Solution of a Sangaku "Tangency" Problem via Geometric Algebra

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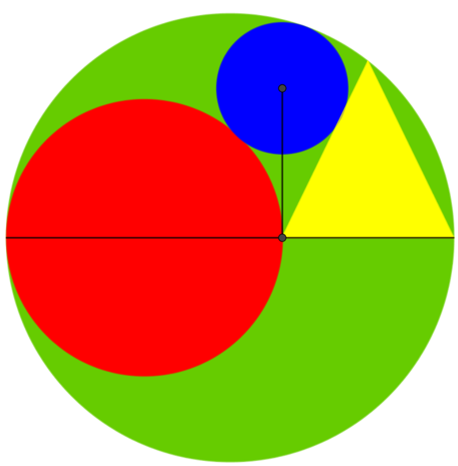
Abstract

Because the shortage of worked-out examples at introductory levels is an obstacle to widespread adoption of Geometric Algebra (GA), we use GA to solve one of the beautiful *sangaku* problems from 19th-Century Japan. Among the GA operations that prove useful is the rotation of vectors via the unit bivector **i**.



"The center of the red circle and the base of the isosceles triangle lie along the same diameter of the green circle. The blue circle is tangent to the other three figures. Prove that the line connecting its center to the point of contact between the red circle and the triangle is perpendicular to the above-mentioned diameter."

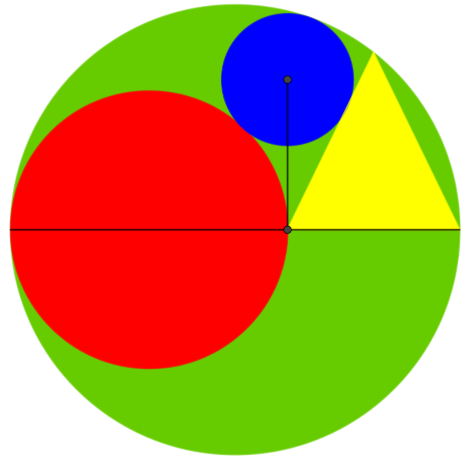


Figure 1: The center of the red circle and the base of the isosceles triangle lie along the same diameter of the green circle. The blue circle is tangent to the other three figures. Prove that the line connecting its center to the point of contact between the red circle and the triangle is perpendicular to the above-mentioned diameter.

1 Problem Statement

> *In Fig. 1, the center of the red circle and the base of the isosceles triangle lie along the same diameter of the green circle. The blue circle is tangent to the other three figures. Prove that the line connecting its center to the point of contact between the red circle and the triangle is perpendicular to the above-mentioned diameter.*

2 Formulation of the Problem in Geometric-Algebra Terms

Fig. 2 defines the vectors that we will use. (Later, we will use an additional, slightly-modified formulation.) Note the notation used to distinguish between points and vectors: for example, $\mathbf{c}_1$ (bolded) is the vector from the origin to the point c_1 (italicized). Also, c_1^2 denotes $\|\mathbf{c}_1\|^2$.

In GA terms, we are to prove that $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$. Other formulations are possible; for example, that $\mathbf{c}_3\hat{\mathbf{b}} = \hat{\mathbf{b}}\mathbf{c}_3$.

