Geometries of O

Hannes Hutzelmeyer

Summary

Geometries of O adhere to Ockham's principle of simplest possible ontology: the only individuals are **points**, there are no straight lines, circles, angles etc. , just as it was was laid down by **Tarski** in the 1920s, when he put forward a set of axioms that only contain two relations, quaternary congruence and ternary betweenness. However, relations are not as intuitive as functions when constructions are concerned. Therefore the **planar geometries** of O contain **only functions** and no relations to start with. Essentially three quaternary functions occur: **appension** for line-joining of two pairs of points, **linisection** representing intersection of straight lines and **circulation** corresponding to intersection of circles. Functions are strictly defined by **composition** of given ones only.

Both, Euclid and Lobachevsky planar geometries are developed using a precise notation for **object-language** and **metalanguage**, that allows for a very broad area of mathematical systems up to theory of types. Some astonishing results are obtained, among them:

(A) Based on a special triangle construction Euclid planar geometry can start with a **less powerful ontological** basis than Lobachevsky geometry.

(B) Usual Lobachevsky planar geometry is **not complete**, there are **nonstandard** planar Lobachevsky geometries. One needs a further axiom, the 'smallest' system is produced by the **proto-octomidial-axiom**.

(C) Real numbers can be abandoned in connection with planar geometry. A very promising **conjecture** is put forward stating that the Euclidean Klein-model of Lobachevsky planar geometry does not contain all points of the constructive Euclidean unit-circle.

Contact: Hutzelmeyer@pai.de
http://pai.de

Content

page

1. Introduction

1.1 Ontological bases of planar geometry

Geometry is not as finished as one usually assumes, it pays to have a new look at an old theory.

Since about 2400 years planar[1)] geometry of Euclid's *Elements (ref. 1)* is the archetype of an axiomatic system. Although it enjoyed some face-lifting since the early 19th century most treatises including Hilbert's *Grundlagen der Geometrie (ref. 3)* essentially talk about the same entities as Euclid: points, lines, circles etc. . More than 11 entities are used explicitly or implicitly in planar geometry:

4 standard geometry sorts: points, infinite straight lines, rays and circles
2 more finite sorts: straight line segments (different from a pair of points), circle segments
3 additional sorts: angles, natural numbers (for Archimedes), real numbers (for measuring)
2 figures: triangles, quadrangles (plus polygons) .

Things get even more complicated when direction comes into play. Archimedes axiom is introduced along with natural numbers, trigonometric real functions appear by magic, the problem of transcendent numbers is widely ignored. This is true for more recent publications as well *(ref. 4 , 5 , 7 , 8 , 12)* , sometime they are a little bit critical about definitions and postulates, but they do not away with the whole zoo of entities. The weirdest of all of these are angles, not even Hilbert does them justice as equivalence classes of triangles with respect to one corner. Pasch is the first to eliminate lines, criticizing their infinity, claiming that segments are enough.

Only in the 1920s the Polish logician Tarski *(ref. 9)* did the decisive step to reduce the entities that have to be put into the axioms (shortly listed in section 1.4) to start with: all you need is **points**. And only two relations, quaternary **congruity**[2)] and ternary **betweenity**[2)] are included in his **ontological basis,** as one calls the basic ingredients of a mathematical theory. From a philosophical point of view this is very satisfying and one has a name for that principle: **Ockham' razor**. In the end period of scholastics of the 15th century Ockham formulated that principle - of course in Latin: "*entia non sunt multiplicanda sine necessitat*e" (a plurality of entities must not be posited without necessity). It can be taken as the begin of nominalism (*Bravo, Ockham 3 points !*) .

Geometries that adhere to Ockham's principle can be called by the catchy name: **geometries of O** .

Tarski's planar geometry certainly is a geometry of O. Are there other planar geometries of O ? Contrary to the general opinion: it seems that planar geometry is not at its end, there are new insights. Tarski has used two **basic relations** . Can one use **basic functions** instead of them as well? Functions have the advantage to be closer to our intuitive notion that geometry is about constructing figures. Tarski treats **R-geometry**, based on relations. In this publication **F-geometries** are put forward that are based on functions. The constructive approach somewhat tries to put into precise language what the Germans mean by untranslatable 'Anschaulichkeit' .

R-geometry short for Ockham-relation-geometry.
F-geometry short for Ockham-function-geometry.

Now one has to specify the essential ingredients of planar F-geometry, i.e. its functions. It uses essentially three quaternary functions (as opposed to R-geometry with its two relations): **appension**[3)] for line-joining of two pairs of points, **linisection** representing intersection of straight lines and **circulation** corresponding to intersection of circles

[1)] Planar is not the same as two-dimensional geometry; two-dimensional **elliptic** or S-geometry is not planar.

[2)] Names of relations are chosen to end on 'ity' ;'congruity' instead of 'congruence', 'betweenity' instead of 'betweenness'

[3)] Names of functions are chosen to end on 'ion'

From these basic-functions all other functions and all relations of F-geometry are obtained. They are strictly defined by **composition** of given ones only. So-called **definition by cases** is not necessary except for special treatment of **one** single point of a function, to which end a purely logically defined quaternary function **decision** is included too. Composition of functions means successive geometric constructions. Constructions like the ones in Perron *(ref. 7)* lend themselves easily into a translation of the function-language of F-geometry.

In the following three chapters so-called absolute (or neutral) geometry, Euclidean geometry and non-Euclidean (Lobachevskyan) geometry will be developed, i.e. N-geometry, E-geometry and L-geometry with corresponding **calcules.** 'Calcule' is the name of a mathematical system with the precise language-metalanguage method **Funcish-Mencish** , as described in sections 1.3 and 1.4 . 'Calcule' is an expression coined by the author in order to avoid confusion. The word 'calculus' is conventionally used for real number mathematics and various logical systems. As a German translation 'Kalkul' is proposed for 'calcule' versus conventional 'Kalkül' that usually corresponds to 'calculus'. Calcules are given names using some convention that relates to the Greek **sort** names of a calcule, e.g. calcule pi with sort π .

In mathematics a **measure** assigns a real number to a subset of elements. A very special measure appears in geometry where one assigns a number to a pair of points, this is done by a metric function (also called distance-function) . Such a metric function has certain properties, a calcule with a metric functions is called metric. As there is a much more general measure theory in mathematics it will rather be talked about **metering** as far as geometry is concerned.

In the following there will appear **no sets** or subsets. Metering appears only for distances between two points, angles at a point with respect to two other points and for areas of triangles given by three points. For clearity these three kinds of measuring will be called **metering**. Metric functions for distance-metering map two points to a number, angle-metering and triangle-area-metering functions map three points to a number. These numbers belong to a meter calcule which is **not** necessarily the calcule of **real** numbers. A meter calcule contains a binary meter function for combinations of entities, e.g. the metric calcule of planar Cartesian geometry has 'addition' as its meter combination function.

An **abstract** calcule is based on axioms as opposed to a **concrete** calcule whose foundation can be put into practice by a machine. The abstract calcules of planar geometry of O will be given by a top-down-axiom systematic. Euclid E-geometry and Lobachevsky L-geometry will also be provided with a metalingual ***Axiom*** mater (usually called 'scheme') that allows for denumerably many ***Axiom*** strings.

1.2 Object-language and metalanguage

One has to specify the logical language that is needed for F-geometries. In order to describe an **object-language** one also needs a **metalanguage**. According to the author's principle metalanguage has to be absolutely precise as well, normal English will not do. There are at least three levels of language as introduced in reference *(10)* where a short sketch of the method is given in chapters 2. and 3. :

English	supralanguage	natural
Mencish	metalanguage	formalized precise
Funcish	object-language	formalized precise.

The essential parts of a language are its sentences. A sentence is a **string** of **characters** of a given **alphabet** that fulfills certain rules. This means that metalanguage talks about the strings of the object-language. The essential parts of the metalanguage are the metasentences (that are strings of characters as well). In supralanguage one talks about the metasentences, just as metalanguage talks about object-language. Here it is not discussed in general what an object-language talks **about**.

On first sight Funcish and Mencish look familiar to what one knows from so-called **predicate-logic** (for geometry f**irst-order logic FOL** is sufficient). However, they are especially adapted to a degree of precision so that they can be used universally for all kind of mathematics. And they lend themselves immediately to a treatment by computers, as they have perfect syntax and semantics. It is not the place to go into details. Both Funcish and Mencish have essentially the same syntax. Mencish, however, has strictly first-order logic, although it can talk about higher-order logic. Notice that Funcish has a context-independent notation, which implies that one can determine the **category** of every object uniquely from its syntax.

planar geometry of O		*sponsor*	*calcule*	**sort**	*comment*	*abstract semiconcrete* [1]	*complete*
R-geometry base relations		Tarski and more	pitau	$\pi\tau$	*Euclidean*	a	Y
F-geometry base functions	N-geometry	Saccheri	pi	π	*circulition*	a	N
	E-geometry	Euclid	piepsilon	$\pi\varepsilon$	*isoscition*	a	Y
		Wantzel	piomega	$\pi\omega$	*with bi-Robinson arithmetic*	a	Y
		Descartes	PIdelta	$\Pi\delta$	*full plane, straight lines*	s	Y
	L-geometry	the author	pichi	$\pi\chi$	*isoscition*	a	?
		Bolyai	pibeta	$\pi\beta$	*circulition, no proto-length*	a	N
		Lobachevsky	pilambda	$\pi\lambda$	*circulition, octomidial*	a	Y
		Nikolai-7	pinualpha	$\pi\nu\alpha$	*ditto, septimidial*	a	Y
		Nikolai-11	pinubeta	$\pi\nu\beta$	*ditto, unidecimidial*	a	Y
		Klein	PIkappa	$\Pi\kappa$	*circular disk, straight lines*	s	Y

The calcules of planar geometry of O

	number	*calcule*	**sort**	*comment*	
discrete arithmetic	natural number	alpha	α	*Robinson*	abstract
		ALPHA	A	*Robinson*	concrete
		ALPHABETA	AB	*bi-Robinson*	concrete
		ALPHAEPSILON	AE	*power-Robinson*	concrete
		LAMBDA	Λ	*recursive functions*	concrete
	integral number	rhoiota	$\rho\iota$		abstract
		RHOIOTA	PI		concrete
rational arithmetic	rational number	rho	ρ		abstract
		RHO	P	*e.g. 2/3*	uniconcrete[2]
	ratio number	RHORHO	PP	*e.g. 24/9*	multiconcrete[2]
algebraic arithmetic	biradical number	deltaalpha	$\delta\alpha$	*+ - . / square root* [3]	abstract
		DELTAalpha	$\Delta\alpha$	*ditto*	semiconcrete[1]
	klein-number	DELTAkappa	$\Delta\kappa$	*see section 4.8*	semiconcrete
	algebraic nr.	etaalpha	$\eta\alpha$		abstract
SOL calculus	real number	zeta	ζ		abstract

[1] semiconcrete is a concrete calcule with strings as individuals, that, however, are not decidable (see section 3.8)
[2] unique, multiple representation of individuals [3] generated from *1* by normal arithmetic functions and square root

Essential survey on the world of number calcules

In supralanguage English one can also metatalk informally about object-language. Precise metalanguage will only be used in this publication when it seems necessary. E.g. sometimes the English word 'axiom' will be used for practical reasons and sometimes the precise word ***Axiom*** of Mencish.

