Accelerated frames of reference without the clock hypothesis: fundamental disagreements with Rindler

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Based on an intuitive generalization of the Lorentz Transformations to non-inertial frames, this study presents new coordinates for a hyperbolically accelerated reference frame. These coordinates are equivalent to the Rindler coordinates exclusively at small times due to the loss of the clock hypothesis. This hypothesis is considered an excellent but fundamentally incorrect approximation for longitudinal motion. The proper acceleration of a hyperbolically accelerated particle is no longer constant and its proper time progressively slows down until becoming constant at the speed of light. This is in agreement with the timeless nature of photons. An event horizon beyond which any information cannot reach the particle is still present and is identical to the Rindler horizon. More importantly, a time dependent factor appears in the metric that could profoundly change our understanding of the space-time dynamic.

Introduction

The Lorentz Transformations are the heart of Special Relativity. They describe how an event (a point in space-time) is seen from two inertial observers moving on a line with a constant velocity. When one of the observers starts to accelerate, these transformations are no longer valid and accelerated frames of reference need to be constructed. For any fixed time, it exists an inertial frame in which the accelerated observer is at rest and shares the same instantaneous velocity. Considering a succession of inertial frames along the path of the accelerated observer, the infinitesimal time dilation relationship derived from the Lorentz Transformations is supposed to be valid over the entire trajectory. This framework is the basis of the Rindler coordinates used for an observer accelerating with a constant proper acceleration (*1*).

However, the above reasoning contains a flaw. For any fixed time, the accelerated observer has also a non-zero instantaneous acceleration. The ”clock hypothesis” claims that this instantaneous acceleration has no effect on the infinitesimal time dilation. This postulate is an important component of the General Relativity.

The clock hypothesis has been verified in circular particle accelerators with accelerations of 10^{18} g (*2*), and has remained well established. However, circular and longitudinal motions have drastically different accelerations. In the first case, the acceleration is centripetal, perpendicular to a constant angular velocity, and in the second case, the acceleration is collinear to an increasing velocity. This study will only focus on longitudinal motions and will quantitatively contest the unique experimental validation of the clock hypothesis for longitudinal motions (*3*) .

Although previous studies have criticized the clock hypothesis, most were primarily based on theoretical considerations without proposed alternatives (*4, 5*). One exception is the development of an extended relativistic dynamic with the hypothesis of a maximum acceleration (*6*). The decay of unstable particles has also been studied (*7*), but the author concluded that the clock hypothesis is true by neglecting a second order correction. More recently, based on an analysis of the energy spectrum of an accelerated system, Dahia and Silva (*8*) have determined that the instantaneous acceleration has an influence on the rate of atomic clock but without a direct violation of the clock hypothesis (the small correction factor vanishes at the limit).

In contrast to the previous studies, this investigation takes a simple, original but also controversial approach. New Generalized Lorentz Transformations between an inertial observer and a non-inertial observer accelerating with a constant coordinate acceleration in one direction are intuited. These transformations are postulated to be a "natural" way of studying accelerated frames of reference. A new infinitesimal time dilation formula that includes the instantaneous acceleration is derived from this system, this contradicts the clock hypothesis for longitudinal motions. Using this new formula in conjunction with the velocity of a Rindler's particle, new coordinates are constructed that describe the same hyperbolically accelerated reference frame as the Rindler coordinates.

Generalized Lorentz Transformations

Let's consider a line on a flat space time. Two observers A and B are associated with two referentials R and R', with R always inertial in this study. Observer B is moving with a velocity v with respect to A and axes are chosen so that $v = -v'$. The notation t (resp. t') will be used for the coordinate time associated with the referential R (resp. R'), while τ (resp. τ') will be the proper time of A (resp. B). Consider that during a time $t < (c - v_0)/a$, B accelerates with a constant coordinate acceleration $a = dv/dt = d^2x_B/dt^2$, with v_0 the initial velocity of B and

c the speed of light. The equation of motion is $x_B = at^2/2 + v_0 t + x_0$. Choosing $x_0 = 0$ and knowing that $v_0 = v - at$, it comes: $x_B = vt - at^2/2$. The novel idea of this study is to replace x by $x + at^2/2$ in the Lorentz Transformations to generalize them to an ideal accelerated frame of reference with the coordinate acceleration a constant. This generalization cannot be logically demonstrated, it is postulated.

Let's consider an event M with coordinates (x, t) in R (inertial) and (x', t') in R' (non-inertial), I *suggest* that the following transformations are correct for $t < (c - v_0)/a$, with $a = dv/dt$ constant and $\gamma = 1/\sqrt{1 - v^2/c^2}$:

$$\begin{aligned} x' &= \gamma \left(x + a\frac{t^2}{2} - vt \right) \\ ct' &= \gamma \left(ct - \frac{v}{c}(x + a\frac{t^2}{2}) \right) \end{aligned} \tag{1}$$

equivalent to:

$$\begin{aligned} x + a\frac{t^2}{2} &= \gamma \left(x' + vt' \right) \\ ct &= \gamma \left(ct' + \frac{v}{c}x' \right) \end{aligned} \tag{2}$$

In the rest frame of B ($x' = 0$, $t' = \tau'$), the motion $x_B = vt - at^2/2$ is valid and $t = \gamma\tau'$, which is the same dilation formula as a constant v. However, in the rest frame of A (x=0, $t = \tau$), the motion is $x'_A/\gamma = -v\tau + a\tau^2/2$ and $t' = \gamma\tau - av\gamma\tau^2/(2c^2)$. The symmetry present in the original Lorentz Transformations between the two observers is therefore broken.