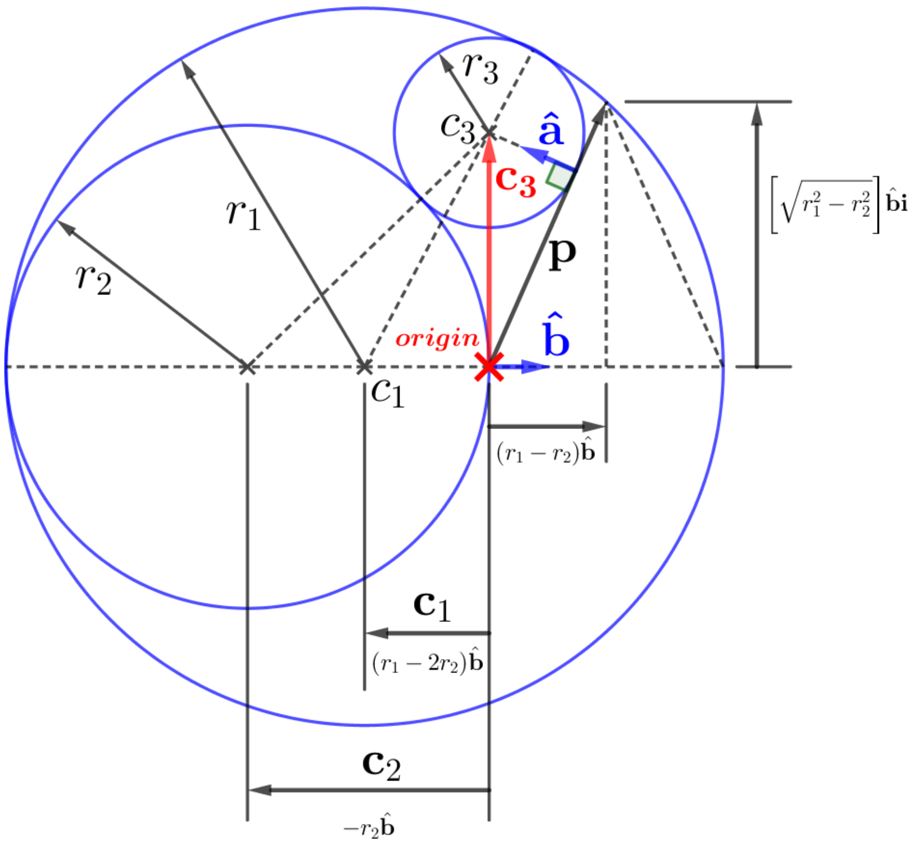


Figure 2: The vectors and frame of reference that we will use in our first solution.

3 Observations

From Fig. 2, two key observations are that

$$r_3 = \mathbf{c}_3 \cdot \hat{\mathbf{a}} \tag{3.1}$$

and that, in turn,

$$\begin{aligned}\hat{\mathbf{a}} &= \hat{\mathbf{p}}\mathbf{i} \\ &= -\left[\frac{\sqrt{r_1^2 - r_2^2}}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right]\hat{\mathbf{b}} \; + \; \left[\frac{r_1 - r_2}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right]\hat{\mathbf{b}}\mathbf{i}\ .\end{aligned} \tag{3.2}$$

We also see that by expressing the distance between c_1 and c_3 as $r_1 - r_3$ and $\|\mathbf{c}_3 - \mathbf{c}_1\|$, we obtain

$$\left(\mathbf{c}_3 - \mathbf{c}_1\right)^2 = \left(r_1 - r_3\right)^2 ,$$

which after simplification becomes

$$c_3^2 + 2\left(2r_2 - r1\right)\mathbf{c}_3 \cdot \hat{\mathbf{b}} + 4r_2\left(r_2 - r_1\right) = r_3^2 - r_3 r_1\ . \tag{3.3}$$

Similarly, because $\|\mathbf{c}_3 - \mathbf{c}_2\| = r_2 + r_3$,

$$c_3^2 + 2r_2\mathbf{c}_3 \cdot \hat{\mathbf{b}} = r_3^2 + 2r_3 r_2\ . \tag{3.4}$$

4 Solutions

For more information on the features of GA that we will use in these solutions, please see References [1] and [2] .

4.1 Solution Strategy

We will derive equations, by two methods, for the center and radius of the smallest circle. We will see that those equations are satisfied by distinct two circles, for one of which $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$.

