 + \frac{a_uv_1}{c}) - u_1(a_0 + \frac{a_uv_0}{c})] \\ &= \frac{e}{\rho^2}\cos\theta - \frac{ea^0}{c^2\rho}\cdot\frac{1-\cos^2\theta}{\gamma\beta}\end{aligned}$$

The following equations are helpful for us:

$$a^1 = \frac{a^0}{\beta}, \ \ \frac{d\gamma}{dv} = \frac{\gamma^3\beta}{c}, \ a^0 = c\frac{d\gamma}{d\tau} = \gamma^3\beta\frac{dv}{d\tau}$$

The velocity term of Θ^{00}_{crs} gives

$$\begin{aligned}\int_P^{P'} dS(-1)\frac{1}{c}\frac{E_0}{4\pi}\frac{e}{\rho^2}\cos\theta &= -\frac{E_0}{4\pi c}\int_0^{2\pi} d\varphi\int_0^{\tau_0} d\tau\int_{-1}^{1} d\cos\theta\cdot\frac{e}{\rho^2}\cos\theta\cdot\frac{c\rho^2}{\gamma(1+\beta\cos\theta)} \\ &= -\frac{eE_0}{2}\int_0^{\tau_0} d\tau\int_{-1}^{1} dx\cdot\gamma\frac{x}{\gamma^2(1+\beta x)} \\ &= \frac{eE_0}{2}\int_0^{\tau_0} d\tau\int_{-1}^{1} dx\cdot\gamma\left[\beta - \frac{x+\beta}{(1+\beta x)}\right]\end{aligned}$$

Then we can immediately write down

$$\int_P^{P'} dS\varepsilon(n)n_\mu \frac{1}{c}\Theta^{\mu\nu}_{crs(v_0)} = \frac{eE_0}{2}\tilde{\tau}_0 \int_{-1}^{1} dx\cdot\gamma_0\left[\beta_0 - \frac{x+\beta_0}{(1+\beta_0 x)}\right]$$
$$= \frac{eE_0}{2}t_0 \int_{-1}^{1} dx\left[\beta_0 - \frac{x+\beta_0}{(1+\beta_0 x)}\right]$$

where (A.7) has been used in the last step. The acceleration term gives

$$\int_P^{P'} dS(-1)\frac{1}{c}\frac{E_0}{4\pi}\frac{-ea^0}{c^2\rho}\frac{1-\cos^2\theta}{\gamma\beta} = \frac{eE_0}{2c}\int_0^{\tau_0}[t_0 - t(\tau)]\frac{dv}{d\tau}d\tau\int_{-1}^{1}dx\frac{1-x^2}{(1+\beta x)^2}$$
$$= \frac{eE_0}{2}\int_0^{\tau_0}[t_0 - t(\tau)]d\left[\int_{-1}^{1}dx\frac{x+\beta}{(1+\beta x)}\right]$$
$$= -\frac{eE_0 t_0}{2}\int_{-1}^{1}dx\frac{x+\beta_0}{(1+\beta_0 x)} + \frac{eE_0}{2}\int_0^{\tau_0}\gamma d\tau\int_{-1}^{1}dx\frac{x+\beta}{(1+\beta x)}$$

Summing up these results according to (A.8) yields

$$\int_\Sigma dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu 0}_{crs} = eE_0\int_0^{\tau_0}\gamma\beta d\tau$$

The treatment for the 1 component similar. Thus it follows that

$$\frac{d}{d\tau_0}\int_\Sigma dS\varepsilon(n)n_\mu\frac{1}{c}\Theta^{\mu\nu}_{crs} = -\frac{e}{c}F^{\nu\lambda}_{ext}v_\lambda$$

After collecting all the results of our calculation, we finally obtain(replace τ_0 with τ)*

$$\frac{\mathrm{d}\mathbf{p}_{mech}}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2}{3}\frac{e^2\gamma\beta}{ca_{\to 0}}\right) = \gamma\left(e\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) - \frac{2}{3}\frac{e^2}{c^5}a^\lambda a_\lambda\gamma\mathbf{v} \tag{A.9a}$$

$$\frac{\mathrm{d}E_{mech}}{c\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\frac{e^2\gamma\ \left(1+\frac{1}{3}\beta^2\right)}{ca_{\to 0}}\right) = \gamma\frac{e\mathbf{v}\cdot\mathbf{E}}{c} - \frac{2}{3}\frac{e^2}{c^4}a^\lambda a_\lambda\gamma \tag{A.9b}$$

Now we apply a cut-off on τ, i.e. we replace $a_{\to 0}$ with a constant a_0:

$$\frac{\mathrm{d}\mathbf{p}_{mech}}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{2}{3}\frac{e^2\gamma\beta}{ca_0}\right) = \gamma\left(e\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) - \frac{2}{3}\frac{e^2}{c^5}a^\lambda a_\lambda\gamma\mathbf{v} \tag{A.10a}$$

$$\frac{\mathrm{d}E_{mech}}{c\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\frac{e^2\gamma\ \left(1+\frac{1}{3}\beta^2\right)}{ca_0}\right) = \gamma\frac{e\mathbf{v}\cdot\mathbf{E}}{c} - \frac{2}{3}\frac{e^2}{c^4}a^\lambda a_\lambda\gamma \tag{A.10b}$$

This procedure is equivalent to digging out a sphere of radius a_0(in the instantaneously rest reference frames) from the volume in which we calculate the energy and momentum of the electromagnetic fields.

*Although they are derived for the case of rectilinear motion, they can apply to the cases of curvilinear motion since there is no limit on the duration of the motion of the electron in our calculation.

References

[1] D. J. Griffiths, Introduction to Electrodynamics, Addison-Wesley, 1999

[2] J. D. Jackson, Classical Electrodynamics , John Wiley and Sons, 1998

[3] J. L. Synge, Relativity: The Special Theory, North-Holland, 1956

[4] F. Rohrlich, Classical Charged Particles, Addison-Wesley, 1965

[5] Julian Schwinger, Foundations of Physics, Vol.13, No.3, 1983

[6] Julian Schwinger, Phys. Rev. 75, 651, 1949

[7] R. P. Feynman, Quantum Electrodynamics, W.A. Benjamin, INC. 1961

[8] V. F. Weisskopf, Phys. Rev. 56, 72, 1939

[9] M. E. Peskin and D. V. Shroeder, An Introduction to Quantum Field Theory, Perseus Books Publishing, 1995: p.193