Differentiating the system (1) and (2) with $dv \neq 0$ and $d\gamma = \gamma^3 v dv/c^2$ it becomes:

$$
\begin{aligned}
&dx' + \frac{\gamma^2}{c} ct' dv = \gamma\,(dx - vdt) + \frac{\gamma}{c} actdt \\
&cdt' + \frac{\gamma^2}{c} x' dv = \gamma \left(cdt - \frac{v}{c} dx \right) - \frac{\gamma}{c} avtdt \\
&dx + atdt - \frac{\gamma^2}{c} ctdv = \gamma\,(dx' + vdt') \\
&cdt - \frac{\gamma^2}{c} \left(x + a\frac{t^2}{2} \right) dv = \gamma \left(cdt' + \frac{v}{c} dx' \right)
\end{aligned}
\tag{3}
$$

For $X = x + at^2/2$ and $T = t$, the metric is:

$$
c^2 dT^2 - (dX)^2 = \left(cdt' + \frac{\gamma^2}{c} x' dv \right)^2 - \left(dx' + \frac{\gamma^2}{c} ct' dv \right)^2 \tag{4}
$$

In the rest frame of B ($x' = 0$, $dx' = 0$, $t' = \tau'$, $v = dx/dt$) the new time-dilation formula appears that contradicts the clock hypothesis:

$$
d\tau' = \tfrac{1}{\gamma}(1 - \tfrac{\gamma^2 avt}{c^2})dt \tag{5}
$$

Eq. (5) is simply the differentiate of $\tau' = t/\gamma$ with $dv \neq 0$. The reader should also notice that $\gamma^2 avt/c^2 \ll 1$ is equivalent here to the clock hypothesis. It is an excellent approximation as high accelerations in one direction cannot be maintained for a long time.

Momentarily Comoving Non-Inertial Reference Frame

The previous section described the behavior of an "ideally" accelerated particle where a is constant for a short time. How is it possible to generalize this to any $a(t)$?

Classically, accelerated frames of reference are constructed based on the clock hypothesis and the existence at each instant of a "Momentarily Comoving Reference Frame" (MCRF) in which the accelerated particle is at rest and shares the same instantaneous velocity as its comoving frame. The time dilation formula $d\tau' = dt/\gamma$ is therefore supposed to hold for $v = v(t)$.

Based on $\alpha' = \frac{d}{dt}\frac{dx_B}{d\tau'}$ with α' the proper acceleration of B, it leads to the relationship $\alpha' = \gamma^3 a$.

Similarly, this study introduces the concept of Momentarily Comoving Non-Inertial Reference Frame (MCNIRF) in which the equations of the Generalized Lorentz Transformations are valid. At each instant, a MCNIRF exists in which the accelerated particle is at rest and shares the same instantaneous velocity and acceleration as its comoving frame. The time dilation formula $d\tau' = dt(1 - \gamma^2 avt/c^2)/\gamma$ therefore holds for $a = a(t)$. Based on $\alpha' = \frac{d}{dt}\frac{dx_B}{d\tau'}$, a different relationship appears:

$$\alpha'(t) = \frac{\left(1-\left(\frac{v}{c}\right)^4+\frac{atv^3}{c^4}\right)a+\left(\frac{v}{c}\right)^2\left(1-\left(\frac{v}{c}\right)^2\right)t\frac{da}{dt}}{\left(1-\left(\frac{v}{c}\right)^2-\frac{atv}{c^2}\right)^2\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{6}$$

The reader should notice that the relationship $\alpha' = \gamma^3 a$ is no longer valid. The two previous sections are summarized in Table 1.

Comparison with the Rindler coordinates

In Special Relativity, Rindler coordinates (*1*) are associated with an inertial observer A and a non-inertial observer B accelerating in one direction with a constant proper acceleration. The velocity of a Rindler particle is $v_{Ri}(t) = \alpha'_0 t/\sqrt{1 + (\alpha'_0 t/c)^2}$ with α'_0 constant. Based on the method using successive MCNIRFs, v_{Ri} is allowed to be plugged in Eq. (5) and Eq. (6) to construct a new proper acceleration and new coordinates similar to the Rindler coordinates (see the Appendix). Results and comparison with Rindler are summarized in Table 2 and Fig. 1.