1.3 Logical languages and syntax, terminology

In the following the object-language **Funcish** and to an even more limited extend the metalanguage **Mencish** will be sketched. The perfection is based on a strict use of an alphabet of 256 characters and the font-principle that allows to distinguish between object-language (Arial and Symbol, normal), metalanguage (Arial and Symbol, boldface italics) and supralanguage **English** (Times New Roman), that is used in **italics** for comments or **heuristically** for shortness or easier understanding of conventional mathematical expressions like *sr(2sr(2)-2)* that are put into italics. The following syntax[1)] of Funcish is used for functions and relations , as shown by examples in a condensed form. Metaproperties should be read in full as indicated in the examples of the next four lines.

sort π πε πλ πνα e.g. ***sort(*** π ***)***

variable π1 π2 ***variable(*** π1 ***)***

referable 1π 2π ***referable(*** 1π ***)***

individual-constant πe πf ***individual-constant (*** πe ***)***

function-constant i.e. name of function, that allows for determination of arity as well, using
′ ◊ ♦ ! ⇑ ⇓ ↓ ⊕ ⊗ ® • ÷ ∤ ∩ ∪ ∗ ∫ ∇ and + − × / √ ↑ ° e.g.

unconditioned (π;π↓π↓π) e.g. decision (quaternary function)
conditioned [2)] (π;π⊕π;π⌊[1π≠2π]∨[[1π=2π]∧[3π=4π]]) appension (quaternary function)
the ***condo*** character ⌊ precedes the condition abbreviated (π;π⊕π;π⌊)

pattern [3)] ***variable*** and ***individual-constant*** as such or put into a ***function-constant*** e.g.
(π1;π2↓π1↓π3)
(πe;πf⊕π1;(π1;π2⊗π3;π4⌊)⌊) and compositions thereof

relation-constant i.e. name of relation, that allows for determination of arity as well, using
| ∤ ⊥ ⎨ ≈ ~ ⌡ ⌠ ≅ ∠ — ∴ … ⊃ ⊆ ⇒ and # < ≤ ∈ and = ≠ e.g.
π;π≈π;π e.g. congruity (quaternary relation)
π —π—π internity (ternary relation)

equitive-phrase ***pattern1*** = ***pattern2*** or ***pattern1*** ≠ ***pattern2***
(πe;πf⊕π1;π2⌊)=π4

relitive-phrase ***variable*** and ***individual-constant*** put into a ***relation-constant*** e.g.
π1;π2≈π6;π5 a quaternary ***formula***
∠πe;π1;π2 a binary ***formula***

sentence is formed from ***equitive-*** and ***relitive-phrase*** strings using quantive logic
with logical ***character*** symbols ∨ ∧ → ↔ ¬ ∃ ∀ [ ]
in a way that all ***variable*** strings are bound by a ***quantor*** .

Some examples must suffice for metalanguage Mencish that talks about the character strings of Funcish referred to by a metavariable e.g. ***πε1*** with essentially the same syntax but notice the boldface italics:

∀πε1[[phrase(πε1)] ↔ [[formula(πε1)] ∨ [sentence(πε1)]]]

∀πε1[[sentence(πε1)] → [[TRUTH(πε1)] ∨ [FALSEHOOD(πε1)]]]

Using a general string-replacement metafunction ***(ϕ ; ϕ ∫ ϕ)*** that replaces all proper appearances of the second argument in the first argument by the third one:

∀πλ1[∀πνα1[[[THEOREM(πλ1)] ∧ [πνα1 = (πλ1; πλ∫πνα***)]] → [THEOREM(πνα1)]]]***

[1)] that is called **Bavarian notation** [2)] conditioned functions are usually called **partial**, unconditioned functions are called **total** ; notice that most of geometric functions are conditioned, e.g. two circles may or may not intersect, whence the function 'circle-circle-intersection' is conditioned. [3)] most of the metaproperties have to be defined recursively

In supralanguage English and metalanguage Mencish new expressions enter and some conventional expressions are given a more precise and sometimes even a somewhat deviant meaning. The essential general ones are listed in the following table. Newly coined special supralingual expressions for geometry appear in appendix B , as e.g. linic, protive, contrive, isoscity, circulity, riscolation etc. .

expression	*description*	*comment*
English		
calcule	mathematical system with the rules of Funcish	*German **Kalkul***
concrete	fit for machine-representation	
uniconcrete	unique individuals as strings of characters	
multiconcrete	multiple representation of the same individual	
semiconcrete	individuals are undecidable strings of characters	
abstract	beyond machine-representation	
metacalcule	metasystem talking in Mencish about its calcule	*German **Metakalkul***
ontological basis	what a calcule needs to start with (basic ingredients)	
junctive logic	for 'propositional logic'	
quantive logic	here for 'predicate first-order-logic with equality'	*higher order too*
mater	for 'schema', giving an infinite set of ***sentence*** strings	***deviant***
conditioned	partial function as opposed to total function	***deviant***
meter	for distances, areas and angles; rather than 'measure'	***new***
proto-	relating to basic pair of points, restricted to proto-line	
cali-	entity calibrated with reference to proto-pair	***new***
radical number	obtained from *1* by + - . / and roots	***new***
biradical number	obtained from *1* by + - . / and square root	***new***
corpus	rather than 'field' (algebraic), German 'Körper'	***deviant***
Mencish		*example ,*
sort	as in theory of types	π , πε , AA , δα , Δα
individual	of a calcule (e.g. number, point)	123
number	***sort*** of ***individual*** in arithmetics calcule	A
point	***sort*** of ***individual*** in geometry calcule	πε
individual-constant	name of an ***individual***	πεe
variable	***sort*** with post-index, used in ***scheme*** and ***phrase***	πε1
referable	***sort*** with pre-index, used in conditions only	***new*** 1πε
entitor, omnitor	characters for quantive logic, ***quantor***	***new*** ∃ ∀
condo	character preceding condition of partial function ⎣...	***new*** ⎣
functum	***function*** or ***relation*** , plural in English use functa	
function-constant	name of a function (notice that the sorts are included)	(δα+δα)
relation-constant	name of a relation (notice that the sorts are included)	πε;πε≈πε;πε
basis-	ingredient of ontological basis	***new***
extra-	defined with the entities of the ontological basis	***new***
pattern	***scheme*** or ***term***	***new***
scheme	contains ***variable***	***deviant*** (δα1+δαu)
term	contains no ***variable***	***deviant*** (δαu+δαu)
equitive-phrase	very simple with one equality symbol =	***new*** πε1=πεe
relitive-phrase	very simple with one relation	***new*** ∠πε1;πε2;πε3
phrase	***formula*** or ***sentence***	***new***
formula	contains free ***variable***	πε1;πε2≈πεe;πε3
sentence	in the sense of mathematical logic, contains no free ***variable***	∀δα1[(δα1+δαn)=δα1]
TRUTH , ***FALSEHOOD***	true ***sentence*** , false ***sentence***	
limbHOOD	***sentence*** that is neither true nor false	***new***

1.4 Planar Euclid R-geometry of Tarski

Taken from reference *(8)* the axioms and the so-called axiom-scheme are put into Funcish/Mencish for abstract calcule pitau of planar Euclid R-geometry with relations only in its ontological basis [1].

sort	πτ	Tarski-point
basis-relation-constant		
pair-congruity	πτ;πτ≈πτ;πτ	notice that betweenity allows for equality of points as
betweenity	—πτ;πτ;πτ—	opposed to internity of F-geometry.

axioms of planar Euclid R-geometry

	pair-congruity	
A1	reflexitivity	∀πτ1[∀πτ2[πτ1;πτ2≈πτ2;πτ1]]
A2	identity	∀πτ1[∀πτ2[∀πτ3[[πτ1;πτ2≈πτ3;πτ3]→[πτ1=πτ2]]]]
A3	transitivity	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∀πτ5[∀πτ6[[[πτ1;πτ2≈πτ3;πτ4]∧[πτ3;πτ4≈πτ5;πτ6]]→[πτ1;πτ2≈πτ5;πτ6]]]]]]]
	betweenity	
A4	reflexitivity	∀πτ1[∀πτ2[[πτ1—πτ2—πτ1]→[πτ1=πτ2]]]
A5	Pasch	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∀πτ5[[[[πτ1—πτ4—πτ3]∧[πτ2—πτ5—πτ3]]→ [∃πτ6[[[πτ4—πτ6—πτ2]∧[πτ5—πτ6—πτ1]]]]]]]]]
A6	lower dimension	∃πτ1[∃πτ2[∃πτ3[[[¬[πτ1—πτ2—πτ3]]∧[¬[πτ2—πτ3—πτ1]]]∧ [¬[πτ3—πτ1—πτ2]]]]]]
	congruity and betweenity	
A7	upper dimension	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∀πτ5[**[[**[[πτ1;πτ4≈πτ1;πτ5]∧ [πτ2;πτ4≈πτ2;πτ5]]∧[πτ3;πτ4≈πτ3;πτ5]]∧[πτ4≠πτ5]]→ [[[—πτ1;πτ2;πτ3—]]∨[—πτ2;πτ3;πτ1—]]∨[—πτ3;πτ1;πτ2—]]]]]]]
A8	Euclid variant A	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∀πτ5[∀πτ6[**[[[**[—πτ1;πτ2;πτ6—]∧ [πτ1;πτ2≈πτ2;πτ6]]∧[[—πτ1;πτ4;πτ5—]∧[πτ1;πτ4≈πτ4;πτ5]]]∧ **[**[—πτ2;πτ4;πτ3—]∧[πτ2;πτ4≈πτ3;πτ4]]]→[πτ2;πτ3≈πτ1;πτ6]]]]]]]
	Euclid: variant C	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∀πτ5[[[—πτ1;πτ4;πτ5—]∧[—πτ2;πτ4;πτ3—]]∧ [πτ1≠πτ4]]→[∃πτ6[∃πτ7[[[—πτ1;πτ2;πτ6—]∧[—πτ1;πτ3;πτ7—]]∧[—πτ6;πτ5;πτ7—]]]]]]]]
A9	five segments	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∀πτ5[∀πτ6[∀πτ7[∀πτ8[[[[[[[[πτ1≠πτ2]∧[—πτ1;πτ2;πτ3—]]∧[—πτ5;πτ6;πτ7—]]∧ [πτ1;πτ2≈πτ5;πτ6]]∧[πτ2;πτ3≈πτ6;πτ7]]∧[πτ1;πτ4≈πτ5;πτ8]]∧ [πτ2;πτ4≈πτ6;πτ8]]→[πτ3;πτ4≈πτ7;πτ8]]]]]]]]]
A10	segment unique [2] existence	∀πτ1[∀πτ2[∀πτ3[∀πτ4[∃πτ5[[[—πτ3;πτ4;πτ5—]∧[πτ4;πτ5≈πτ1;πτ2]]∧ [∀πτ6[[[—πτ3;πτ4;πτ6—]∧[πτ4πτ6≈πτ1πτ2]]→[πτ5=πτ6]]]]]]]

axiom mater of continuity (for so-called axiom scheme)

∀πτ1[∀πτ2[[[[[¬[πτ1⊃ πτ3***]]∧[¬[πτ2⊃*** πτ3***]]]∧[sentence(*** ∀πτ1[***πτ1***] ***)]]∧ [sentence(*** ∀πτ2[***πτ2***] ***)]]→[Truth(*** [∃πτ3[∀πτ1[∀πτ2[[[***πτ1***]∧[***πτ2***]]→[—πτ3;πτ1;πτ2—]]]]]→ [∃πτ3[∀πτ1[∀πτ2[[[***πτ1***]∧[***πτ2***]]→[—πτ1;πτ3;πτ2—]]]]] ***)]]]***

[1] However, I am not sure that this basis and these axioms are sufficient to cover **all** of planar Euclidean geometry. How about the unique existence of the top of a triangle with given base and the length of two sides? How about the two orientation of triangles? I have not gone into calcule pitau any deeper - it does not lend itself easily to constructions. How about Archimedes? [2] Uniqueness is missing at Tarski

1.5 Straightedge and roundedge construction

Plausibility is enhanced by heuristic sketches. It is often said that planar geometry is based one compass and ruler. Let's be a little more precise.

A **divider** is a **compass** with two needles, rather than one needle and a drawing-lead. No lines have to drawn on paper as tracks of lead in the heuristic approach, as there are no straight or circular lines in F-geometry, but only points, and a divider can punch holes - that is sufficient. A ruler is not a measuring rod, just a straight rod. A **straightedge** is an idealized ruler of infinite length (with a definite start), a **roundedge** is an idealized divider with arbitrary length of its legs.

A **decisor** is an instrument with four needles that is applied to four points. In case that the first two points are equal it selects the third point, otherwise it selects the fourth point.

Let's be precise: one needs **two** straightedges, a **needle** (for straightedge-intersection), **two** roundedges (for circulation) and a **decisor** for F-geometry. Construction of isoscelic triangles (isoscelition) can be done by the two roundedges that are used with equal spacing, but one can also construct a special **isoscelator**, that allows for marking the tip of an isoscelic triangle with a given base and given legs (at least half of the base).This will be useful in chapter 3 ; of course it is an idealized **isoscelator** for arbitrary lengths.