4.2 First Solution

4.2.1 Derivation of the Equation that We Seek

We begin by subtracting Eq. (3.4) from Eq. (3.3), then solving for r_3:

$$r_3 = \left(\frac{r_1 - r_2}{r_1 + r_2}\right)\left(2r_2 + \mathbf{c}_3 \cdot \hat{\mathbf{b}}\right). \tag{4.1}$$

Substituting that expression for r_3 in Eq. (3.4), then simplifying,

$$c_3^2 - \left(\frac{r_1 - r_2}{r_1 + r_2}\right)^2 \left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 - 4r_1r_2 \left[\frac{r_1 - r_2}{\left(r_1 + r_2\right)^2}\right] \mathbf{c}_3 \cdot \hat{\mathbf{b}} = \frac{8r_1r_2^2\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2}.$$

Now, we write c_3^2 as $\left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 + \left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2$, obtaining

$$\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2 + \left[\frac{4r_1r_2}{\left(r_1 + r_2\right)^2}\right] \left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 - 4r_1r_2 \left[\frac{r_1 - r_2}{\left(r_1 + r_2\right)^2}\right] \mathbf{c}_3 \cdot \hat{\mathbf{b}} = \frac{8r_1r_2^2\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2}. \tag{4.2}$$

An expression for $\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2$ in terms of $\mathbf{c}_3 \cdot \hat{\mathbf{b}}$ by equating the expressions for r_3 given by Eqs. (3.1) and (4.1),

$$\mathbf{c}_3 \cdot \hat{\mathbf{a}} = \left(\frac{r_1 - r_2}{r_1 + r_2}\right) \left(2r_2 + \mathbf{c}_3 \cdot \hat{\mathbf{b}}\right),$$

then expressing $\hat{\mathbf{a}}$ via Eq. (3.2):

$$\mathbf{c}_3 \cdot \left\{ -\left[\frac{\sqrt{r_1^2 - r_2^2}}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right] \hat{\mathbf{b}} + \left[\frac{r_1 - r_2}{\sqrt{2r_1\left(r_1 - r_2\right)}}\right] \hat{\mathbf{b}}\mathbf{i} \right\} = \left(\frac{r_1 - r_2}{r_1 + r_2}\right) \left(2r_2 + \mathbf{c}_3 \cdot \hat{\mathbf{b}}\right). \tag{4.3}$$

Thus,

$$\begin{aligned} \left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2 = & \left[\frac{\sqrt{2r_1\left(r_1 - r_2\right)}}{r_1 + r_2}\right]^2 \left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 \\ & + 4r_2 \left[\frac{2r_1\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2} + \sqrt{\frac{2r_1}{r_1 + r_2}}\right] \mathbf{c}_3 \cdot \hat{\mathbf{b}} \\ & + \frac{8r_1r_2\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2}. \end{aligned} \tag{4.4}$$

Substituting that expression for $\left[\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right)\right]^2$ in Eq. (5.2),

$$\begin{aligned} 0 = & \left\{ \left[\frac{\sqrt{2r_1\left(r_1 - r_2\right)}}{r_1 + r_2} + \sqrt{\frac{2r_1}{r_1 + r_2}}\right]^2 + \frac{4r_1r_2}{\left(r_1 + r_2\right)^2} \right\} \left(\mathbf{c}_3 \cdot \hat{\mathbf{b}}\right)^2 \\ & + 4r_2 \left\{ \frac{3r_1\left(r_1 - r_2\right)}{\left(r_1 + r_2\right)^2} + \sqrt{\frac{2r_1}{r_1 + r_2}} \right\} \mathbf{c}_3 \cdot \hat{\mathbf{b}}. \end{aligned} \tag{4.5}$$

The two roots are

1. $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$, with $\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right) = \dfrac{2r_2\sqrt{2r_1\left(r_1 - r_2\right)}}{r_1 + r_2}$ and $r_3 = \dfrac{2r_2\left(r_1 - r_2\right)}{r_1 + r_2}$; and