Current theory	This study
Lorentz Transformations Event with coordinates (x,t) in R (inertial) and (x',t') in R' (inertial) For $v=\frac{dx_B}{dt}$ constant: $x'=\gamma\left(x-vt\right)$ $ct'=\gamma\left(ct-\frac{v}{c}x\right)$ **In the rest frame of B** $d\tau'=\frac{dt}{\gamma}$	**Generalized Lorentz Transformations** Event with coordinates (x,t) in R (inertial) and (x',t') in R' (non inertial) For $a=\frac{d^2x_B}{dt^2}$ constant and $t<\frac{c-v_0}{a}$: $x'=\gamma\left(x+a\frac{t^2}{2}-vt\right)$ $ct'=\gamma\left(ct-\frac{v}{c}(x+a\frac{t^2}{2})\right)$ **In the rest frame of B** $d\tau'=\frac{1}{\gamma}\left(1-\frac{\gamma^2 avt}{c^2}\right)dt$
B is now accelerating randomly Clock hypothesis + Existence of a succession of Momentarily Comoving Inertial Reference Frames with the same instantaneous velocity than B $\rightarrow d\tau'=\frac{dt}{\gamma}$ holds for $v=v(t)$	**B is now accelerating randomly** Existence of a succession of Momentarily Comoving Non-Inertial Reference Frames with the same instantaneous velocity and acceleration than B $\rightarrow d\tau'=\frac{1}{\gamma}\left(1-\frac{\gamma^2 avt}{c^2}\right)dt$ holds for $a=a(t)$
Proper acceleration $\alpha'(t)=\frac{d}{dt}\frac{dx_B}{d\tau'}$ $\alpha'(t)=a\gamma^3$	**Proper acceleration** $\alpha'(t)=\frac{d}{dt}\frac{dx_B}{d\tau'}$ $\alpha'(t)=\frac{\left(1-\left(\frac{v}{c}\right)^4+\frac{atv^3}{c^4}\right)a+\left(\frac{v}{c}\right)^2\left(1-\left(\frac{v}{c}\right)^2\right)t\frac{da}{dt}}{\left(1-\left(\frac{v}{c}\right)^2-\frac{atv}{c^2}\right)^2\sqrt{1-\left(\frac{v}{c}\right)^2}}$

Table 1: Comparison of the method between the current theory and this study to construct an accelerated frame of reference. In both cases, $\gamma=1/\sqrt{1-(v/c)^2}$.

Rindler	This study
$v_{Ri}(t) = \frac{\alpha'_0 t}{\sqrt{1+\left(\frac{\alpha'_0 t}{c}\right)^2}}$ $\alpha'(t) = \alpha'_0$ $x = \left(\frac{c^2}{\alpha'_0} + x'\right)\cosh\left(\frac{\alpha'_0 t'}{c}\right) - \frac{c^2}{\alpha'_0}$ $ct = \left(\frac{c^2}{\alpha'_0} + x'\right)\sinh\left(\frac{\alpha'_0 t'}{c}\right)$ $(ds)^2 = \left(1 + \frac{\alpha'_0 x'}{c^2}\right)^2 (cdt')^2 - (dx')^2$ **In the rest frame of B** $\left(1 + \frac{\alpha'_0 x}{c^2}\right)^2 - \left(\frac{\alpha'_0 t}{c}\right)^2 = 1$ $\tau' = \frac{c}{\alpha'_0}\ln\left(\frac{\alpha'_0 t}{c} + \sqrt{1 + \left(\frac{\alpha'_0 t}{c}\right)^2}\right)$	$v_{Ri}(t) = \frac{\alpha'_0 t}{\sqrt{1+\left(\frac{\alpha'_0 t}{c}\right)^2}}$ $\alpha'(t) = \alpha'_0\left(1 + 3\left(\frac{\alpha'_0 t}{c}\right)^2\right)$ $x = \left(\frac{c^2}{\alpha'_0} + x'\right)\tilde{\gamma} - \frac{c^2}{\alpha'_0}$ $ct = \left(\frac{c^2}{\alpha'_0} + x'\right)\tilde{\gamma}\frac{\alpha'_0}{c}t'$ with $\tilde{\gamma} = \frac{1}{\sqrt{1-\left(\frac{\alpha'_0 t'}{c}\right)^2}}$ $(ds)^2 = \left(\frac{1+\frac{\alpha'_0 x'}{c^2}}{1-\left(\frac{\alpha'_0 t'}{c}\right)^2}\right)^2 (cdt')^2 - (dx')^2$ **In the rest frame of B** $\left(1 + \frac{\alpha'_0 x}{c^2}\right)^2 - \left(\frac{\alpha'_0 t}{c}\right)^2 = 1$ $\tau' = \frac{t}{\sqrt{1+\left(\frac{\alpha'_0 t}{c}\right)^2}}$
Behavior for $\frac{\alpha'_0 t'}{c} << 1$ **or** $\frac{\alpha'_0 t}{c} << 1$ **:** identical	
Behavior for $t = +\infty$ $v = c$ $a = 0$ $\alpha' = \alpha'_0$ $\tau' = +\infty$	**Behavior for** $t = +\infty$ $v = c$ $a = 0$ $\alpha' = +\infty$ $\tau' = \frac{c}{\alpha'_0}$

Table 2: Comparison between the Rindler coordinates and this study for the same velocity $v_{Ri}(t)$. $\alpha'(t)$ is the proper acceleration of B, α'_0 a constant and $ds^2 = (cdt)^2 - dx^2$.