In section 4.8 it will be useful to have two **ellipsedges**, i.e.an idealized ellipsograph with two needles and axes of arbitrary length.

With these tools one can simulate the functions of F-geometry in the real physical world: applying of functions means drawing and vice versa, from a drawing recipe one gets a function.

1.6 Results

This publication is not a usual mathematics textbook or a handbook. Very few proofs will be given in this publication, as most of the theorems of planar geometries are well established. After all, the aim of this publication is categorical clearity. As far a proofs are concerned there are some proof ideas given. Actually, most of the usual proofs - some of them with an age of more than 2.000 years - can be simply transferred into the language of this publication, and there is no need to do this in most cases, and it is boring, too. Furthermore proofs get **very** lengthy when one uses a perfect logical language. This publication is about the basis of geometry, it is about ontology, it is about levels of languages, it shows the differences between various systems of planar geometry, it points out the problems and shortcomings of the conventional treatises and it gives the proper solution thereto. **And it leads to some partially astonishing results.**

In the following three chapters calcules (of planar geometries) for neutral N-geometry, Euclidean E-geometry and Lobachevskyan L-geometry will be developed. N-geometry is sort of **mother of all geometries** of O. It is well-known that N-geometry is not **complete**, as one can add the parallelity axiom for Euclid E-geometry or its negation for Lobachevsky L-geometry. After all, that's where the interesting story of planar geometry started in the early 19th century.

The usual features of unique and multiple parallels are encountered, as well as the difference in metering areas (either 'base times height' or angle defect), the common impossibility of dividing of angles except successive bisection, the impossibility of higher dividings of segments but the bisection in L-geometry, a different kind of 'Pythagoras' and so on.

Twelve important observations show common features and differences of planar F-geometries:

(1) The concept of **angles** can already be developed in N-geometry although angles are not entities of F-geometry calcules. But still one can talk about equality and minority of angles, right angles, acute angles, combining of angles and so on. There is a way to talk about **metering** of angles without the reference to a calcule of numbers. No numbers (whatsoever) are included in the language of the calcule pi of N-geometry. They do not belong to the ontological basis of the calcule.

(2) In E-geometry and L-geometry there is a way to meter segments (given by pairs of points), angles (given by a triangle with a marked tip) and areas of triangles (given by three points). There is a way to talk about **metering** without reference to a calcule of numbers. **No numbers** (whatsoever) have to be included in the language of the calcules of E- and L-geometry. They are not necessary in the ontological basis of the calcules. Actually it is the other way round: numbers can be **produced** by E-geometry in a certain sense.

(3) In E-geometry and L-geometry **junctive logic** is sufficient for conditions of partial functions.

(4) In E-geometry and L-geometry ***Axiom*** **maters of inductivity** take care that all models are **isomorphic** (a model of a calcule is either another calcule or a subcalcule of a calcule - see section 4.6).

(5) Based on a special triangle construction E-geometry can start with a **less powerful ontological** basis than L-geometry. **Isoscition**, i.e. the construction of isoscelic triangles replaces circulition in the ontological basis. Circulition in E-geometry can be constructed with the use of isoscition. For the heuristic sketches one does not need two roundedges, but only one isoscelator (of section 1.4).

(6) If one extends the calcules E-geometry by numbers for conventional metering **no real numbers** are necessary. So-called **biradical** numbers are sufficient, i.e. numbers that are generated from *1* (one) by arithmetical functions addition, negativation, multiplication, reciprocation and biradication (square root). No kind of limit feature has to be included in E-geometry and L-geometry.

(7) There is a fundamental difference in the choice of proto-pair πo;πe in E-geometry and L-geometry. In E-geometry it has no properties; actually one cannot even attribute properties to pairs of points in E-geometry without reference to the proto-pair. The situation in L-geometry is totally different. In connection with angles one can define properties of pairs; this fact is mirrored in an additional axiom ***A26*** that is needed in L-geometry. The standard choice is the so-called ***Proto-octomidial-axiom*** .

(8) There are many non-standard L-geometries depending on the choice for ***A26*** e.g. one can choose the ***Proto-septimidial-axiom*** . Without such an axiom L-geometry is **incomplete**.

(9) The Euclidean Klein-model of Lobachevsky planar geometry necessitates some kind of **recursion** (for the introduction of the property of points to be 'part of klein-circle').

(10) The Euclidean Klein-model of Lobachevsky planar geometry supposedly does not contain the full unit-circle. The simplest form of a very plausible **conjecture** says: there is no point with the coordinates *(1/2 , 0)* . Biradical numbers supposedly contain a closed
genuine subset of newly defined **klein-numbers** between *-1* and *1*

(11) E-geometry can be extended for a full treatment of **Archimedes** axiom and for **polygons** . This inclusion of natural numbers is done conform to the used system. Whereas E-geometry has regular polygons with a count of corners that is a product of Fermat-primes and powers of *2* standard L-geometry has only trivial regular polygons with a count of powers of *2* starting with the octagon .

(12) The **undecidability** of planar geometries is straightforward, it is essentially the same as the undecidability for biradical numbers (whether a construction leads to a positive number or not).

2. Planar N-geometry

2.1 Ontological basis of calcule pi of planar N-geometry

Absolute geometry was given its name by Bolyai. However, 'absolute' seems a bad choice as it is leaves ways open for completion. Greenberg proposed to rather call it **neutral**. As **Saccheri** thought that it suffices for E-geometry, his name is given to the **abstract calcule** pi **of Saccheri planar N-geometry**. There are three basic geometrical functions, all of them are partial, i.e. they carry a condition and can only applied when they are preceded by a clause that guarantees, that the condition is fulfilled. E.g. one can append a pair of points by another pair only if the two points are different, (heuristically speaking: otherwise one would not know in which direction the new segment is to be added). Properly written: in
∀π1[∀π2[[π1≠π2]→[∀π3[∀π4[... (π1;π2⊕π3;π4∟π1≠π2)... or in
∃π1[∃π2[[[π1≠π2]∧[∀π3[∀π4[... (π1;π2⊕π3;π4∟π1≠π2)... it is shortened to: ... (π1;π2⊕π3;π4∟)...

Furthermore there is the trivial basic logical projection function **decision** (π;π↓π↓π) marked by ***log-arrow*** ↓ . It is a total quaternary function, applicable in **every calcule**. Values are the third and fourth argument, depending on equality and inequality of the first two arguments.

[[π1=π2]→[(π1;π2↓π3↓π4)=π3]]∧[[π1≠π2]→[(π1;π2↓π3↓π4)=π4]]

The **ontological basis** of abstract calcule pi comprises the following **ingredients**. There are only points, no line segments, no lines, no rays, no circle segments and no circles! No ***basis-relation-constant*** strings appear. For the Archimedes feature the function **entiration** is included. The first function **appension** corresponds directly to Tarski's ***Axiom A10*** ' segment unique existence' . New names are chosen for the functions and relations, so that they cannot be confused with similar conventional ones.

sort	π		*point*
basis-individual-constant	***proto-origin*** πo	***proto-end*** πe	*no special properties*
basis-function-constant	*condition*		*conventional description*

appension [1]	(π;π⊕π;π∟	1π;2π;3π;4π)	*segment-segment-attachment*
circulition	(π;π∇π;π∟	1π;2π. . . .∴3π;4π)	*circle-circle-intersection*
isoscition	(π;π∇π∟	1π;2π. . . .∴3π)	*isosceles construction*
linisection	(π;π⊗π;π∟	1π;2π⫲3π;4π)	*line-line-intersection*
entiration	(π;π•π∟	[1π≠2π]∧[1π≠3π])	*maximum of multiples of a pair*

extra-relation-constant strings appearing in conditions:

appensity [2]	π;π;π;π	[1π≠2π]∨[[1π=2π]∧[3π=4π]]	
circulity	π;π. . . .∴π;π		*distances form triangle or linic*[3] *triple*
isoscity	π;π. . . .∴π		*greater half base*
linisectivity	π;π⫲π;π		

Circulity is necessary for - conventionally speaking - two circles to intersect. **Isoscition** and **isoscity** for intersection of circles of same radius has been added to the ontological basis in order to make things easier when E-geometry is introduced, as many ***extra-relation-constant*** strings can already be defined using isoscition.

Linisectivity is necessary for intersection of lines, as one allows for cases where - conventionally speaking - two lines do not intersect (they are called parallels).

[1] by convention supralingual names of functions are chosen to end on 'ion' [2] names of relations are chosen to end on 'ity'

Observe that there are two kind of line intersections, that are called transsectivity and cissectivity. In the first case of transsectivity intersection is straightforward whereas in the second case of cissectivity the intersection has to be checked in some fashion. It will turn out that cissectivity is different in E-geometry and L-geometry, whereas transsectivity coincides. Transsectivity is described with junctive logic, whereas cissectivity is nonjunctive in N-geometry although junctive both in E-geometry and L-geometry. Definitions are developed further down. They get much clearer when the concept of distance has been introduced in section 2.3 . For writing down the ***Axiom*** strings a few ***extra-functum-constant*** strings come handy, that are defined below on the base of ***basis-function-constant*** strings only:

extra-relation-constant strings:

pair-congruity	π;π≈π;π	
pair-minority	π;π≺π;π	
pair-conminority	π;π≺≈π;π	
internity	π —π—π	*no equalities allowed*
linicity	—π;π;π	*at most one equality allowed on line*
line-equality	π;π—π;π	*two genuine pairs on same kind*
tri-angularity	∠π;π;π	*different points, not linic* [1]
tria-isoscity	π;π∴π	*genuine isosceles-triangular condition*
lina-isoscity	π;π…π	*linic isosceles-triangular condition*
tria-circulity	π;π∴π;π	*three distances form circular triple*
lina-circulity	π;π…π;π	*three distances form linic triple*
protivity	⎛π;π;π	*protive triangle*
lini-protivity	⎛—π;π;π	*protive triangle or linic*
regular-transsectivity	π;π╫╫π;π	*guaranteed intersection of two lines*
fringe[2]-transsectivity	π;π|╫|π;π	*fringe case thereof*

Definition of ***extra-relation-constant*** strings:

pair-congruity [π1;π2≈π3;π4]↔[[[π1=π2]∧[π3=π4]]∨[[π1≠π2]∧[(π1;π2⊕π1;π2⎿)=(π1;π2⊕π3;π4⎿)]]]

pair-minority [π1;π2≺π3;π4]↔[[[π1≠π2]∧[(π1;π2⊕π1;π2⎿)≠(π1;π2⊕π3;π4⎿)]]∧
[(π1;π2⊕π3;π4⎿)=(π1;(π1;π2⊕π1;π2⎿)⊕(π1;π2⊕π1;π2⎿);(π1;π2⊕π3;π4⎿)⎿)]]

pair-conminority [π1;π2≺≈π3;π4]↔[[π1=π2]∨[(π1;π2⊕π3;π4
[[π1≠π2]∧[(π1;π2⊕π3;π4⎿)=(π1;(π1;π2⊕π1;π2⎿)⊕(π1;π2⊕π1;π2⎿);(π1;π2⊕π3;π4⎿))]]]

internity [π1 —π2—π3]↔[[[π1≠π2]∧[π2≠π3]]∧[(π1;π2⊕π2;π3⎿)=π3]]

linicity [—π1;π2;π3]↔[[[[[π1=π2]∧[π2≠π3]]∨[[π2=π3]∧[π3≠π1]]]∨[[π3=π1]∧[[π1≠π2]]]∨
[[[π1 —π2—π3]∨[π2 —π3—π1]]∨[π3 —π1—π2]]]

line-equality [π1;π2—π3;π4]↔[[[π1≠π2]∧[π3≠π4]]∧[[[[[(π1;π2⊕π2;π3⎿)=π3]∨
[(π2;π1⊕π1;π3⎿)=π3]]∨[π1=π3]]∨[[π1≠π3]∧[(π1;π3⊕π3;π2⎿)=π2]]]∧
[[[[(π1;π2⊕π2;π4⎿)=π4]∨[(π2;π1⊕π1;π4⎿)=π4]]∨[π1=π4]]∨
[[π1≠π4]∧[(π1;π4⊕π4;π2⎿)=π2]]]]]

tri-angularity [∠π1;π2;π3]↔[[[[π1≠π2]∧[π2≠π3]]∧[π3≠π1]]∧[¬[—π1;π2;π3]]]

tria-isoscity [π1;π2∴π3]↔[[π1≠π2]∧[π1;π2≺π1;(π1;π3⊕⎿)]]

lina-isoscity [π1;π2…π3]↔[[π1≠π2]∧[π1;π2≈π1;(π1;π3⊕)]]

