Positivity of the Fourier Transform of the Shortest Maximal Order Convolution Mask for Cardinal B-splines

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In [2] approximations of functions into manifolds were studied. For the transformation of function values to B-spline coefficients convolution masks were considered. Some of the proofs required that the convolution mask had a positive Fourier transform. This property was used to show that the inverse of the convolution exists and that the spatial dependency decays exponentially. For splines of degree m the existence of such convolution masks of length $m^2/2$ were constructively proven. It was posed as an open question if families with shorter sequences could satisfy this property. For $m \leq 21$ it was computationally verified that the shortest possible sequence that satisfies the polynomial reproduction property also has a positive Fourier transform. This sequence has length m. It was conjectured that this holds true for all odd $m > 0$. In Section 1 of this work we will prove this fact. In Section 2 we describe how the convolution masks can be computed.

1 Theory

We start by defining cardinal B-splines

Definition 1. *Cardinal B-splines can recursively be defined by*

$$B_0 = 1_{\left[-\frac{1}{2},\frac{1}{2}\right]} \text{ and } B_m = B_{m-1} * B_0 \text{ for all } m \geq 1$$

where $1_{\left[-\frac{1}{2},\frac{1}{2}\right]}$ *denotes the indicator function on the interval* $\left[-\frac{1}{2},\frac{1}{2}\right]$ *and* $*$ *denotes the convolution.*

Next we consider the polynomial reproduction property

$$\sum_{i\in\mathbb{Z}} \sum_{j=-S}^{S} B_m(x-i)p(i-j)\lambda_j = p(x) \text{ for all } x \in \mathbb{R} \tag{1}$$

and for all polynomials p of degree $\leq m$. As in [2] we will use the functions $N_m, \Lambda\colon \mathbb{C} \to \mathbb{C}$ defined by

$$N_m(z) = \sum_{j=-(m-1)/2}^{(m-1)/2} B_m(j) z^j, \qquad \Lambda(z) = \sum_{j=-S}^{S} \lambda_j z^j. \tag{2}$$

We will now proof an equivalent formulation of the polynomial reproduction property.

Lemma 1. *The polynomial reproduction property* (1) *is equivalent to*

$$\frac{d^l}{dz^l}(N_m(z)\Lambda(z))|_{z=1} = \begin{cases} 1 & l = 0 \\ 0 & l \in \{1, \dots, m\} \end{cases}. \tag{3}$$

Proof. We have

$$\begin{aligned}
& \frac{d^l}{dz^l}(N_m(z)\Lambda(z))|_{z=1} & (4)\\
= \quad & \frac{d^l}{dz^l}\left(\sum_{k=-(m-1)/2}^{(m-1)/2} B_m(k) z^k\right)\left(\sum_{j=-S}^{S} \lambda_j z^j\right)|_{z=1} & (5)\\
= \quad & \frac{d^l}{dz^l}\left(\sum_{k=-S-(m-1)/2}^{S+(m-1)/2} \sum_{j=-S}^{S} B_m(k-j)\lambda_j z^k\right)|_{z=1} & (6)\\
= \quad & \left(\sum_{k=-S-(m-1)/2}^{S+(m-1)/2} \sum_{j=-S}^{S} B_m(k-j)\lambda_j k \dots (k-(l-1)) z^{k-l}\right)|_{z=1} & (7)\\
= \quad & \left(\sum_{k=-S-(m-1)/2}^{S+(m-1)/2} \sum_{j=-S}^{S} B_m(k-j)\lambda_j p_l(-k)\right) & (8)
\end{aligned}$$

where $p_0 = 1, p_l(x) = (-x)(-x-1)\dots(-x-(l-1))$. If the polynomial reproduction property (1) holds Expression (8) is equal to

$$p_l(0) = \begin{cases} 1 & l = 0 \\ 0 & l \in \{1, \dots, m\}. \end{cases}$$

On the other hand if (3) holds the polynomial reproduction property (1) holds for $x = 0$ and $p = p_l$ for all $l \in \{0, \dots, m\}$. As the polynomials $p_0, \dots, p_m$ build a basis for the space of polynomials of degree $\leq m$ the polynomial reproduction property (1) holds for all polynomials p of degree $\leq m$ at $x = 0$. By replacing x resp. i by $x+1$ resp. $i+1$ it follows that the polynomial reproduction property holds at all integer points. Now consider the function

$$f(x) = \sum_{i \in \mathbb{Z}} \sum_{j=-S}^{S} B_m(x-i) p(i-j) \lambda_j.$$

Since $B'_m(x) = B_{m-1}(x+\frac{1}{2}) - B_{m-1}(x-\frac{1}{2}) = \delta B_{m-1}(x-\frac{1}{2})$ where δ is the discrete difference operator $\delta u(x) = u(x+1) - u(x)$ it follows inductively that $B_m^{(m)}(x) = \delta^m B_0(x - \frac{m}{2})$. Since δ commutes with the convolution it follows that