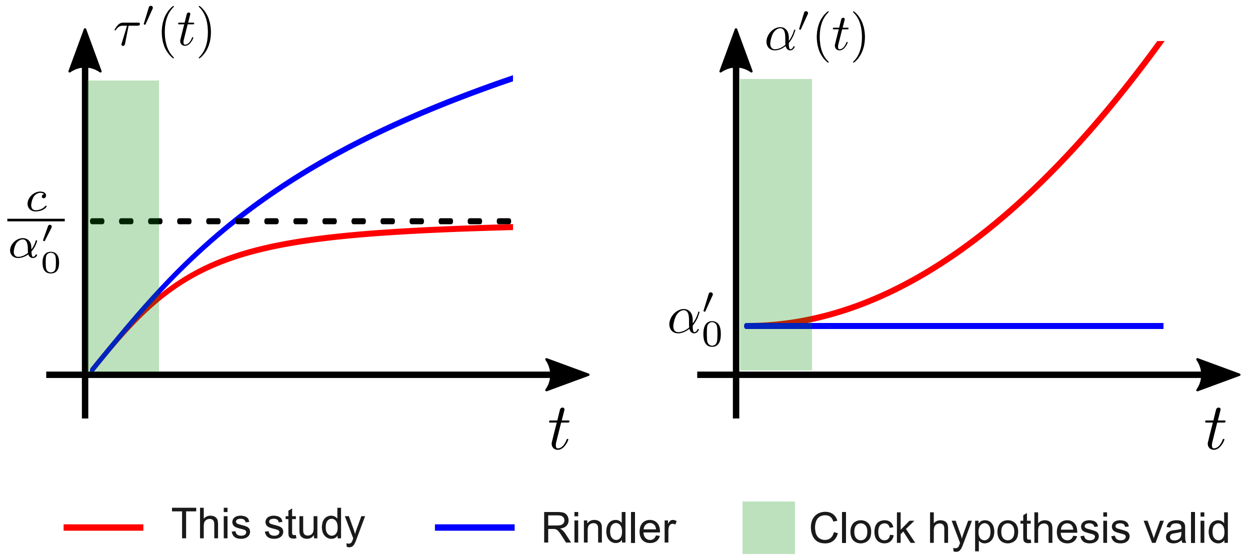

Figure 1: Proper time (left panel) and proper acceleration (right panel) versus coordinate time of a particle associated with a velocity $v_{Ri}(t) = \alpha'_0 t/\sqrt{1+(\alpha'_0 t/c)^2}$.

The hyperbolic motion is only derived from the velocity v_{Ri}, there is therefore a strict equivalence between Rindler and this study for the motion, velocity and coordinate acceleration of an accelerated particle. However, the coordinates, proper acceleration, proper time and metric have changed. The terms $\cosh$ and $\sinh$ in the expression of the Rindler coordinates have been replaced by a factor that is more consistent with the rest of the mathematical framework: $\tilde{\gamma} = 1/\sqrt{1-(\alpha'_0 t'/c)^2}$, analogous to γ. The proper acceleration α' is now increasing with time, and its quadratic expression is surprisingly simple knowing the convoluted differential equation (Eq. (6)) it comes from. The proper time progressively freezes until becoming constant (c/α'_0) when the particle reaches the speed of light (Fig. 1). This is in agreement with the idea that photons do not experience time. A temporal factor $1/(1-(\alpha'_0 t'/c)^2)^2$ has appeared in the metric of the system and originates from the loss of the clock hypothesis. Additionally, an event horizon x'_h can be calculated in the proper frame of the accelerated particle (no information beyond this limit can reach the particle), with $x'_h = -c^2/\alpha'_0$, identical to the Rindler horizon (see the

Appendix).

Rindler and this study behave identically for $\alpha'_0 t/c \ll 1$ using a Taylor expansion. The clock hypothesis is restored in this regime: $d\tau' = dt(1 - \gamma^2 avt/c^2)/\gamma$ becomes equivalent to $d\tau' = dt/\gamma$. The metric of Rindler is known to be locally equivalent to the metric of Schwarzschild (*9*). This result can be easily explained by the fact that the Equivalence Principle requires the acceleration to be constant, and here $\alpha'(t) = \alpha'_0$ only for $\alpha'_0 t/c \ll 1$.

To the best of my knowledge, only one study has experimentally verified the clock hypothesis for longitudinal motion (*3*), and the average acceleration goes up to 10^{16} g. Roos et al. did not explicitly specify the time during which this acceleration is maintained, but they gave maximum values of $c\tau' \sim 0.05$ m. Therefore $\tau' \sim 10^{-10}$ s, $v \sim a\tau' \sim 10^7$ m.s^{-1}. For these order of magnitudes, $\gamma^2 avt/c^2 \sim 10^{-3}$ and the relative difference between $d\tau' = dt/\gamma$ and $d\tau' = dt(1 - \gamma^2 avt/c^2)/\gamma$ is only 0.1 %: any discrepancies are impossible to detect.

The easiest way to experimentally validate or invalidate this study would be to reproduce the measurements at two different heights of the Muons (*10*) or Mesons (*11*) traversing the atmosphere. By estimating the high deceleration experienced on their trajectory, we can predict the impact of the time dilation on their half life and see which prediction is the closest to the observation: the one with the clock hypothesis, or the one without?