[1] 'linic' means that points belong to a straight line, for good reasons the 'linear' is avoided, it usually has other meanings

[2] 'fringe' refers to cases at the edges of a condition, e.g. equality is the fringe part of equal-minority

isoscity	[π1;π2…∴π3]↔[[[π1≠π2]∧[π1;π2≀≈π1;(π1;π3⊕)]]∨[[π1=π2]∧[π1=π3]]]

tria-circulity [π1;π2∴π3;π4]↔[[[[[[π1≠π2]∧[π1≠π3]]∧[π2≠π4]]∧
smaller pro-contra-sum [(π1;((π2;π1⊕π1;π2└);π1⊕π1;π3└)⊕π2;π4└)=
(π1;π2⊕π2;(π1;((π2;π1⊕π1;π2└);π1⊕π1;π3└)⊕π2;π4└)└)]]∧
smaller pro-sum [(π1;((π2;π1⊕π1;π2└);π1⊕π1;π3└)⊕((π2;π1⊕π1;π2└);π1⊕π1;π3└);(π1;π2⊕π2;π4└)└)=
(π1;π2⊕π2;π4└)]]∧
smaller contra-sum [(π2;((π1;π2⊕π1;π2└);π2⊕π2;π4└)⊕((π1;π2⊕π1;π2└);π2⊕π2;π4└);(π2;π1⊕π1;π3└)└)=
(π2;π1⊕π1;π3└)]]

lina-circulity [π1;π2…π3;π4]↔[[π1≠π2]∧
[[[((π2;π1⊕π1;π2└);π1⊕π1;π3└)=((π1;π2⊕π1;π2└);π2⊕π2;π4└)]∨
[((π2;π1⊕π1;π2└);π1⊕π1;π3└)=(π1;π2⊕π2;π4└)]]∨
[(π2;π1⊕π1;π3└)=((π1;π2⊕π1;π2└);π2⊕π2;π4└)]]]

With these auxiliary ***extra-functum-constant*** strings the necessary ***extra-relation-constant*** strings that are necessary for the conditions in the ***basis-function-constant*** strings.

circulity	[π1;π2…∴π3;π4]↔ [[[π1;π2…π3;π4]∨[π1;π2∴π3;π4]]∨[[[π1=π2]∧[π1=π3]]∧[π1=π4]]]

protivity [⎛π1;π2;π3]↔[[∠π1;π2;π3]∧[π1=((π1⊕π2;π1;π3└);π3∇π1└)]]

this definition for a triangle with a definite orientation (that will be called clockwise) is chosen as compared to the simpler with π3=(π1;π2∇π3;π3└)
in order to be best prepared for E-geometry

lini-protivity [⎛—π1;π2;π3]↔[[⎛π1;π2;π3]∨[—π1;π2;π3]]

regular-transsectivity [π1;π2┼|┼π3;π4]↔[[[π1≠π2]∧[π3≠π4]]∧ *genuine pairs*
[[[[[⎛—π1;π2;π3]∧[⎛π2;π1;π4]]∨[[⎛—π2;π1;π3]∧[⎛π1;π2;π4]]]∨ *opposite sense*
[[⎛—π3;π4;π1]∧[⎛π4;π3;π2]]]∨[[⎛—π4;π3;π1]∧[⎛π3;π4;π2]]]]] *of orientation*

fringe-transsectivity [π1;π2|┼|π3;π4]↔[[[[π1=π2]∧[π1=π3]]∧[π1=π4]]∨
[[[[π1≠π2]∧[π3=π4]]∧[—π1;π2;π3]]∨[[[π1=π2]∧[π3≠π4]]∧[—π3;π4;π1]]]]

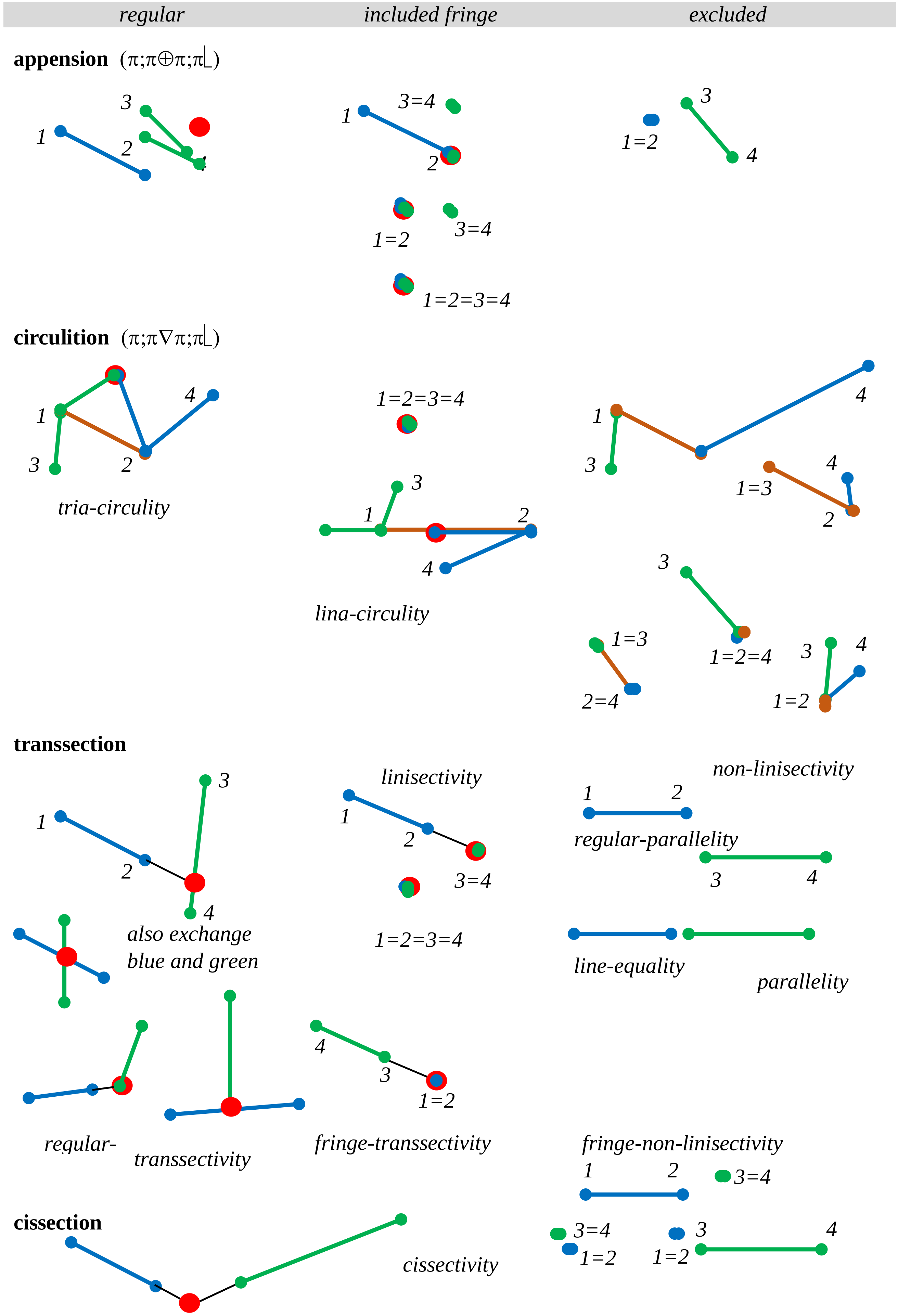
transsectivity [π1;π2┼π3;π4]↔[[[π1;π2┼|┼π3;π4]∨[π1;π2|┼|π3;π4]]∧ *regular or fringe- ,*
[¬[π1;π2—π3;π4]]] *but not line-equal*

Notice that only this definition is **junctive**. It uses appension and isoscition (for sense of orientation by protivity) without ***entitor*** ∃ as opposed to the following **nonjunctive** definition. For the application of nonjunctive ***Syniom***[1)] strings one has to remember that ***variable*** collisions have to be avoided at insertions.

cissectivity [π1;π2╫π3;π4]↔
[∃π5[[[π1—π2—π5]∨[π5—π1—π2]]∧[[π3—π4—π5]∨[π5—π3—π4]]]]

lini-sectivity	[π1;π2⫼π3;π4]↔[[π1;π2┼π3;π4]∨[π1;π2╫π3;π4]]

1) ***Syniom*** strings are synonymous ***sentence*** strings



Conditions and principal arrangements for basic functions in the various possible cases

2.2 Axioms of planar N-geometry

The following ***Axiom*** strings are listed in a top-down fashion so that the conditions can be expressed with the preceding functions.

<u>appension</u>

A1 vanities appension
$\forall\pi1[\forall\pi2[[[(\pi1;\pi1\oplus\pi1;\pi2\llcorner)=\pi2]\wedge[(\pi1;\pi1\oplus\pi2;\pi1\llcorner)=\pi2]]\wedge[\forall\pi3[(\pi1;\pi2\oplus\pi3;\pi3\llcorner)=\pi2]]]]$

A2 symmetry appension
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[\pi1\neq\pi2]\rightarrow[(\pi1;\pi2\oplus\pi3;\pi4\llcorner)=(\pi1;\pi2\oplus\pi4;\pi3\llcorner)]]]]]$

A3 independence appension
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[\forall\pi5[\forall\pi6[[[\pi1\neq\pi2]\wedge[\pi3\neq\pi4]]\rightarrow[(\pi1;(\pi1;\pi2\oplus\pi3;\pi4\llcorner)\oplus\pi5;\pi6\llcorner)=(\pi2;(\pi1;\pi2\oplus\pi3;\pi4\llcorner)\oplus\pi5;\pi6\llcorner)]]]]]]]$

A4 associativity appension
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[\forall\pi5[\forall\pi6[[[\pi1\neq\pi2]\wedge[\pi3\neq\pi4]]\rightarrow[(\pi1;\pi2\oplus\pi3;(\pi3;\pi4\oplus\pi5;\pi6\llcorner)\llcorner)=(\pi1;(\pi1;\pi2\oplus\pi3;\pi4\llcorner)\oplus\pi5;\pi6\llcorner)]]]]]]]$

A5 commutativity appension
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[\forall\pi5[\forall\pi6[[\pi1\neq\pi2]\rightarrow[(\pi1;(\pi1;\pi2\oplus\pi3;\pi4\llcorner)\oplus\pi5;\pi6\llcorner)=(\pi1;(\pi1;\pi2\oplus\pi5;\pi6\llcorner)\oplus\pi3;\pi4\llcorner)]]]]]]]$

A6 suspensivity appension
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[[\pi1\neq\pi2]\wedge[\pi3\neq\pi4]]\rightarrow[((\pi1;\pi2\oplus\pi3;\pi4\llcorner);\pi2\oplus\pi1;\pi2\llcorner)=\pi1]]]]]$

A7 directivity[1)] appension
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[\pi1\neq\pi2]\rightarrow[(\pi1;\pi2\oplus\pi3;\pi4)\neq\pi1]]]]]$

A8 triangularity appension
$\forall\pi1[\forall\pi2[\forall\pi3[[\angle\pi1;\pi2;\pi3]\rightarrow[\pi1;\pi2\prec\pi1;(\pi1;\pi3\oplus\pi3;\pi2\llcorner)]]]]$

<u>circulition</u>

A9 auto-circulition
$\forall\pi1[(\pi1;\pi1\nabla\pi1;\pi1\llcorner)=\pi1]$

A10 congruities tria-circulition
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[\pi1;\pi2\therefore\pi3;\pi4]\rightarrow[[\pi1;(\pi1;\pi2\nabla\pi3;\pi4\llcorner)\approx\pi1;\pi3]\wedge[\pi2;(\pi1;\pi2\nabla\pi3;\pi4\llcorner)\approx\pi2;\pi4]]]]]]$