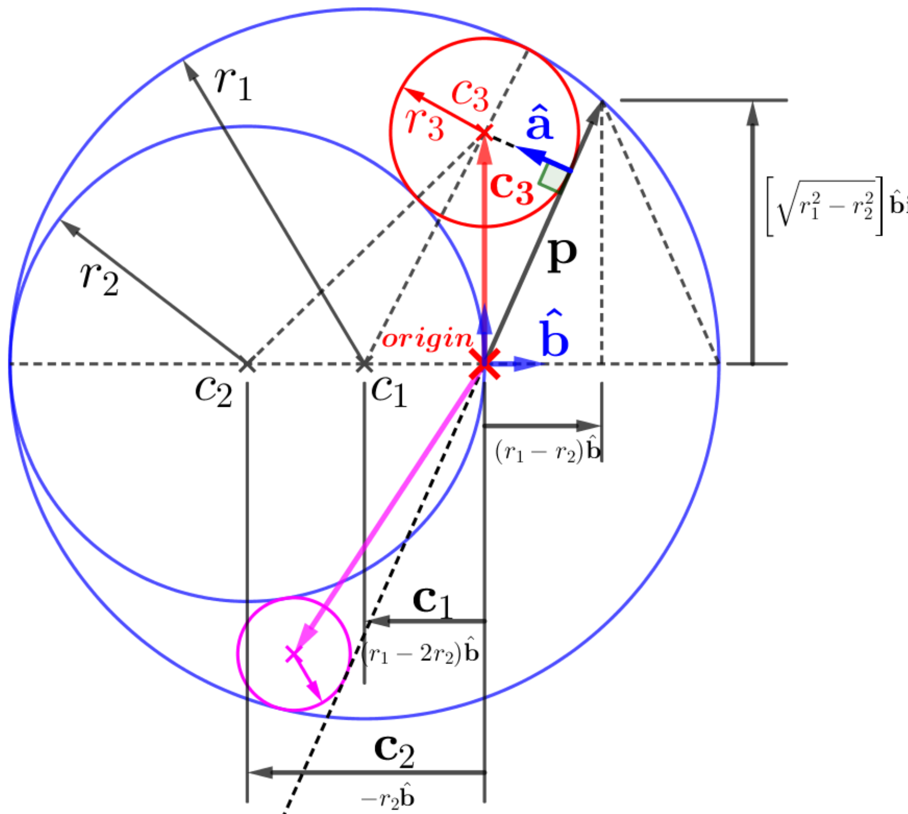


Figure 3: The two solutions to Eq. (4.5). For our purposes, the magenta circle is extraneous: it is tangent to the extension of the side of the isosceles triangle, but not to the triangle itself.

2. $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = -4r_2\left(r_1 - r_2\right)\left[\dfrac{r_1 + \sqrt{2r_1\left(r_1 + r_2\right)}}{2\left(r_1 - r_2\right)\sqrt{2r_1\left(r_1 + r_2\right)} + 3r_1^2 + r_2^2}\right]$,

$\mathbf{c}_3 \cdot \left(\hat{\mathbf{b}}\mathbf{i}\right) = -\dfrac{2r_2\left(r_1 + r_2\right)\sqrt{2r_1\left(r_1 - r_2\right)} + 4r_1r_2\sqrt{r_1^2 - r_2^2}}{2\left(r_1 - r_2\right)\sqrt{2r_1\left(r_1 + r_2\right)} + 3r_1^2 + r_2^2}$, and

$r_3 = \dfrac{2r_1r_2\left(r_1 - r_2\right)}{2\left(r_1 - r_2\right)\sqrt{2r_1\left(r_1 + r_2\right)} + 3r_1^2 + r_2^2}$.

The circle for which $\mathbf{c}_3 \cdot \hat{\mathbf{b}} = 0$ is in red in Fig. 3; the magenta circle is extraneous. It is tangent to the extension of the side of the isosceles triangle, but not to the triangle itself.

4.3 Second Solution

This approach is more conventional than the first, and (arguably) makes better use of GA. We begin by modifying Fig. 2 slightly, to produce Fig. 4. The unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular; their product $\mathbf{i} = \mathbf{e}_1\mathbf{e}_2$ is the unit bivector

