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A Proposed Proof of The ABC Conjecture

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Abstract In this paper, from a, b, c positive integers relatively prime with $c = a + b$, we consider a bounded of c depending of a, b, then we do a choice of $K(\epsilon)$ and finally we obtain that the ABC conjecture is true. Four numerical examples confirm our proof.

Keywords Elementary number theory · real functions of one variable.

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To the memory of my Father who taught me arithmetic.

1 Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given above:

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Conjecture 1 (***ABC*** **Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students.

2 Proof of the conjecture (1)

Let a, b, c positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$, we can write that a verifies:

$$c = a + b \Rightarrow c(a - b) = a^2 - b^2 < a^2 \Longrightarrow c < \frac{a^2}{a - b} \tag{4}$$

We can write also:

$$c < \frac{a^2}{a - b}.\frac{K(\epsilon).rad(abc)^{1+\epsilon}}{K(\epsilon).rad(abc)^{1+\epsilon}} \tag{5}$$

We propose the constant $K(\epsilon)$ depending of ϵ as :

$$K(\epsilon) = \frac{2}{\epsilon^2} \tag{6}$$

it is a decreasing function so that $lim_{\epsilon \longrightarrow 0} K(\epsilon) = +\infty$ and $lim_{\epsilon \longrightarrow +\infty} K(\epsilon) = 0$. We write (5) as:

$$c < \frac{a^2 \epsilon^2}{a - b}.\frac{1}{2R(abc)^{1+\epsilon}}.K(\epsilon).rad(abc)^{1+\epsilon} \tag{7}$$

It is known that $2 \leq rad(q)$ for $\forall\, q$ a positive integer, then $2^3 \leq rad(abc) \Longrightarrow \frac{1}{rad(abc)} \leq \frac{1}{2^3}$. As $1 + \epsilon < 1 + a^2\epsilon$, we obtain:

$$c < \frac{a^2 \epsilon^2}{a - b}.\frac{1}{2^{4+3a^2\epsilon}}.K(\epsilon).rad(abc)^{1+\epsilon} \tag{8}$$

Let:

$$G(\epsilon, a, b) = \frac{a^2 \epsilon^2}{a - b}.\frac{1}{2^{4+3a^2\epsilon}} \tag{9}$$

Then, equation (8) is written as:

$$c < G(\epsilon, a, b).K(\epsilon).rad(abc)^{1+\epsilon} \tag{10}$$

If we can give a proof that $G(\epsilon, a, b) < 1$ independently of a, b, ϵ, we will obtain:

$$c < G(\epsilon, a, b).K(\epsilon).rad(abc)^{1+\epsilon} < K(\epsilon).rad(abc)^{1+\epsilon} \tag{11}$$

then the ABC conjecture holds with proposing the expression of the constant $K(\epsilon) = \frac{2}{\epsilon^2}$.

2.1 The Proof

We write:

$$G(\epsilon,a,b)=\frac{a^2\epsilon^2}{a-b}.\frac{1}{2^{4+3a^2\epsilon}}\overbrace{<}^{?}1\Rightarrow(a-b)2^{4+3a^2\epsilon}-a^2\epsilon^2\overbrace{>}^{?}0$$

As $a>b$, the minimum value of $a-b$ is equal to 1, then we must verify if :

$$16\times 2^{3a^2\epsilon}-a^2\epsilon^2\overbrace{>}^{?}0\quad or\quad 16e^{(3a^2\epsilon)Log2}-a^2\epsilon^2\overbrace{>}^{?}0 \tag{12}$$

We call:

$$\phi(\epsilon)=16e^{(3a^2\epsilon)Log2}-a^2\epsilon^2\Rightarrow\phi'(\epsilon)=2a^2(24e^{3a^2\epsilon Log2}Log2-\epsilon) \tag{13}$$

$$\phi"(\epsilon)=2a^2(72a^2(Log2)^2e^{3a^2\epsilon Log2}-1)>0\quad\forall\epsilon>0\quad and\ a\geq 2 \tag{14}$$

If we write the table of variations of the function ϕ when $\epsilon\in[0,+\infty[$, we obtain successively $\phi"(\epsilon)>0,\phi'(\epsilon)>0$ and $\phi(\epsilon)>0$ for $\forall a\geq 2$, we deduce that $\forall\epsilon>0,a\geq 2$:

$$16e^{(3a^2\epsilon)Log2}-a^2\epsilon^2>0\Longrightarrow(a-b)2^{4+3a^2\epsilon}-a^2\epsilon^2>0\Longrightarrow$$
$$(a-b)2^{4+3a^2\epsilon}>a^2\epsilon^2\Longrightarrow 1>\frac{a^2\epsilon^2}{(a-b)2^{4+3a^2\epsilon}}\Longrightarrow G(\epsilon,a,b)<1 \tag{15}$$

Then we obtain the important result of the paper:

$$\begin{aligned}&c<K(\epsilon).rad(abc)^{1+\epsilon}\quad\forall\epsilon>0\\&\text{with the constant } K(\epsilon)=\frac{2}{\epsilon^2}\end{aligned} \tag{16}$$

Q.E.D

3 Examples

In this section, we are going to verify some numerical examples.

3.1 Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10}\times 109+2=23^5=6436343 \tag{17}$$

$a=3^{10}.109\Rightarrow\mu_a=3^9=19683$ and $rad(a)=3\times 109$,
$b=2\Rightarrow\mu_b=1$ and $rad(b)=2$,

$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} \tag{18}$$

Let us verify (11):

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{\epsilon} \Longrightarrow c = 6436343 \stackrel{?}{<} 2.10^4.(3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 < 186142827.83 \tag{19}$$

Hence (11) is verified.

3.2 Example of A. Nitaj

3.2.1 Case 1

The example of Nitaj about the ABC conjecture [3] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{20}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{21}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{22}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{23}$$

we take $\epsilon = 100$ we have:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2.10^{-4}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 5.5310368633286126480363 8e + 1355 \tag{24}$$

then (11) is verified.

3.2.2 Case 2

We take $\epsilon = 0.000001 = 10^{-6}$, then:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} 2.10^{12}.(2.3.5.7.11.13.41.79.311.953)^{1.000001} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 57\,658\,458\,237\,370\,924\,700\,998\,757.17498 \tag{25}$$

We obtain that (11) is verified.

3.2.3 Case 3

We take $\epsilon = 1$, then

$$\begin{gathered} c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \\ 613\,474\,845\,886\,230\,468\,750 \overset{?}{<} 2.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow \\ 613\,474\,845\,886\,230\,468\,750 < 1\,662\,145\,872\,249\,552\,942\,316\,264\,200 \end{gathered} \tag{26}$$

Ouf!

4 Conclusion

This is an elementary proof of the *ABC* conjecture, confirmed by four numerical examples. We can announce the important theorem:

Theorem 1 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2018) Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{27}$$

where $K(\epsilon)$ is a constant depending of ϵ equal to $\dfrac{2}{\epsilon^2}$.

References

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