A11 uniqueness tria-circulition
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[\forall\pi5[[[[\pi1;\pi2\therefore\pi3;\pi4]\wedge[\pi1;\pi2\therefore\pi5;\pi4]]\wedge[\pi1;\pi5\approx\pi1;\pi3]]\rightarrow[(\pi1;\pi2\nabla\pi3;\pi4\llcorner)=(\pi1;\pi2\nabla\pi5;\pi4\llcorner)]]]]]]$

A12 duplicity orientation tria-circulition
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[\pi1;\pi2\therefore\pi3;\pi4]\rightarrow[(\pi1;\pi2\nabla\pi3;\pi4\llcorner)\neq(\pi2;\pi1\nabla\pi4;\pi3\llcorner)]]]]]$

A13 lina-circulition
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[\pi1;\pi2\ldots\pi3;\pi4]\rightarrow[[[[((\pi2;\pi1\oplus\pi1;\pi2\llcorner);\pi1\oplus\pi1;\pi3\llcorner)=((\pi1;\pi2\oplus\pi1;\pi2\llcorner);\pi2\oplus\pi2;\pi4\llcorner)]\wedge[(\pi1;\pi2\nabla\pi3;\pi4\llcorner)=((\pi2;\pi1\oplus\pi1;\pi2\llcorner);\pi1\oplus\pi1;\pi3\llcorner)]]\vee[[((\pi2;\pi1\oplus\pi1;\pi2\llcorner);\pi1\oplus\pi1;\pi3\llcorner)=(\pi1;\pi2\oplus\pi2;\pi4\llcorner)]\wedge[(\pi1;\pi2\nabla\pi3;\pi4\llcorner)=\pi4]]]\vee[[(\pi2;\pi1\oplus\pi1;\pi3\llcorner)=((\pi1;\pi2\oplus\pi1;\pi2\llcorner);\pi2\oplus\pi2;\pi4\llcorner)]\wedge[(\pi1;\pi2\nabla\pi3;\pi4\llcorner)=\pi3]]]]]]]$

A14 isoscition [2)]
$\forall\pi1[\forall\pi2[\forall\pi3[(\pi1;\pi2\nabla\pi3\llcorner)=(\pi1;\pi2\nabla\pi3;(\pi1;\pi2\oplus\pi1;\pi3\llcorner)\llcorner)]]]$

<u>linisection</u>

A15 linisection Pasch
$\forall\pi1[\forall\pi2[\forall\pi3[\forall\pi4[[[[[\frown\pi1;\pi2;\pi3]\wedge[\frown\pi1;\pi2;\pi4]]\wedge[\frown\pi4;\pi2;\pi3]]\wedge[\frown\pi1;\pi4;\pi3]]\rightarrow[\exists\pi5[[\pi1\text{—}\pi4\text{—}\pi5]\wedge[\pi2\text{—}\pi5\text{—}\pi3]]]]]]]$

A16 auto-section linisection
$\forall\pi1[[(\pi1;\pi1\otimes\pi1;\pi1\llcorner)=\pi1]\wedge[\forall\pi2[\forall\pi3[[[\pi1\neq\pi2]\wedge[\text{—}\pi1;\pi2;\pi3]]\rightarrow[(\pi1;\pi2\otimes\pi3;\pi3\llcorner)=\pi3]]]]]$

[1)] ***A7*** excludes elliptic geometries

[2)] ***A14*** is just a definition. however, it is included trivially as an ***Axiom*** in order to simplify definitions of functions, that do not need circulition, but for which isoscition i.e. the construction of isosceles triangles is sufficient. This will become essential in chapter 3 for E-geometry where one can keep the same definitions

A17	incidence linisection	$\forall\pi_1[\forall\pi_2[\forall\pi_3[\forall\pi_4[[[[\pi_1;\pi_2\nparallel\pi_3;\pi_4]\wedge[\pi_1\neq\pi_2]]\wedge[\pi_3\neq\pi_4]]\rightarrow [[\text{—}\pi_1;\pi_2;(\pi_1;\pi_2\otimes\pi_3;\pi_4\llcorner)]\wedge[\text{—}\pi_3;\pi_4;(\pi_1;\pi_2\otimes\pi_3;\pi_4\llcorner)]]]]]]$
A18	uniqueness linisection	$\forall\pi_1[\forall\pi_2[\forall\pi_3[\forall\pi_4[[[[\pi_1;\pi_2\nparallel\pi_3;\pi_4]\wedge[\pi_1\neq\pi_2]]\wedge[\pi_3\neq\pi_4]]\rightarrow [\forall\pi_5[[[\text{—}\pi_1;\pi_2;\pi_5]\wedge[\text{—}\pi_3;\pi_4;\pi_5]]\rightarrow[\pi_5=(\pi_1;\pi_2\otimes\pi_3;\pi_4\llcorner)]]]]]]]$
A19	commutativity linisection	$\forall\pi_1[\forall\pi_2[\forall\pi_3[\forall\pi_4[[[[\pi_1;\pi_2\nparallel\pi_3;\pi_4]\wedge[\pi_1\neq\pi_2]]\wedge[\pi_3\neq\pi_4]]\rightarrow [(\pi_1;\pi_2\otimes\pi_3;\pi_4\llcorner)=(\pi_3;\pi_4\otimes\pi_1;\pi_2\llcorner)]]]]]$
A20	symmetry linisection	$\forall\pi_1[\forall\pi_2[\forall\pi_3[\forall\pi_4[[[\pi_1;\pi_2\nparallel\pi_3;\pi_4]\wedge[\pi_1\neq\pi_2]]\rightarrow [(\pi_1;\pi_2\otimes\pi_3;\pi_4\llcorner)=(\pi_2;\pi_1\otimes\pi_3;\pi_4\llcorner)]]]]]$

[1]

<u>Archimedes entiration and other</u>

A22	entiration initiality	$\forall\pi_1[\forall\pi_2[\forall\pi_3[[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]\rightarrow[[[\pi_1;\pi_2\prec\pi_1;\pi_3]\rightarrow[(\pi_1;\pi_2\bullet\pi_3\llcorner)=\pi_1]]\wedge [[\pi_1;\pi_2\approx\pi_1;\pi_3]\rightarrow[(\pi_1;\pi_2\bullet\pi_3\llcorner)=\pi_2]]]]]]$
A23	entiration commutativity	$\forall\pi_1[\forall\pi_2[\forall\pi_3[[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]\rightarrow [(\pi_1;(\pi_1;\pi_2\oplus\pi_1;\pi_3\llcorner)\bullet\pi_3\llcorner)=(\pi_1;(\pi_1;\pi_2\bullet\pi_3\llcorner)\oplus\pi_1;\pi_3\llcorner)]]]]$
A24	entiration maximality	$\forall\pi_1[\forall\pi_2[\forall\pi_3[[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]\rightarrow [(\pi_1;(\pi_1;\pi_2\oplus\pi_1;\pi_3\llcorner)\bullet\pi_3\llcorner)=(\pi_1;(\pi_1;\pi_2\bullet\pi_3\llcorner)\oplus\pi_1;\pi_3\llcorner)]]]]$
A25	non-triviality	$\pi o\neq\pi e$

For so-called Archimedes-ordering there are two alternatives of ontological bases (see section 5.1). However, the above choice is preferred for the moment as it is simpler and better adapted to construction. There are three important comments:

a) Notice that use is made of relations internity, linicity, linisectivity, tria-circulity, lina-circulity, pair-congruity, pair-minority , pair-conminority and tri-angularity. However, it was done in a **top-down** way so that no self-references appear.

b) In the calcules of geometry of O the introduction of functions through value definition is not admissible by our own self-imposed requirement: functions can only be defined by **composition** of given ones. By the use of the decision function some logic enters the definition of functions, e.g. for emination and emaxation (see section 2.3) .

c) Another remark relates to the analysis of geometrical proofs. Sometimes it is said 'take any point on this line' but with strict logic it is not possible to pick any point without further qualification. One has to **specify a point**, and there rests no problem, as soon as a certain point is chosen, everything goes along smoothly. As an example: when constructing the parallel-angle at a given point Perron (p. 42) asks you to pick any point on a perpendicular. One rather has to take a specific point, e.g. the point that has the distance of the perpendicular pair. Only in this way one can fulfil the above composition-requirement.

If the above ***Axiom*** strings contain an ***Axiom*** that is not independent of the others (meaning that it can be proven from the other ones as a ***THEOREM***) just drop it, no harm is done.

[1] number gap on purpose, this is where parallelity-related axiom ***A21*** will fit in

2.3 Fundamental extrafuncta

Definitions of some convenient ***extra-function-constant*** strings | *condition* $\llcorner$

pair-doublition	$(\pi_1;\pi_2\oplus)=(\pi_1;\pi_2\oplus\pi_1;\pi_2\llcorner)$	*none*
dispension	$(\pi_1\oplus\pi_2;\pi_3;\pi_4\llcorner)=((\pi_2;\pi_1\oplus);\pi_1\oplus\pi_3;\pi_4\llcorner)$	$[1\pi\neq2\pi]\vee[[1\pi=2\pi]\wedge[3\pi=4\pi]]$
suspension	$(\pi_1;\pi_2;\pi_3\oplus\pi_4\llcorner)=(\pi_2\oplus\pi_1;\pi_3;\pi_4\llcorner)$	*ditto*
adipension	$(\oplus\pi_1;\pi_2;\pi_3;\pi_4\llcorner)=(\pi_1\oplus\pi_2;\pi_1;(\pi_2\oplus\pi_1;\pi_3;\pi_4\llcorner)\llcorner)$	*ditto*

Now for the first time use is made of logical function $(\pi;\pi\downarrow\pi\downarrow\pi)$ for determination of the smaller and greater distance:

emination $(\pi_1\cap\pi_2;\pi_3)=$ *none*

$(\pi_3;\pi_1\downarrow\pi_1\downarrow(\pi_1;(\pi_3;\pi_2\downarrow\pi_1\downarrow(\pi_3;(\pi_3\oplus\pi_1;\pi_3;\pi_2\llcorner)\oplus(\pi_3\oplus\pi_1;\pi_3;\pi_2\llcorner);\pi_1\llcorner))\downarrow\pi_2\downarrow\pi_1))$

smaller or equal distance to last — *if same distance* π_2 *is taken*

the internal conditions are not relevant due to preceding decision — *fringe cases are included*

emaxation $(\pi_1\cup\pi_2;\pi_3)=((\pi_1\cap\pi_2;\pi_3);\pi_1\downarrow\pi_2\downarrow\pi_1)$ *none*

$(\pi_3;\pi_1\downarrow\pi_1\downarrow(\pi_1;(\pi_3;\pi_2\downarrow\pi_1\downarrow(\pi_3;(\pi_3\oplus\pi_1;\pi_3;\pi_2\llcorner)\oplus(\pi_3\oplus\pi_1;\pi_3;\pi_2\llcorner);\pi_1\llcorner))\downarrow\pi_1\downarrow\pi_2))$

$=((\pi_1\cap\pi_2;\pi_3);\pi_1\downarrow\pi_2\downarrow\pi_1)$

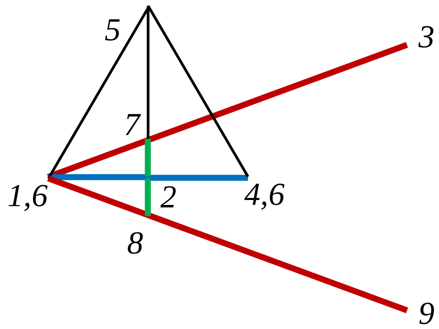
greater or equal distance to last — *if same distance* π_1 *is taken*

fringe cases are included

reflection $(\pi;\pi\div\pi\llcorner 1\pi\neq2\pi)$

$(\pi_1;\pi_2\div\pi_3\llcorner)=$ $\pi_6\oplus(((\pi_1\cup(\pi_1;\pi_2\llcorner);\pi_3);\pi_3\otimes\pi_2;(\pi_1;(\pi_1;\pi_2\oplus)\nabla(\pi_1;\pi_2\oplus)\llcorner)\llcorner);\pi_2\oplus);\pi_1;\pi_3\llcorner)$