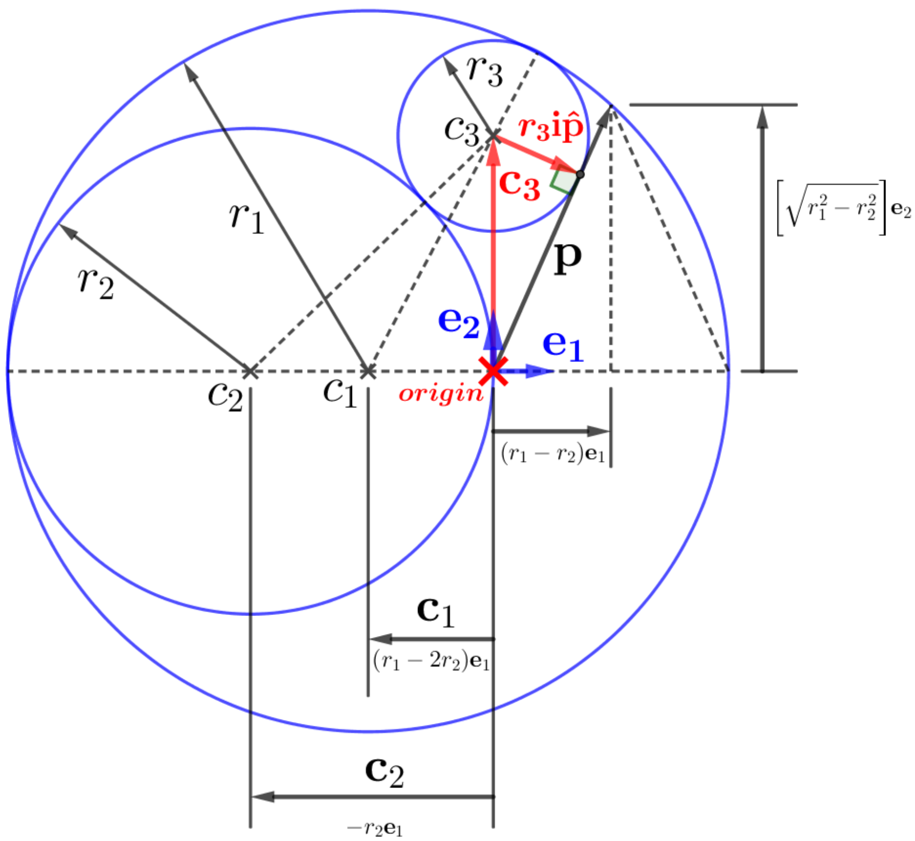


Figure 4: The vectors and frame of reference that we will use in our second solution. The vector $\hat{\mathbf{b}}$ from Fig. 2 has been renamed $\mathbf{e}_1$. Together, $\mathbf{e}_1$ and $\mathbf{e}_2$ form an orthornormal basis for the plane that contains the triangle and the three circles. The unit bivector for said plane is $\mathbf{e}_1\mathbf{e}_2$. Thus, the vector $\mathbf{i}\hat{\mathbf{p}}$ is the clockwise rotation, through $\pi/2$ radians, of $\hat{\mathbf{p}}$.

for the plane that contains the triangle and the three circles.

Now, we'll obtain expressions for the radii of the two solution circles, and the corresponding values of $\mathbf{c}_3 \cdot \mathbf{e}_1$ (which is identical to $\mathbf{c}_3 \cdot \hat{\mathbf{b}}$ in the first solution.) First, we return to Eq. (3.4),

$$c_3^2 + 2r_2\mathbf{c} \cdot \hat{\mathbf{b}} = r_3^2 + 2r_2r_3,$$

and make the substitutions $\mathbf{c_3} \cdot \hat{\mathbf{b}} = \mathbf{c}_3 \cdot \mathbf{e}_1$ and $\mathbf{c}_3 = (\mathbf{c}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{c}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2$. Thus,

$$\left[(\mathbf{c}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{c}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2\right]^2 + 2r_2\,(\mathbf{c}_3 \cdot \mathbf{e}_1) = r_3^2 + 2r_2r_3,$$
$$\therefore (\mathbf{c}_3 \cdot \mathbf{e}_1)^2 + (\mathbf{c}_3 \cdot \mathbf{e}_2)^2 + 2r_2\,(\mathbf{c}_3 \cdot \mathbf{e}_1) = r_3^2 + 2r_2r_3.$$

From Eq. (4.5),

$$\mathbf{c}_3 \cdot \mathbf{e}_1 = \left[\frac{r_1 + r_2}{r_1 - r_2}\right] r_3 - 2r_1. \qquad (4.6)$$