$$\begin{aligned} f^{(m)}(x) &= \sum_{i\in\mathbb{Z}} \sum_{j=-S}^{S} B_m^{(m)}(x-i)p(i-j)\lambda_j && (9) \\ &= \sum_{i\in\mathbb{Z}} \sum_{j=-S}^{S} \delta^m B_0\left(x - i - \frac{m}{2}\right) p(i-j)\lambda_j && (10) \\ &= \sum_{i\in\mathbb{Z}} \sum_{j=-S}^{S} B_0\left(x - i - \frac{m}{2}\right) \delta^m p(i-j)\lambda_j && (11) \end{aligned}$$

Note that every time δ is applied the polynomial degree decreases by 1. Hence $\delta^m p(i-j)$ and therefore $f^{(m)}$ is constant. It follows that f is a polynomial and since p coincides with f on all integers that $f = p$. Hence the polynomial reproduction property holds for all polynomials of degree $\leq m$ and all $x \in \mathbb{R}$. $\square$

We can now formulate and proof our theorem.

Theorem 1. *Let $k \geq 0$ be an nonnegative integer. Then there exist a unique symmetric (i.e. $\lambda_{-i} = \lambda_i$) sequence $(\lambda_j)_{j=-k}^{k} \subset \mathbb{R}$ that satisfies the polynomial reproduction property* (1) *for $m = 2k+1$. Furthermore we have*

$$\sum_{j=-k}^{k} \lambda_j e^{2\pi i j\omega} \geq 1 > 0 \text{ for all } \omega \in \mathbb{R}.$$

Proof. Since λ and B_m are symmetric, i.e. $\lambda_{-i} = \lambda_i$ and $B_m(-x) = B_m(x)$, both $N_m(z)$ and $\Lambda(z)$ can be written as polynomials of degree k in

$$x = z + z^{-1} - 2 = \frac{(z-1)^2}{z}, \text{ i.e. } N_m(z) = p\left(\frac{(z-1)^2}{z}\right), \quad \Lambda(z) = q\left(\frac{(z-1)^2}{z}\right)$$

for polynomials p, q of degree k. Condition (3) is by Lemma 1 equivalent to

$$p(x)q(x) = 1 + x^{k+1}(\dots), \tag{12}$$

i.e. that the constant coefficient of $p(x)q(x)$ is one and coefficients of order 1 up to order k are zero. We first prove uniqueness of q and therefore of $(\lambda_j)_{j=-k}^{k} \subset \mathbb{R}$. Let q_1, q_2 be two polynomials of degree k satisfying (12). Then it follows that

$$p(x)(q_1(x) - q_2(x)) = x^{k+1}(\dots)$$

and since $p(0) = 1 \neq 0$ that $q_1(x) - q_2(x) = x^{k+1}(\dots)$ and since q_1 and q_2 are polynomials of degree k that $q_1 = q_2$.

The polynomial $N_m(z)z^{(m-1)/2}m!$ is known as the Eulerian polynomial. By [1] the Eulerian polynomial and therefore N_m has only negative and simple real roots. If z_1 is a root of N_m then $x_i = z_i + z_i^{-1} - 2 \leq -4$ is a root of p. Furthermore all k roots $x_1, \dots, x_k$ of p can be constructed in this way. Therefore the roots of p are all smaller or equal to -4. Note that for $|x| < 4$ we have

$$\frac{1}{p(x)} = \frac{1}{\prod_{i=1}^{k}\left(1 - \frac{x}{x_i}\right)} = \prod_{i=1}^{k}\sum_{j=0}^{\infty}\frac{x^j}{x_i^j}.$$

Define q by the truncating this power series at order $k+1$, i.e. such that

$$\frac{1}{p(x)} = q(x) + x^{k+1}(\dots)$$

Then we have

$$p(x)q(x) = p(x)\left(\frac{1}{p(x)} + x^{k+1}(\dots)\right) = 1 + x^{k+1}(\dots),$$

which shows that (12) is satisfied, i.e. the sequence $(\lambda_j)_{j=-k}^{k}$ corresponding to q satisfies the polynomial reproduction property.