$\pi_4=(\pi_1;\pi_2\oplus)$
$\pi_5=(\pi_1;\pi_4\nabla\pi_4\llcorner)$
$\pi_6=(\pi_1\cup\pi_4;\pi_3)$
$\pi_7=(\pi_6;\pi_3\otimes\pi_2;\pi_5\llcorner)$
$\pi_8=(\pi_7;\pi_2\oplus)$
$\pi_9=(\pi_6\oplus\pi_8;\pi_1;\pi_3\llcorner)$



it is also valid for fringe cases,
π_3 *on* π_2 π_5 *or on* π_1 π_2

By the way, reflection is a symmetry, i.e. a rigid motion of polygons.

perpendiculation $(\pi;\pi;\pi!\llcorner 1\pi\neq2\pi)$
$(\pi_1;\pi_2;\pi_3!\llcorner)=\pi_{11}=(\pi_1;\pi_2\otimes\pi_3;(\pi_1;\pi_2\div\pi_3\llcorner)\llcorner)$

One can express these function very easy with circulation; by two ***THEOREM*** strings (isoscition was introduced in the ontological basis in foresight to E-geometry).

perpendiculation $(\pi_1;\pi_2;\pi_3!\llcorner)=(\pi_1;\pi_2\otimes(\pi_1;\pi_2\nabla\pi_3;\pi_3\llcorner);(\pi_2;\pi_1\nabla\pi_3;\pi_3\llcorner)\llcorner)$
reflection $(\pi_1;\pi_2\div\pi_3\llcorner)=(\pi_3;(\pi_1;\pi_2;\pi_3!\llcorner)\oplus)$

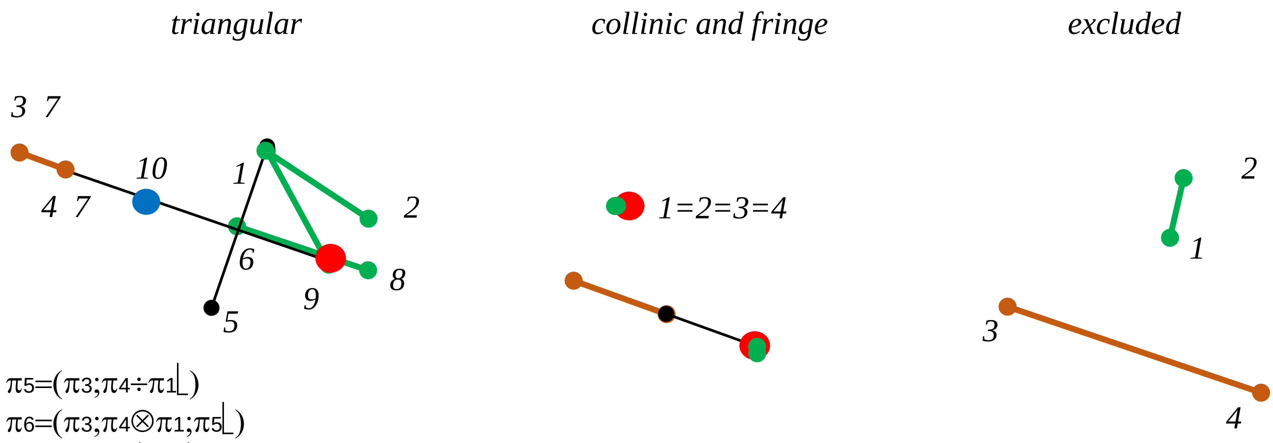
So far six ***extra-relation-constant*** strings were introduced that were convenient for the conditions of the three ***basic-function-constant*** and the ***Axiom*** strings. Now a great number of more ***extra-relation-constant*** strings are defined. Notice that except for cissectivity and parallelity (both containing ***entitor***) all definitions are **mere junctive abbreviations**.

betweenity	$[\text{—}\pi_1;\pi_2;\pi_3\text{—}]\leftrightarrow[[[[\pi_1\neq\pi_2]\wedge[\pi_2\neq\pi_3]]\wedge[(\pi_1;\pi_2\oplus\pi_2;\pi_3\llcorner)=\pi_3]]\vee$ $[[\pi_1\neq\pi_3]\wedge[[\pi_2=\pi_1]\vee[\pi_2=\pi_3]]]]$	
collinicity	$[\pi_1;\pi_2;\pi_3\text{—}]\leftrightarrow[[[[\pi_1\neq\pi_2]\wedge[\pi_2\neq\pi_3]]\wedge[\pi_3\neq\pi_1]]\wedge$ $[[[(\pi_1;\pi_2\oplus\pi_2;\pi_3\llcorner)=\pi_3]\vee[(\pi_2;\pi_3\oplus\pi_3;\pi_1\llcorner)=\pi_1]]\vee[(\pi_3;\pi_1\oplus\pi_1;\pi_2\llcorner)=\pi_2]]]$	
zero-angularity	$[\pi_1\text{—}\pi_2;\pi_3]\leftrightarrow[[[\pi_1\text{—}\pi_2\text{—}\pi_3]\vee[\pi_1\text{—}\pi_3\text{—}\pi_2]]\vee[[\pi_1\neq\pi_2]\wedge[\pi_2=\pi_3]]]$	
straight-angularity	$[\pi_1;\pi_2\text{—\,—}\pi_3]\leftrightarrow[\pi_2\text{—}\pi_1\text{—}\pi_3]$	
line-angularity	$[\text{—\,—}\pi_1;\pi_2;\pi_3]\leftrightarrow[[\pi_1\text{—}\pi_2;\pi_3]\vee[\pi_1;\pi_2\text{—\,—}\pi_3]]$	
angularity	$[\angle\pi_1;\pi_2;\pi_3\text{—}]\leftrightarrow[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]$	*tri- or line-angularity*
contrivity	$[\text{⎠}\pi_1;\pi_2;\pi_3]\leftrightarrow[\text{⎛}\pi_1;\pi_2;\pi_3]$	
lini-contrivity	$[\text{⎠—}\pi;\pi;\pi]\leftrightarrow[\text{⎛—}\pi_2;\pi_1;\pi_3]$	
linisectivity	$[\pi_1;\pi_2\text{╫}\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\text{┼}\pi_3;\pi_4]\vee[\pi_1;\pi_2\text{╫}\pi_3;\pi_4]]$	*implicit* **entitor**
fringe-non-linisectivity	$[\pi_1;\pi_2|\pi_3;\pi_4]\leftrightarrow[[[[\pi_1=\pi_2]\wedge[[[\pi_3=\pi_4]]\wedge[\pi_1\neq\pi_3]]\vee$ $[[\pi_3\neq\pi_4]\wedge[\neg[\text{—}\pi_1;\pi_3;\pi_4]]]]\vee[[[\pi_3=\pi_4]]\wedge[\pi_1\neq\pi_2]]\wedge[\neg[\text{—}\pi_1;\pi_2;\pi_3]]]]]$	*no* **entitor**
parallelity	$[\pi_1;\pi_2\text{├─┤}\pi_3;\pi_4]\leftrightarrow[\neg[[\pi_1;\pi_2\text{╫}\pi_3;\pi_4]\vee[\pi_1;\pi_2|\pi_3;\pi_4]]]$	*implicit* **entitor**
regular-parallelity	$[\pi_1;\pi_2\|\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\text{├─┤}\pi_3;\pi_4]\wedge[\neg[\pi_1;\pi_2\text{—}\pi_3;\pi_4]]]$	*parallel, not line-equal*
non-linisectivity	$[\pi_1;\pi_2|\|\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\text{├─┤}\pi_3;\pi_4]\vee[\pi_1;\pi_2|\pi_3;\pi_4]]$	*implicit* **entitor**
syn-parallelity	$[\pi_1;\pi_2\wr\wr\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\|\pi_3;\pi_4]\wedge[[[\text{⎛}\pi_1;\pi_2;\pi_3]\wedge[\text{⎠}\pi_1;\pi_3;\pi_4]]\vee$ $[[\text{⎠}\pi_1;\pi_2;\pi_3]\wedge[\text{⎛}\pi_1;\pi_3;\pi_4]]]]]$	*same direction of points*
anti-parallelity	$[\pi_1;\pi_2\wr\wr\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\|\pi_3;\pi_4]\wedge[[[\text{⎛}\pi_1;\pi_2;\pi_3]\wedge[\text{⎛}\pi_1;\pi_3;\pi_4]]\vee$ $[[\text{⎠}\pi_1;\pi_2;\pi_3]\wedge[\text{⎠}\pi_1;\pi_3;\pi_4]]]]]$	*opposite direction*
syn-line-equality	$[\pi_1;\pi_2\wr\text{—}\wr\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\text{—}\pi_3;\pi_4]\wedge[\pi_2;\pi_3\approx\pi_4;(\pi_1;\pi_2\oplus\pi_3;\pi_4\llcorner)]]$ *or* $\wedge[(\pi_2;\pi_3\oplus)=(\pi_2;\pi_3\oplus\pi_4;(\pi_1;\pi_2\oplus\pi_3;\pi_4\llcorner)\llcorner)]]$	
anti-line-equality	$[\pi_1;\pi_2\wr\text{—}\wr\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\text{—}\pi_3;\pi_4]\wedge[\pi_2;\pi_4\approx\pi_3;(\pi_1;\pi_2\oplus\pi_3;\pi_4\llcorner)]]$ *or* $\wedge[(\pi_2;\pi_4\oplus)=(\pi_2;\pi_4\oplus\pi_3;(\pi_1;\pi_2\oplus\pi_3;\pi_4\llcorner)\llcorner)]]$	
pro-parallelity	$[\pi_1;\pi_2\text{⎛⎛}\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\|\pi_3;\pi_4]\wedge[\text{⎛}\pi_1;\pi_2;\pi_3]]$	*protive side of* $\pi_1;\pi_2$
contra-parallelity	$[\pi_1;\pi_2\text{⎠⎠}\pi_3;\pi_4]\leftrightarrow[[\pi_1;\pi_2\|\pi_3;\pi_4]\wedge[\text{⎠}\pi_1;\pi_2;\pi_3]]$	*contrive side of* $\pi_1;\pi_2$
equilaterality	$[\pi_1\approx\pi_2\approx\pi_3]\leftrightarrow[[\pi_1\neq\pi_2]\wedge[[\pi_1;\pi_2\approx\pi_2;\pi_3]]\wedge[\pi_2;\pi_3\approx\pi_3;\pi_1]]$	
isoscelity	$[\pi_1;\pi_2\approx\pi_3]\leftrightarrow[[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]\wedge[\pi_1;\pi_2\approx\pi_1;\pi_3]]$	
right-angularity	$[\pi_1;\pi_2\perp\pi_3]\leftrightarrow[[\angle\pi_1;\pi_2;\pi_3]\wedge[(\pi_2;\pi_1\oplus\pi_2;\pi_1\llcorner);\pi_3\approx\pi_2;\pi_3]]$	
riso-angularity	$[\pi_1\perp\pi_2\approx\pi_3]\leftrightarrow[[\pi_1;\pi_2\perp\pi_3]\wedge[\pi_1\approx\pi_2;\pi_3]]$	
triangle-congruity	$[\pi_1;\pi_2;\pi_3\cong\pi_4;\pi_5;\pi_6]\leftrightarrow[[[[[[\pi_1;\pi_2;\pi_3\cong\approx\pi_4;\pi_5;\pi_6]\vee[\pi_1;\pi_2;\pi_3\cong\approx\pi_5;\pi_6;\pi_4]]\vee$ $[\pi_1;\pi_2;\pi_3\cong\approx\pi_6;\pi_4;\pi_5]]\vee[\pi_1;\pi_2;\pi_3\cong\approx\pi_5;\pi_4;\pi_6]]\vee[\pi_1;\pi_2;\pi_3\cong\approx\pi_6;\pi_5;\pi_4]]\vee[\pi_1;\pi_2;\pi_3\cong\approx\pi_4;\pi_6;\pi_5]]$	
triangle-anchor-congr.	$[\pi_1;\pi_2;\pi_3\cong\approx\pi_4;\pi_5;\pi_6]\leftrightarrow[[[\pi_1;\pi_2\approx\pi_4;\pi_5]\wedge[\pi_2;\pi_3\approx\pi_5;\pi_6]]\wedge[\pi_3;\pi_1\approx\pi_6;\pi_4]]$	
triangle-anchor-sense-congruity	$[\pi_1;\pi_2;\pi_3\angle\cong\pi_4;\pi_5;\pi_6]\leftrightarrow[[\pi_1;\pi_2;\pi_3\cong\approx\pi_4;\pi_5;\pi_6]\wedge$ $[[[\text{⎛}\pi_1;\pi_2;\pi_3]\wedge[\text{⎛}\pi_4;\pi_5;\pi_6]]\vee[[\text{⎠}\pi_1;\pi_2;\pi_3]\wedge[\text{⎠}\pi_4;\pi_5;\pi_6]]]]$	
triangle-sense-congruity	$[\pi_1;\pi_2;\pi_3\cong\angle\pi_4;\pi_5;\pi_6]\leftrightarrow[[\pi_1;\pi_2;\pi_3\cong\pi_4;\pi_5;\pi_6]\wedge$ $[[[\text{⎛}\pi_1;\pi_2;\pi_3]\wedge[\text{⎛}\pi_4;\pi_5;\pi_6]]\vee[[\text{⎠}\pi_1;\pi_2;\pi_3]\wedge[\text{⎠}\pi_4;\pi_5;\pi_6]]]]$	

circle-12- line-34-intersections

Here one meets the problem to select one of the two possible solutions in the normal case which is easy as one can require the condition that the two solutions have the same direction as the pair of points that define the line. One meets other problems if the center of the the lies on the line or even worse if it coincides with one the two points that define the line. But with the tool of decision one can manage.