In this study, particles with a velocity v_{Ri} have no constant proper acceleration. Apparently, no analytical (or even numerical) expression of $v(t)$ from the Equation (6) exists considering $\alpha'(t)$ constant. What happens to the Equivalence Principle if trajectories with constant proper acceleration do not exist anymore? A possible idea is to consider a constant coordinate acceleration instead, knowing that $t = \gamma\tau'$ is still valid when $a = dv/dt = d^2x_B/dt^2$ constant. By choosing an initial velocity of zero, the particle motion is therefore $x = a\frac{t^2}{2}$, leading to $v^2 = 2ax$. The time dilation formula becomes:

$$\tau' = t\sqrt{1 - \frac{2ax}{c^2}} \tag{7}$$

Now consider a Schwarzschild metric, with t the coordinate time of an observer A far away from a massive object (mass M) and τ' the proper time of an observer B within the gravitational field. Let's define $g = GM/r^2$ with r as the radial coordinate of the observer B and G the Gravitational constant. If both A and B are at rest with the massive body, it becomes:

$$\tau' = t\sqrt{1 - \frac{2gr}{c^2}} \tag{8}$$

The problem now is that Eq (8) holds for an arbitrarily long time, while (7) is evolving due to an increasing $x(t)$. This can be solved by an interesting idea: g can stay constant with different M and r. By following the particle at three successive instants t_1, t_2 and t_3 so that $\tau_i' = t_i\sqrt{1 - \frac{2ax_i}{c^2}}$ with $i = 1, 2$ or 3, we have $x_1 < x_2 < x_3$. Similarly, three sets of (M_i, r_i) are chosen so that g stays constant: $g = \frac{GM_1}{r_1^2} = \frac{GM_2}{r_2^2} = \frac{GM_3}{r_3^2}$ with $r_1 < r_2 < r_3$ and $M_1 < M_2 < M_3$. For each t_i, with $x_i = r_i$, a perfect analogy exists between an accelerated frame with a constant and a gravitational field with g constant, at least in terms of time dilation. When the velocity of the particle reaches the speed of light, a black hole forms in the comparison ($2gr/c^2 = 1$ because r_i was increasing). There is therefore an analogy between the existence of black holes and the existence of superluminal particles, for which the proper time is an imaginary time.

Conclusion

This study claims that the clock hypothesis is incorrect for longitudinal motions and the results presented here have no consequences for circular motions with a constant angular velocity. The starting assumption is to consider as ”unnatural” the way accelerated frames of reference are constructed in Special Relativity. The Lorentz transformations are only valid for constant velocities between two inertial observers, and they have been forced with the clock hypothesis to construct the Rindler coordinates.

The Lorentz Transformations first need to be generalized to non-inertial frames to construct accelerated frames of reference. The idea of replacing x by $x + at^2/2$ in the Lorentz Transformations to create a ”natural” framework in which accelerated referentials can be studied for $t < (c - v_0)/a$ is intuitive and cannot be logically demonstrated. The relationship $\tau' = t/\gamma$ stays valid for a constant, and the differentiation of this formula leads to $d\tau' = dt\,(1 - \gamma^2 avt/c^2)\,/\gamma$. The clock hypothesis is now equivalent to $\gamma^2 avt/c^2 \ll 1$. The use of successive MCNIRFs permits to construct new coordinates for a particle with a velocity $v_{Ri}(t) = \alpha_0' t/\sqrt{1 + (\alpha_0' t/c)^2}$, which is the velocity of a Rindler particle. Comparison with the Rindler coordinates is fruitful: the mathematical framework of the new coordinates is remarkable as $\cosh$ and $\sinh$ have disappeared without the loss of the hyberbolic motion or the horizon event. The proper time becomes constant for light speed particles, and this new system is equivalent to Rindler for small times and accelerations (Table 2, Fig. 1). Additionally, this study explains why the Rindler metric and the Schwarzschild metric are equivalent only locally: the proper acceleration of a hyperbolically accelerated particle is actually not constant.

The most important change is in the metric of the system. The term $1/(1 - (\alpha_0' t'/c)^2)^2$

cannot be found in the General Relativity as Einstein built his theory with the clock hypothesis. If this study is correct, a similar temporal term must exist in General Relativity and therefore the Einstein field equations must be modified to accept the loss of the clock hypothesis when the magnitude of the velocity changes.

It has been quantitatively proved that the only experiment testing for the clock hypothesis for longitudinal motions (*3*) could not detect the factor $\gamma^2 avt/c^2$ present in the new time dilation formula. Modern and accurate measurements of the flux of atmospheric Muons or Mesons performed at two different heights will validate or invalidate this study. Finally, this work is missing many updates compared to the current theory: the relativistic Doppler effect for accelerated bodies, the Unruh effect, the development of a new relativistic dynamic and a Group theory study on the asymmetrical Generalized Lorentz Transformations. If these new transformations are correct, fundamental changes are expected to occur in our understanding of acceleration and space-time dynamic.