After making that substitution in the previous result, then solving for $\mathbf{c}_3 \cdot \mathbf{e}_2$, we find that

$$\mathbf{c}_3 \cdot \mathbf{e}_2 = \frac{2\sqrt{r_1 r_2 r_3 \left(r_1 - r_2 - r_3\right)}}{r_1 - r_2}. \tag{4.7}$$

Using that result, and Eq. (4.6), we can write $\mathbf{c}_3$ in terms of r_3:

$$\mathbf{c}_3 = \underbrace{\left\{ \left[\frac{r_1 + r_2}{r_1 - r_2} \right] r_3 - 2r_1 \right\}}_{\mathbf{c}_3 \cdot \mathbf{e}_1} \mathbf{e}_1 + \underbrace{\left[\frac{2\sqrt{r_1 r_2 r_3 \left(r_1 - r_2 - r_3\right)}}{r_1 - r_2} \right]}_{\mathbf{c}_3 \cdot \mathbf{e}_2} \mathbf{e}_2. \tag{4.8}$$

Next, we note, from Fig. 4, that the vector $\mathbf{c}_3 + r_3 \left(\mathbf{i}\hat{\mathbf{p}}\right)$ is a scalar multiple of $\mathbf{p}$. Therefore,

$$\left\{\mathbf{c}_3 + r_3 \left(\mathbf{i}\hat{\mathbf{p}}\right)\right\} \wedge \mathbf{p} = 0.$$

Using $\mathbf{c}_3 = \left(\mathbf{c}_3 \cdot \mathbf{e}_1\right) \mathbf{e}_1 + \left(\mathbf{c}_3 \cdot \mathbf{e}_2\right) \mathbf{e}_2$, then rearranging,

$$\begin{aligned} \left(\mathbf{c}_3 \cdot \mathbf{e}_1\right) \mathbf{e}_1 \wedge \mathbf{p} + \left(\mathbf{c}_3 \cdot \mathbf{e}_2\right) \mathbf{e}_2 \wedge \mathbf{p} &= -r_3 \left(\mathbf{i}\hat{\mathbf{p}}\right) \wedge \mathbf{p} \\ &= -r_3 \mathbf{i} \|\mathbf{p}\|, \end{aligned} \tag{4.9}$$

because $\left(\mathbf{i}\hat{\mathbf{p}}\right) \wedge \mathbf{p} = \left\langle \left(\mathbf{i}\hat{\mathbf{p}}\right) \mathbf{p} \right\rangle_2 = \left\langle \mathbf{i} \left(\hat{\mathbf{p}}\mathbf{p}\right) \right\rangle_2 = \left\langle \mathbf{i} \|\mathbf{p}\| \right\rangle_2 = \mathbf{i} \|\mathbf{p}\|$.

From Fig. 4, $\mathbf{p} = \left(r_1 - r_2\right) \mathbf{e}_1 + \left[\sqrt{r_1^2 - r_2^2}\right] \mathbf{e}_2$. Thus, $\|\mathbf{p}\| = \sqrt{\mathbf{2r_1 \left(r_1 - r_2\right)}}$. Making those substitutions, and using Eqs. (4.6) and (4.7), plus $\mathbf{i} = \mathbf{e}_1 \mathbf{e}_2$, Eq. (4.3) becomes (after considerable simplification),

$$\begin{aligned} 2\sqrt{r_1 r_2 r_3 \left(r_1 - r_2 - r_3\right)} &= 2r_2 \sqrt{r_1^2 - r_2^2} \\ &\quad + r_3 \left[\sqrt{2r_1 \left(r_1 - r_2\right)} - \left(\frac{r_2 + r_2}{r_1 - r_2} \right) \sqrt{r_1^2 - r_2^2} \right]. \end{aligned} \tag{4.10}$$

The two values of r_3 that satisfy that equation, and the value of $\mathbf{c}_3 \cdot \mathbf{e}_1$, are as given at the end of Section 4.2.1. As we saw there, $\mathbf{c}_3 \cdot \mathbf{e}_1 = 0$ for the small circle shown in the problem statement.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] J. Smith, 2016, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054.