syn-resection (π;π®⊕π;π└[[3π≠4π]∧[π1;(π3;π4÷π1└)⦃≈π1;(π1;π2⊕)]]∨[[[1π=2π]∧[1π=3π]]∧[1π=4π]])
anti-resection (π;π⊕®π;π└)



π5=(π3;π4÷π1└)
π6=(π3;π4⊗π1;π5└)
π7=(π6;π3↓π4↓π3)
π8=(π7;π6⊕π1;π2└)
π9=(π1;π6↓π8;(π1;π5∇π2└))
π10=(π9;π6⊕)

(π1;π2®⊕π3;π4└)=((π9;π3⊕);(π9;π3⊕π4;(π10;π9⊕π3;π4└)└)↓π9↓π10)
(π1;π2⊕®π3;π4└)=((π1;π2®⊕π3;π4└);π6⊕)

(π1;π2®⊕π3;π4└);(π1;π2⊕®π3;π4└) *have same direction as* π3;π4

rectification (triangle) (!π1;π2;π3└)=(π1;(!π1;π2└);π3!└) └1π≠2π cathetus from π1

Notice that the following definitions also only need isoscition, and not circulition:

equi-triangulation	(π1∇π2)=(π1;π2∇π2└)	*via equilateral triangle*
pair-dichotomition	(π1⫽π2)=(π1;π2⊗(π1∇π2);(π2∇π1))	*(with character **trip** ⫽)*
anti-perculation	(!π1;π2└)=((π2;π1⊕);π2∇π2└)	*perculation at 1π via equilateral triangle*
syn-perculation	(π1;π2!└)=(π1;(π1;π2⊕)∇(π1;π2⊕)└)	*perculation at 2π via equilateral triangle*
anti-riscolation	(!!π1;π2└)=(π1⊕(!π1;π2└);π1;π2└)	*right-isoscelation at 1π*
syn-riscolation	(π1;π2!!└)=(π2⊕(π1;π2!└);π1;π2└)	*right-isoscelation at 2π*
opposition	(π1÷π2;π3└)=(π3;(π1⫽π2)⊕)	*rotation by straight angle around center*

Existence of parallel can be proven. However, uniqueness is a different question that will be treated in ***Axiom*** strings ***A21e*** and ***A21l*** of E- and L-geometry resp. .

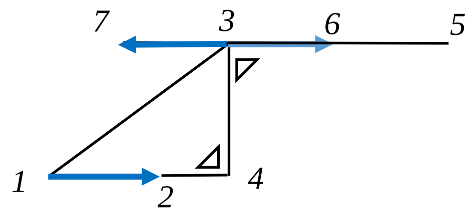
THEOREM existence of parallel ∀π1[∀π2[∀π3[[∠π1;π2;π3]→[∃π4[π1;π2||π3;π4]]]]]

The following function **syn-ortho-parallelation** (π;π⇑⇓π└1π≠2π) can be interpreted as some kind of **translation** (producing a rigid motion of polygons) in planar N-geometry: the segment π1;π2 is moved along π1;π3 so that π1 is put to π3 and π2 to (π1;π2⇑⇓π3└): for a given π1;π2 it constitutes a bijective mapping of points π3 .Translations in N-geometry are not necessarily commutative, in general: (π1;π4⇑⇓(π1;π2⇑⇓π3└)└)≠(π1;π2⇑⇓(π1;π4⇑⇓π3└)└)

syn-ortho-parallelation $(\pi;\pi\Uparrow\Downarrow\pi\llcorner\angle 1\pi;2\pi;3\pi)$ *parallel thru π3 with syn-parallel point*
anti-ortho-parallelation $(\pi;\pi\Downarrow\Uparrow\pi\llcorner\angle 1\pi;2\pi;3\pi)$ *parallel thru π3 with anti-parallel point*

$\pi 4=(\pi 1;\pi 2;\pi 3!\llcorner)$ perpendiculation
$\pi 5=(!\pi 3;\pi 4\llcorner)$ anti-perculation
$\pi 6=(\pi 3\oplus\pi 5;\pi 1;\pi 2\llcorner)$
$\pi 7=(\pi 6;\pi 3\oplus)$

one has to pick the point with the proper direction:



$(\pi 1;\pi 2\Uparrow\Downarrow\pi 3\llcorner)=((\pi 1;\pi 2\nabla\pi 3;\pi 3\llcorner);(\pi 2;\pi 6\nabla\pi 3;\pi 3\llcorner)\downarrow\pi 6\downarrow((\pi 2;\pi 1\nabla\pi 3;\pi 3\llcorner);(\pi 6;\pi 2\nabla\pi 3;\pi 3\llcorner)\downarrow\pi 6\downarrow\pi 7\llcorner)\llcorner)$
$(\pi 1;\pi 2\Downarrow\Uparrow\pi 3\llcorner)=((\pi 1;\pi 2\nabla\pi 3;\pi 3\llcorner);(\pi 2;\pi 6\nabla\pi 3;\pi 3\llcorner)\downarrow\pi 7\downarrow((\pi 2;\pi 1\nabla\pi 3;\pi 3\llcorner);(\pi 7;\pi 2\nabla\pi 3;\pi 3\llcorner)\downarrow\pi 7\downarrow\pi 6\llcorner)\llcorner)$

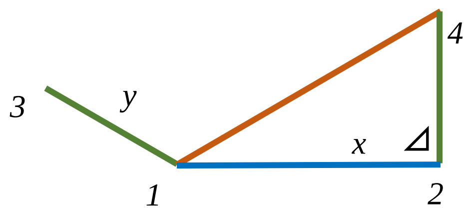
pro-hypothetion $(\pi!!\pi;\pi\llcorner 1\pi\neq 2\pi)$

distance to π1 of protive right triangle π1 π2 π4

in E-geometry $sr(x^2+y^2)$ [1)]

$(\pi 1!!\pi 2;\pi 3\llcorner)=(\pi 2\oplus(\pi 1;\pi 2!\llcorner);\pi 1;\pi 3\llcorner)$

cathetus-right-cathetus-protive-triangulation

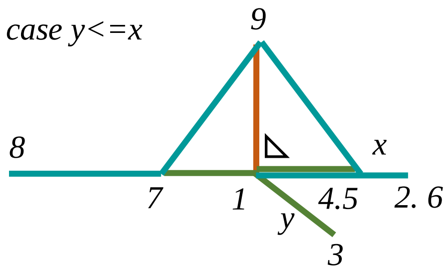


side-side-right-triangulation

pro-cathetetion $(\pi!\pi;\pi\llcorner[1\pi\neq 2\pi]\wedge[1\pi\neq 3\pi])$

$\pi 4=(\pi 1\oplus\pi 2;\pi 1;\pi 3\llcorner)$ *pair minority not necessary*
$\pi 5=(\pi 1\cap\pi 2;\pi 4)$ *picking smaller distance*
$\pi 6=(\pi 1\cup\pi 2;\pi 4)$ *picking greater distance*
$\pi 7=(\pi 5;\pi 1\oplus)$
$\pi 8=(\pi 1;\pi 7\oplus\pi 1;\pi 6\llcorner)$
$(\pi 1!\pi 2!\pi 3\llcorner)=\pi 9=(\pi 7;\pi 5\nabla\pi 8\llcorner)$
distance to π1 of protive right triangle π1π5 π9
in E-geometry $sr(abs(x^2-y^2))$ [2)]

illustration
case y<=x



One can also introduce **contra-hypothetion** $(\pi;\pi!!\pi\llcorner 1\pi\neq 2\pi)$ and **contra-cathetetion** $(\pi;\pi!\pi\llcorner 1\pi\neq 2\pi)$ giving contrive triangles.

Calibrated functions refer to the proto-pair πo;πe (what is called 'calibration'), they are marked by ***function-symbol apostrophe*** . The following triangle constructions are important examples:

cali-adipension $(\oplus\pi;\pi'\llcorner 1\pi\neq 2\pi)$ gives the calibrated absolute difference

$(\oplus\pi 1;\pi 2'\llcorner)=(\oplus\pi 1;\pi 2;\pi o;\pi e\llcorner[1\pi\neq 2\pi]\vee[[1\pi=2\pi]\wedge[[1\pi=\pi o]]\vee[1\pi=\pi e]]]$

cali-pro-hypothetion $('\pi;\pi!\llcorner 1\pi\neq 2\pi)$ *giving a protive triangle, in E-geometry for* $sr(1+x^2)$
$('\pi 1;\pi 2!\llcorner)=(\pi 2\oplus(\pi 1;\pi 2!\llcorner);\pi o;\pi e\llcorner)$

cali-pro-cathetetion $(\pi'!\pi\llcorner 1\pi\neq 2\pi)$ *giving a protive triangle, in E-geometry for* $sr(abs(1-x^2))$
$(\pi 1'!\pi 2\llcorner)=(((\pi 1\cap\pi 2;(\pi 1\oplus\pi 2;\pi o;\pi e\llcorner));\pi 1\oplus);(\pi 1\cap\pi 2;(\pi 1\oplus\pi 2;\pi o;\pi e\llcorner))\nabla$ *distance to π1 of*
$(\pi 1;((\pi 1\cap\pi 2;(\pi 1\oplus\pi 2;\pi o;\pi e\llcorner));\pi 1\oplus)\oplus\pi 1;(\pi 1\cup\pi 2;(\pi 1\oplus\pi 2;\pi o;\pi e\llcorner))\llcorner)\llcorner)$ *protive right triangle π1 π2* $(\pi 1'!\pi 2\llcorner)$

One can also introduce **cali-contra-hypothetion** $(!\pi;\pi'\llcorner 1\pi\neq 2\pi)$ and **cali-contra-cathetetion** $(\pi!'\pi\llcorner 1\pi\neq 2\pi)$ giving contrive triangles.