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Appendix

Proof of Eq. 3

Let's calculate the differentiate of Eq. 1 and Eq. 2 with a constant. It comes:

$$
\begin{aligned}
dx' &= \gamma\left(dx - vdt\right) + \gamma atdt - \gamma tdv + \frac{\gamma^3 vdv}{c^2}\left(x + a\frac{t^2}{2} - vt\right)\\
dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) - \gamma\frac{v}{c^2}atdt - \frac{\gamma}{c^2}\left(x + a\frac{t^2}{2}\right)dv + \frac{\gamma^3 vdv}{c^2}\left(t - \frac{v}{c^2}\left(x + a\frac{t^2}{2}\right)\right)\\
dx + atdt &= \gamma\left(dx' + vdt'\right) + \gamma t'dv + \frac{\gamma^3 vdv}{c^2}\left(x' + vt'\right)\\
dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) + \gamma x'\frac{dv}{c^2} + \frac{\gamma^3 vdv}{c^2}\left(t' + \frac{v}{c^2}x'\right)
\end{aligned}
$$

Knowing that $\gamma^2 = 1 + (\gamma v/c)^2$, the first line needs to be factorized by $(-\gamma tdv)$, the second line by $(-\gamma xdv/c^2)$, the third line by $(\gamma t'dv)$ and the fourth by $(\gamma x'dv/c^2)$. It comes:

$$
\begin{aligned}
dx' &= \gamma\left(dx - vdt\right) + \gamma atdt - \gamma^3 tdv + \frac{\gamma^3 vdv}{c^2}\left(x + a\frac{t^2}{2}\right)\\
dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) - \gamma\frac{v}{c^2}atdt - \frac{\gamma^3 xdv}{c^2} - \frac{\gamma}{c^2}a\frac{t^2}{2}dv + \frac{\gamma^3 vdv}{c^2}\left(t - \frac{v}{c^2}a\frac{t^2}{2}\right)\\
dx + atdt &= \gamma\left(dx' + vdt'\right) + \gamma^3 t'dv + \frac{\gamma^3 vdv}{c^2}x'\\
dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) + \frac{\gamma^3 x'dv}{c^2} + \frac{\gamma^3 vdv}{c^2}t'
\end{aligned}
$$

Now, the first line needs to be factorized by $(-\gamma^2 dv/c)$, the third line by $(\gamma^3 dv)$ and the fourth line by $(\gamma^3 dv/c^2)$. The second line needs first to be factorized by $(-at^2\gamma^3 dv/(2c^2))$ and simplified with $\gamma^{-2} + (v/c)^2 = 1$. Then, it needs to be factorized by $(-\gamma^2 dv/c^2)$. Using Eq. 1 and 2, it comes:

$$
\begin{aligned}
dx' &= \gamma\left(dx - vdt\right) + \gamma atdt - \frac{\gamma^2}{c}ct'dv\\
dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) - \frac{\gamma}{c^2}avtdt - \frac{\gamma^2}{c^2}x'dv\\
dx + atdt &= \gamma\left(dx' + vdt'\right) + \gamma^2 tdv\\
dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) + \frac{\gamma^2}{c^2}\left(x + a\frac{t^2}{2}\right)dv
\end{aligned}
$$

Which leads to Eq. 3.

Proof of Eq. 4

Eq. 4 comes the difference of the square of the second and first line of Eq. 3. So let's calculate:

$$W = \left(\gamma\left(cdt - \frac{v}{c}dx\right) - \frac{\gamma}{c}avtdt\right)^2 - \left(\gamma\left(dx - vdt\right) + \frac{\gamma}{c}actdt\right)^2$$

It is trivial that $(\gamma(cdt - vdx/c))^2 - (\gamma(dx - vdt))^2 = c^2dt^2 - dx^2$. W becomes:

$$W = c^2dt^2 - dx^2 + \frac{\gamma^2}{c^2}a^2v^2t^2dt^2 - 2\frac{\gamma^2}{c}\left(cdt - \frac{v}{c}dx\right)avtdt - \gamma^2a^2t^2dt^2 - 2\gamma^2(dx - vdt)atdt$$

Knowing that $\gamma^2 - (\gamma v/c)^2 = 1$ and factorizing by $(-a^2t^2dt^2)$, it comes:

$$W = c^2dt^2 - dx^2 - 2\frac{\gamma^2}{c}\left(cdt - \frac{v}{c}dx\right)avtdt - a^2t^2dt^2 - 2\gamma^2(dx - vdt)atdt$$

Then it needs to be factorized by $(-2\gamma^2atdt/c)$.

$$\begin{aligned}
W &= c^2dt^2 - dx^2 - 2\frac{\gamma^2}{c}atdt\left(cdx - \frac{v^2}{c}dx\right) - a^2t^2dt^2 \\
&= c^2dt^2 - dx^2 - 2atdtdx - a^2t^2dt^2 \\
&= c^2dt^2 - (dx + atdt)^2 \\
&= c^2dt^2 - \left(d\left(x + a\frac{t^2}{2}\right)\right)^2 \\
&= c^2dT^2 - dX^2
\end{aligned}$$

with $T = t$ and $X = x + at^2/2$.

