

ANOTHER PROOF FOR CATALAN'S CONJECTURE

JULIAN BEAUCHAMP

ABSTRACT. In 2002 Preda Mihailescu used the theory of cyclotomic fields and Galois modules to prove Catalan's Conjecture. In this short paper, we give a very simple proof. We first prove that no solutions exist for $a^x - b^y = 1$ for $a, b > 0$ and $x, y > 2$. Then we prove that when $x = 2$ the only solution for a is $a = 3$ and the only solution for y is $y = 3$.

Introduction Catalan's Conjecture was first made by Belgian mathematician Eugne Charles Catalan in 1844, and states that 8 and 9 (2^3 and 3^2) are the only consecutive powers, excluding 0 and 1. That is to say, that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b > 0$, $x, y > 1$ is $a = 3$, $x = 2$, $b = 2$, $y = 3$. In other words, Catalan conjectured that $3^2 - 2^3 = 1$ is the only nontrivial solution. It was finally proved in 2002 by number theorist Preda Mihailescu making extensive use of the theory of cyclotomic fields and Galois modules.

Theorem 0.1. *To demonstrate that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b > 0$, $x, y > 1$ is $a = 3$, $x = 2$, $b = 2$, $y = 3$.*

Lemma 0.2. *To demonstrate that no solutions in the natural numbers exist for $a^x - b^y = 1$ for $a, b > 0$, $x, y > 2$.*

Proof. We first assume that solutions do exist to this equation for $a, b > 0$, $x, y > 2$.

Then we observe that:

$$(0.1) \quad a^x - b^y = (a + b)(a^{x-1} - b^{y-1}) - ab(a^{x-2} - b^{y-2}).$$

We can therefore state our equation as follows:

$$(0.2) \quad (a + b)(a^{x-1} - b^{y-1}) - 1 = ab(a^{x-2} - b^{y-2}).$$

But note that $(a^{x-1} - b^{y-1})$ and $(a^{x-2} - b^{y-2})$ are both divisible by $(a - b)$. Therefore, as long as all four exponents are greater than zero, i.e. $x, y > 2$, it follows that neither side is divisible by $(a - b)$ unless $a - b = 1$, for $(a + b)$ is then divisible by 1.

So let $b = a - 1$, such that from (0.2):

$$(0.3) \quad (2a - 1)(a^{x-1} - (a - 1)^{y-1}) - 1 = a(a - 1)(a^{x-2} - (a - 1)^{y-2}).$$

$$(0.4) \quad \rightarrow 2a^x - a^{x-1} - 2a(a - 1)^{y-1} + (a - 1)^{y-1} - 1 = a^x - a^{x-1} - a^2(a - 1)^{y-2} + a(a - 1)^{y-2},$$

Date: Dec 2018.

2010 Mathematics Subject Classification. Primary 11D41.

Key words and phrases. Number Theory, Catalan's Conjecture, Mihailescu's Theorem.

$$(0.5) \quad \rightarrow a^x - 1 = (a - 1)^{y-2}(a - a^2) + (a - 1)^{y-1}(2a - 1),$$

$$(0.6) \quad \rightarrow a^x - 1 = (a - 1)^{y-2}[(a - a^2) + (a - 1)(2a - 1)],$$

$$(0.7) \quad \rightarrow a^x - 1 = (a - 1)^{y-2}(3a - a^2 - 1).$$

But we know that $a^x - 1 = b^y$, from which it follows that:

$$(0.8) \quad b^y = (a - 1)^{y-2}(3a - a^2 - 1),$$

And since $b = (a - 1)$, it follows that:

$$(0.9) \quad (a - 1)^y = (a - 1)^{y-2}(3a - a^2 - 1),$$

$$(0.10) \quad \rightarrow (3a - a^2 - 1) = (a - 1)^2,$$

$$(0.11) \quad \rightarrow (3a - a^2 - 1) = (a^2 - 2a + 1)$$

$$(0.12) \quad \rightarrow 2a^2 - 5a + 2 = 0,$$

$$(0.13) \quad \rightarrow (2a - 1)(a - 2) = 0,$$

From this it follows either that:

$$(0.14) \quad a = \frac{1}{2},$$

which is not an integer solution, or that:

$$(0.15) \quad a = 2.$$

And since $b = a - 1$ it follows that:

$$(0.16) \quad b = 1.$$

So we get:

$$(0.17) \quad 2^x - 1^y = 1.$$

The only solution for x is $x = 1$. But this is disallowed by the conditions that require that $x, y > 1$.

Therefore there can be no solutions for exponents when both exponents are greater than 2 for the equation $a^x - b^y = 1$, leading to a contradiction in our initial assumption. Therefore the lemma is true.

It follows from this that at least x or y must equal 2 (and its respective base be square-free).

Lemma 0.3. *To demonstrate that when $x = 2$ no solutions in the natural numbers exist for $a^x - b^y = 1$ for $a, b > 0$, $x, y > 1$ other than $a = 3$, $x = 2$, $b = 2$, $y = 3$.*

We have shown that at least x or y must be a square (and its base be square-free). Therefore let $x = 2$. Using our equation in (0.2), we can state:

$$(0.18) \quad (a + b)(a - b^{y-1}) - 1 = ab(1 - b^{y-2}).$$

It is still true that $b = a - 1$, so it follows that:

$$(0.19) \quad a^2 + a(a - 1) - a(a - 1)^{y-1} - (a - 1)^y - 1 = a(a - 1) - a(a - 1)^{y-1}.$$

$$(0.20) \quad \rightarrow a^2 - 1 = (a - 1)^y.$$

$$(0.21) \quad \rightarrow (a + 1)(a - 1) = (a - 1)^y.$$

$$(0.22) \quad \rightarrow (a + 1) = (a - 1)^{y-1}.$$

However, since the bases on both sides differ in value by just 2, there are only two non-trivial values for a that avoid different prime factors on both sides, namely $a = 2$ and $a = 3$. But if $a = 2$, then $(2 + 1) \neq 1^{y-1}$. On the other hand, if $a = 3$, then it follows that:

$$(0.23) \quad 4 = 2^{y-1}.$$

$$(0.24) \quad \rightarrow y = 3.$$

And from the original equation, we can now say that:

$$(0.25) \quad 3^2 - 1 = b^3.$$

We know anyway that if $a = 3$ and $b = a - 1$ it follows that $b = 2$, all of which satisfies the original equation:

$$(0.26) \quad 3^2 - 2^3 = 1.$$

Thus, it follows that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b > 0, x, y > 1$ is $a = 3, x = 2, b = 2, y = 3$. \square

REFERENCES

THE RECTORY, VILLAGE ROAD, WAVERTON, CHESTER CH3 7QN, UK
E-mail address: julianbeauchamp47@gmail.com