The statement for the Fourier transform is equivalent to

$$\Lambda(z) \geq 1 \text{ for all } |z| = 1$$

which is equivalent to

$$q(x) \geq 1 \text{ for all } -4 \leq x \leq 0.$$

Since all roots $x_1, \dots, x_k$ of p are negative all terms $\frac{x^j}{x_i^j}$ in the power series of $\frac{1}{p(x)}$ are positive for $x \in (-4, 0)$. Therefore also all terms of the power series of $\frac{1}{p(x)}$ and therefore also all polynomial terms of $q(x)$ are positive and since the zero-order term is one we have $q(x) \geq 1$. The special cases $x = -4$ and $x = 0$ follow from continuity. □

2 Construction

In this section, we describe how to construct the coefficients. It is tedious even for small m to compute the roots of the polynomial p and the power series of $1/p$ used in the proof in the previous section. To get the coefficients it is easier to determine the polynomial coefficients the polynomial q recursively. To compute the coefficients of the N_m, i.e. the values of the B-splines at integers de Boors 3-point recursion can be used which for our case of uniform grids reads

$$B_m(x) = \frac{\left(\frac{m+1}{2} + x\right) B_{m-1}(x + \frac{1}{2}) + \left(\frac{m+1}{2} - x\right) B_{m-1}(x - \frac{1}{2})}{m}. \tag{13}$$

2.1 The cases $m = 3$ and $m = 5$

For $m = 3$ we have

$$N_m(z) = \frac{1}{6}z^{-1} + \frac{4}{6} + \frac{1}{6}z^1 = 1 + \frac{1}{6}\left(z + z^{-1} - 2\right) \tag{14}$$

$$p(x) = 1 + \frac{x}{6} \tag{15}$$

$$\frac{1}{p(x)} = 1 - \frac{x}{6} + \left(\frac{x}{6}\right)^2 - \cdots \tag{16}$$

$$q(x) = 1 - \frac{1}{6}x \tag{17}$$

$$\Lambda(z) = 1 - \frac{1}{6}\left(z + z^{-1} - 2\right) = -\frac{1}{6}z^{-1} + \frac{4}{3} - \frac{1}{6}z^1 \tag{18}$$

$$(\lambda_j)_{j=-k}^{k} = \left(-\frac{1}{6}, \frac{4}{3}, \frac{1}{6}\right) \tag{19}$$

For $m = 5$ we have

$$N_m(z) = \frac{1}{120}z^{-2} + \frac{26}{120}z^{-1} + \frac{66}{120} + \frac{26}{120}z^1 + \frac{1}{120}z^2 \tag{20}$$

$$= 1 + \frac{1}{4}\left(z + z^{-1} - 2\right) + \frac{1}{120}\left(z + z^{-1} - 2\right)^2 \tag{21}$$

$$p(x) = 1 + \frac{x}{4} + \frac{x^2}{120} \tag{22}$$

$$q(x) = 1 - \frac{x}{4} + \frac{13x^2}{240} \tag{23}$$

$$\Lambda(z) = 1 - \frac{1}{4}\left(z + z^{-1} - 2\right) + \frac{13}{240}\left(z + z^{-1} - 2\right)^2 \tag{24}$$

$$= \frac{13}{240}z^{-2} - \frac{7}{15}z^{-1} + \frac{73}{40} - \frac{7}{15}z^1 + \frac{13}{240}z^2 \tag{25}$$

$$(\lambda_j)_{j=-k}^{k} = \left(\frac{13}{240}, -\frac{7}{15}, \frac{73}{40}, -\frac{7}{15}, \frac{13}{240}\right) \tag{26}$$

2.2 Code to construct convolution mask for arbitrary m

Below an octave/matlab code for computing the convolution mask.

```
function lambda = conv_mask(m)

assert(mod(m,2)==1,'m must be an odd positive integer')
k=(m-1)/2;

%compute B-spline at integer by de Boor three point recursion
b=1;
for j=2:m
   x=(j-1)/2*linspace(-1,1,j);
```

```
   b=(((j+1)/2+x).*[b 0]+((j+1)/2-x).*[0 b])/j;
end

%coefficient wrt z of powers of x=1/z+z-2
powx=cell(1,k);
powx{1}=[1 -2 1];
for j=2:k
   powx{j}=conv(powx{1},powx{j-1});
end

%determine recursively coeeficients of p(x)
cp=zeros(1,k+1);
for j=1:k
   cp(j)=b(1)/powx{k+1-j}(end);
   b=b-b(1)*powx{k+1-j};
   %remove first and last coefficient which is zero
   b([1 end])=[];
end
cp(k+1)=b(1);

%determine coefficients of q
cq=zeros(1,k+1);
cq(k+1)=1;
for j=k:-1:1
   cq(j)=-sum(cq(j+1:end).*cp(end-1:-1:j));
end

%determine coefficients of lambda(z)
lambda=cq(end);
for j=1:k
   lambda=[0 lambda 0]+cq(end-j)*powx{j};
end

end
```

References

[1] G. Frobenius. Über die Bernoullischen Zahlen und die Eulerschen Polynome, 1910.

[2] P. Grohs and M. Sprecher. Projection-based quasiinterpolation in manifolds. Technical Report 2013-23, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2013.