Proof of Eq. 6

Eq. 6 comes from $\alpha' = \frac{d}{dt}\frac{dx_B}{d\tau'}$ and $d\tau' = dt(1-\gamma^2 avt/c^2)/\gamma$ with $a = a(t)$ and $v = dx_B/dt$. It comes:

$$\begin{aligned}
\alpha' &= \frac{d}{dt}\left(\gamma v \frac{1}{1-\frac{\gamma^2 avt}{c^2}}\right) \\
&= \frac{d}{dt}\left(\frac{\gamma v}{U}\right)
\end{aligned}$$

Let's calculate first dU/dt:

$$\begin{aligned}
\frac{dU}{dt} &= -\frac{\gamma^2}{c^2}\left(\frac{2\gamma^2 a^2 v^2 t}{c^2} + vt\frac{da}{dt} + a^2 t + av\right) \\
&= -\frac{\gamma^2}{c^2} V
\end{aligned}$$

It comes:

$$\begin{aligned}
\alpha' &= \gamma U^{-2}\left(\gamma^2 \left(\frac{v}{c}\right)^2 Ua + Ua + v\frac{\gamma^2}{c^2}V\right) \\
&= \gamma U^{-2}\left(\gamma^2 Ua + v\frac{\gamma^2}{c^2}V\right) \\
&= \gamma^5 U^{-2}\left(\frac{Ua}{\gamma^2} + \frac{v}{c^2\gamma^2}V\right)
\end{aligned}$$

Then Ua/γ^2 and $vV/(c^2\gamma^2)$ need to be calculated:

$$\begin{aligned}
\frac{Ua}{\gamma^2} &= a\left(\frac{1}{\gamma^2} - \frac{avt}{c^2}\right) \\
\frac{vV}{c^2\gamma^2} &= a\left(\frac{2av^3 t}{c^4} + \frac{avt}{\gamma^2 c^2} + \frac{v^2}{\gamma^2 c^2}\right) + \left(\frac{v}{c}\right)^2 \frac{t}{\gamma^2}\frac{da}{dt}
\end{aligned}$$

Finally:

$$\begin{aligned}
\alpha' &= \gamma^5 U^{-2}\left(a\left(\frac{1}{\gamma^2}-\frac{avt}{c^2}+\frac{2av^3t}{c^4}+\frac{avt}{\gamma^2c^2}+\frac{v^2}{\gamma^2c^2}\right)+\left(\frac{v}{c}\right)^2\frac{t}{\gamma^2}\frac{da}{dt}\right)\\
&= \gamma^5 U^{-2}\left(a\left(1-\left(\frac{v}{c}\right)^4-\frac{avt}{c^2}+\frac{2av^3t}{c^4}+\frac{avt}{\gamma^2c^2}\right)+\left(\frac{v}{c}\right)^2\frac{t}{\gamma^2}\frac{da}{dt}\right)\\
&= \gamma^5 U^{-2}\left(a\left(1-\left(\frac{v}{c}\right)^4+\frac{atv^3}{c^4}\left(\frac{c^2}{\gamma^2v^2}-\frac{c^2}{v^2}+2\right)\right)+\left(\frac{v}{c}\right)^2\frac{t}{\gamma^2}\frac{da}{dt}\right)\\
&= \gamma^5 U^{-2}\left(a\left(1-\left(\frac{v}{c}\right)^4+\frac{atv^3}{c^4}\right)+\left(\frac{v}{c}\right)^2\frac{t}{\gamma^2}\frac{da}{dt}\right)
\end{aligned}$$

The final expression of Eq. 6 is found with:

$$\gamma^5 U^{-2} = \gamma\left(1-\left(\frac{v}{c}\right)^2-\frac{avt}{c^2}\right)^{-2}$$

Proof of $\alpha'(t) = \alpha'_0(1+3(\alpha'_0 t/c)^2)$ for $v = v_{Ri}(t)$

For $v_{Ri}(t) = \alpha'_0 t/\sqrt{1+(\alpha'_0 t/c)^2}$, the first and second derivatives are:

$$\begin{aligned}
a &= \frac{\alpha'_0}{\left(1+\left(\frac{\alpha'_0 t}{c}\right)^2\right)^{3/2}}\\
\frac{da}{dt} &= \frac{-3\frac{(\alpha'_0)^3 t}{c^2}}{\left(1+\left(\frac{\alpha'_0 t}{c}\right)^2\right)^{5/2}}
\end{aligned}$$

Now I define $\epsilon = \alpha'_0 t/c$, and all the following relationships are correct:

$$\begin{aligned}
\left(\frac{v}{c}\right)^2 &= \frac{\epsilon^2}{1+\epsilon^2}\\
1-\left(\frac{v}{c}\right)^2 &= \frac{1}{1+\epsilon^2}\\
a &= \frac{\alpha'_0}{(1+\epsilon^2)^{3/2}}\\
\frac{atv}{c^2} &= \frac{\epsilon^2}{(1+\epsilon^2)^2}\\
t\frac{da}{dt} &= -3\alpha'_0\frac{\epsilon^2}{(1+\epsilon^2)^{5/2}}
\end{aligned}$$

The term in $(1+\epsilon^2)^{9/2}$ cancels itself in the denominator and numerator of Eq. 6 and the simplification is trivial at the end.

Construction of the coordinates

The velocity $v_{Ri}(t) = \alpha'_0 t/\sqrt{1+(\alpha'_0 t/c)^2}$ needs first to be plugged into Eq. 5:

$$d\tau' = \frac{1}{\left(1+\left(\frac{\alpha'_0 t}{c}\right)^2\right)^{3/2}} dt$$

Integration gives:

$$\tau'(t) = \frac{t}{\sqrt{1+(\frac{\alpha'_0 t}{c})^2}}$$

Now, let's reverse the equation:

$$t(\tau') = \frac{\tau'}{\sqrt{1-(\frac{\alpha'_0 \tau'}{c})^2}}$$

$x(t)$ is given by integrating $v_{Ri}(t) = dx/dt$, and choosing $x(0) = 0$:

$$x(t) = \frac{c^2}{\alpha'_0}\sqrt{1+(\frac{\alpha'_0 t}{c})^2} - \frac{c^2}{\alpha'_0}$$

Replacing with the expression of $t(\tau')$, it comes:

$$x = \frac{c^2}{\alpha'_0}\frac{1}{\sqrt{1-(\frac{\alpha'_0 \tau'}{c})^2}} - \frac{c^2}{\alpha'_0}$$
$$ct = \frac{c\tau'}{\sqrt{1-(\frac{\alpha'_0 \tau'}{c})^2}}$$

And these two equations verify the same hyperbolic relationship than Rindler in the proper frame of B:

$$\left(1+\frac{\alpha'_0 x}{c^2}\right)^2 - \left(\frac{\alpha'_0 t}{c}\right)^2 = 1$$

The last step is to leave the rest frame of B to make the x' appear (τ' becomes t'). A simple analogy with the Rindler coordinates gives the final expression:

$$x = \left(\frac{c^2}{\alpha'_0} + x'\right)\tilde{\gamma} - \frac{c^2}{\alpha'_0}$$
$$ct = \left(\frac{c^2}{\alpha'_0} + x'\right)\tilde{\gamma}\frac{\alpha'_0}{c}t'$$

with $\tilde{\gamma} = 1/\sqrt{1 - (\alpha'_0 t'/c)^2}$.

Metric of the new coordinates

Knowing that $d\tilde{\gamma} = \alpha'_0 t' \tilde{\gamma}^3 dt'/c^2$ and $(\alpha'_0 t' \tilde{\gamma}/c)^2 + 1 = \tilde{\gamma}^2$, the differentiation of the coordinates x and ct gives:

$$dx = \tilde{\gamma}\left(dx' + \left(1 + \frac{\alpha'_0 x'}{c^2}\right)\tilde{\gamma}^2 \frac{\alpha'_0 t'}{c} cdt'\right)$$
$$cdt = \tilde{\gamma}\left(\frac{\alpha'_0 t'}{c} dx' + \left(1 + \frac{\alpha'_0 x'}{c^2}\right)\tilde{\gamma}^2 cdt'\right)$$

The difference of the squares gives:

$$(cdt)^2 - dx^2 = -(dx')^2 + (cdt')^2 \tilde{\gamma}^4 \left(1 + \frac{\alpha'_0 x'}{c^2}\right)^2$$

which leads to the final expression.

Event horizon

Exactly like for the Rindler coordinates, in the proper frame of B ($x' = 0$, $\tau' = t'$) is found the following hyperbolic relationship:

$$\left(1 + \frac{\alpha'_0 x}{c^2}\right)^2 - \left(\frac{\alpha'_0 t}{c}\right)^2 = 1$$

For $x > 0$ and $t > 0$, this hyperbolic motion is delimited by the asymptote:

$$ct = x + \frac{c^2}{\alpha'_0}$$

Any information sent from an event located above this asymptote will never reach the accelerated particle. Where is this horizon located in the proper frame of the particle ? Coming back to the new coordinates, with $\tilde{\gamma} = 1/\sqrt{1-(\alpha'_0 t'/c)^2}$:

$$\begin{array}{l} x = \left(\frac{c^2}{\alpha'_0} + x'\right)\tilde{\gamma} - \frac{c^2}{\alpha'_0} \\ ct = \left(\frac{c^2}{\alpha'_0} + x'\right)\tilde{\gamma}\frac{\alpha'_0 t'}{c} \end{array}$$

The equation of the asymptote needs to be replaced in the first line. It gives:

$$\begin{array}{l} ct = \left(\frac{c^2}{\alpha'_0} + x'_h\right)\tilde{\gamma} \\ ct = \left(\frac{c^2}{\alpha'_0} + x'_h\right)\tilde{\gamma}\frac{\alpha'_0 t'}{c} \end{array}$$

Let's calculate the square of the first line and the second line. The difference gives:

$$0 = \left(\frac{c^2}{\alpha'_0} + x'_h\right)^2$$

And the event horizon is seen at a constant distance $x'_h = -c^2/\alpha'_0$ by the accelerated particle, which is an identical result compared to the Rindler coordinates.