[1)] *sr(...)* is the square root function - in English we have to stick to Times-Roman characters, no 'root' character available
[2)] *abs(...)* is the absolute value function - no vertical bar symbol available in Times-Roman

Calibrated relations refer to the proto-pair πo;πe as well. One can define ***extra-relation-constant*** **cali-circlity** (short for 'cali-circle-interiority') ⊂π⊃ by [⊂π1⊃]↔[πo;π1⦔πo;πe] that applies for all points inside the cali-circle-line πo;π1≈πo;πe . This will be a useful property in connection with the Klein-model of L-geometry (section 4.6). In addition to proto-pair of ***basis-point-constant*** πo and πe one can define a **reference proto-line** by the property **proto-linity** ⊆π and ***extra-point-constant*** strings. These points will be made use of for so-called proto-extrafunctions.

proto-linity	⊆π	[⊆π1]↔[—πo;πe;π1]	
proto-minus-end	πm	(πe;πo⊕πo;πe⌊)	on proto-line
proto-full	πf	(πo;πe⊕πo;πe⌊)	on proto-line double
proto-dimi	πc	(πo⁄πe)	on proto-line center (half)
proto-threehalf	πg	(πo;πe⊕πo;πc⌊)	on proto-line three half
proto-quarter	πv	(πo⁄πc)	on proto-line quarter (half of half)
proto-right	πr	(πm⊕πo;πo;(πi⁄πe)⌊)	on proto-line for right angle
proto-halfright	πw	(πm⊕πo;πo;((πo⊕(πi⁄πe);πo;πe⌊)⁄πe)⌊)	on proto-l. for halfright angle
cali-imago	πi	(πo⊕(πe;πo!⌊);πo;πe⌊)	outside proto-line, above origin (just a name)
cali-low	πl	((πe;πo!⌊);πo⊕πo;πe⌊)	outside proto-line, below origin
cali-above	πa	(πo⊕(πe;πo!⌊);πo;πe⌊)	outside proto-line, above unit
cali-halfright	πx	(πo⁄πa)	halfright-angle protive triangle to base πo;πe

The **ortho-line** is the line orthogonal to proto-line through πεo , i.e. the line containing πεo and πεi . So there is a heuristic way of taking about proto-axis and ortho-axis, abscissa and ordinate (see section 3.9).

Look at **triangles:** there are 111 ways that one can classify a triple of points:

no triangle		triangle	protive and contraprotive
1	three equal points	18	obtuse, isoscelic, smaller, greater
6	two equal points	18	right, isoscelic, smaller, greater
18	internity , bisecting, smaller or greater	2	acute, equilateral
		12	acute, isoscelic
		36	acute, not Gauss, small-medium-great, sgm, msg, mgs, gsm, gms

Besides triangles one can talk about **quadrangles**, **quintangles** and so on. However, one cannot immediately talk about **polygons**, as they necessitate numbers and a way to include tuples into the language - which can be done by extending the calcule, but not in a simple fashion (see section 5.1).

A **quadrangle** has 4 sides (edges) and 4 corners (vertices, angles). It can be convex (diagonals intersect inside), concave (diagonals intersect outside) or pervex (diagonals lie outside).

A **quadrisymmetrical** is a quadrangle with certain symmetries, essentially the following ones:

kite	2 pairs of adjacent equal sides
parallelogram	2 pairs of opposite equal sides (does not need the relation *parallelity*)
isosceles trapezoid	two opposite equal sides and symmetry
rhombus	4 equal sides
equangle	4 equal angles
rectangle	4 right angles (only in E-geometry)
square	4 equal sides, 4 right angles (only in E-geometry)
quadrilateral	in L-geometry, see section 4.4
quadrate	in L-geometry, see section 4.4

2.4 Proto-extrafuncta

Further properties are introduced along the proto-line. Notice that proto-nonpositivity is not the negation of proto-positivity, as points outside the proto-line are neither positive nor negative.

proto-positivity	<π	[<π1]↔[[πo—π1;πe]∨[πo—πe;π1]]	
proto-nonnegativity	≤π	[≤π1]↔[[<π1]∨[π1=πo]]	defining the **proto-ray**
proto-negativity	π<	[π1<]↔[[πo—π1;πm]∨[πo—πm;π1]]	
proto-nonpositivity	π≤	[≤π1]↔[[π1<]∨[π1=πo]]	
proto-minority	π<π	[π1<π2]↔[[[⊆π1]∧[⊆π2]]∧[πo—π1—π2]]∨[[π1=πo]∧[<π2]]]	
proto-equal-minority	π≤π	[π1≤π2]↔[[[⊆π1]∧[π1=π2]]∨[π1<π2]]	
proto-radity	≤π#	[≤π1<]↔[[≤π1]∧[π1<πe]]	
proto-cyclicity	≤π<	[≤π1<]↔[[≤π1]∧[π1<πf]]	
proto-diametrity	#π#	[#π1#]↔[[πm<π1]∧[π1<πe]]	*inside proto-diameter-segment* πm;πe

Along the proto-line one can define ***extra-function-constant*** strings that are only applicable along the line. They are marked with ***function-constant*** characters + – that are also used in arithmetic calcules. Only in combination with the proto-pair one can include **direction** along the proto-line. Proto-functions are only defined on the proto-line and their values lie on the proto-line. The proto-pair πo;πe constitutes sort of a measuring rod. **Proto-addition** (π+π└) is a little bit complicated, as appension produces sort of **absolute value** strings. One can define some more proto-functions connected to appension:

proto-addition	(π+π└[—1π]∧[—2π]]	(π1+π2└)=(πo;(π1⁄π2└)⊕)
proto-duplication	(π+└—1π)	(π1+└)=(π1+π1└)
proto-negativation	(−π└—1π)	(−π1└)=((πm∩πe;π1);πo⊕πo;π1└)
proto-subtraction	(π−π└[—1π]∧[—2π])	(π1−π2└)=(π1+(−π2└)└)
proto-absolution	(+π└—1π)	(+π1└)=(πo⊕πe;πo;π1└)
proto-absolute-subtraction	(+π−π└[—1π]∧[—2π])	(+π1−π2└)=(+(π1−π2└)└)
proto-bisection	(π/└—1π)	(π1/└)=(πo⁄π1) ***(slash*** / *for bisection and* ***trip*** ⁄ *for dichotomition)*

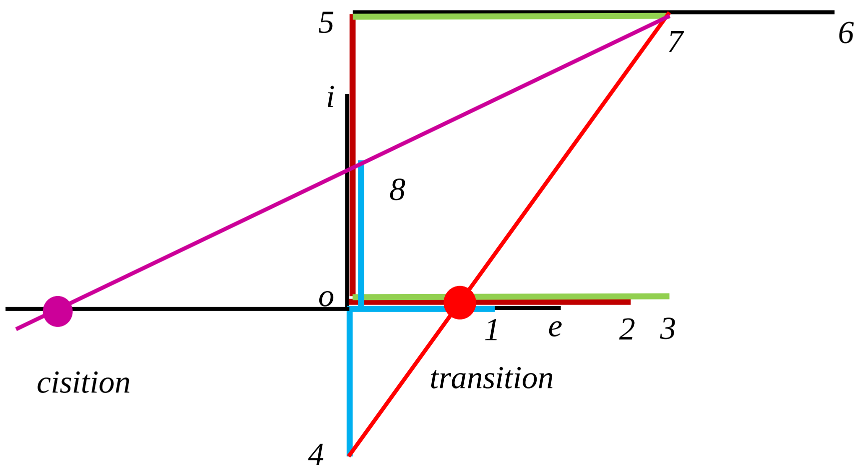
The following lination ***THEOREM*** strings justify the naming of the above functa: with proto-unit, proto-minority, proto-addition and -negativation along the proto-line there is an **ordered unlimited group**.

	<u>proto-minority</u>	
TPM1	non-reflexitivity	∀π1[[⊆π1]→[¬[π1<π1]]
TPM2	antisymmetry	∀π1[∀π2[[[⊆π1]∧[⊆π2]]→[[π1<π2]→[¬[π2<π1]]]]]
TPM3	transitivity	∀π1[∀π2[∀π3[[[[⊆π1]∧[⊆π2]]∧[⊆π3]]→[[[π1<π2]∧[π2<π3]]→[π1<π3]]]]]
	<u>proto-addition and -negativation</u>	
TPL1	conclusity	∀π1[∀π2[[[⊆π1]∧[⊆π2]]→[⊆(π1+π2└)]]]
TPL2	neutrivity	∀π1[[⊆π1]→[(π1+πo└)=π1]]
TPL3	commutativity	∀π1[∀π2[[[⊆π1]∧[⊆π2]]→[(π1+π2└)=(π2+π1└)]]]
TPL4	associativity	∀π1[∀π2[∀π3[[[[⊆π1]∧[⊆π2]]∧[⊆π3]]→[(π1+(π2+π3└)└)=((π1+π2└)+π3└)]]]
TPL5	invertivity	∀π1[[⊆π1]→[(π1+(−π1└)=πo]]
TPL6	monotony	∀π1[∀π2[[[⊆π1]∧[<π2]]→[π1<(π1+π2└)]]]

Furthermore one can transfer the triangle constructions of hypothetion and cathetetion to the proto-line: In E-geometry they are connected with square roots - that is why ***function-symbol*** √ was picked

proto-hypotion	(π√+π└[—1π]∧[—2π])	(π1√+π2└)=(πo⊕πe;π1;(πo⊕πi;πo;π2└)└)
proto-cathetion	(π√−π└2π≤1π)	(π1√−π2└)=(πo⊕πe;πo;(πo;(πo;π1⊕)∇(πo⊕πi;πo;π2└)└)└)
proto-hypocation	(√+π└—1π)	(√+π1└)=(πo⊕πe;πe;(πo⊕πi;πo;π1└)└)
proto-cathecation	(√−π└[—1π]∧[1π≤πe])	(√−π1└)=(πo⊕πe;πo;(πo;(πo;π1⊕)∇πe└)└).

Preparing for 'proportition' in E-geometry (section 3.3) and line intersection in the Klein model of L-geometry (section 4.7) two functions proto-transition (π/×π;π⌊[[≤1π]]∧[≤2π]]∧[≤3π]) and proto-cisition (π×/π;π⌊1π⊆2π⊆3π) with condition proto-cisity π⊆π⊆π are introduced.



There is no need for perpendiculation due the existence of πi , therefore one can allow for all distances to be zero. The distances o1 and o2 have been put to 4 and 5 , i.e. opposite sides of the proto-line, thus guaranteeing transsection. If one puts them on the same side, to 8 and 5 , it is cissection - if there is intersection at all, as given by cissectivity π;π⧻π;π .

Starting with three points on the nonnegative proto-ray one gets:

π4=(πi;πo⊕πo;π1⌊)
π5=(πo⊕πi;πo;π2⌊)
π6=(!πo;π5⌊)
π7=(π5⊕π6;πo;π3⌊)= ((πo⊕πi;πo;π2⌊)⊕(!πo;(πo⊕πi;πo;π2⌊)⌊);πo;π3⌊)
π8=(πo⊕πi;πo;π1⌊)

proto-transition

(π1/×π2;π3⌊)= (πo;πe⊗π7;π4⌊)=
(πo;πe⊗((πo⊕πi;πo;π2⌊)⊕(!πo;(πo⊕πi;πo;π2⌊)⌊);πo;π3⌊);(πi;πo⊕πo;π1⌊)4⌊)

proto-cisity

[π1⊆π2⊆π3]↔[[[[≤π1]∧[≤π2]]∧[≤π3]]∧[πo;πe⧻(πi;πo⊕π7;π8 ⌊)]]↔
[[[[≤π1]∧[≤π2]]∧[≤π3]]∧
[πo;πe⧻(πi;πo⊕((πo⊕πi;πo;π2⌊)⊕(!πo;(πo⊕πi;πo;π2⌊)⌊);πo;π3⌊);(πo⊕πi;πo;π1⌊) ⌊)]]

proto-cisition

[π1⊆π2⊆π3]→[(π1×/π2;π3⌊)=
(πo;πe⊗π7;π8⌊)=
(πo;πe⊗((πo⊕πi;πo;π2⌊)⊕(!πo;(πo⊕πi;πo;π2⌊)⌊);πo;π3⌊);(πo⊕πi;πo;π1⌊)⌊)

The following definitions reach outside the methods employed so far and they will not be made use of in this chapter. However, as they will be used for some detours in other connection it seems appropriate to introduce them now, without any further, leave alone, deeper discussion.

Find the dyadic points on the proto-line, starting from πe with successive proto-bisection (π/⌊⊆1π) , proto-addition (π+π⌊[⊆1π]∧[⊆2π]) and proto-negativation (−π⌊⊆1π) . By this **recursive definition** one gets **proto-dyadicity** π∈ and if combined with cyclity the property **proto-dyadic-cyclicity** ≤π∈<

[π1∈]↔[[π1=πe]∨[∃π2[[π2∈]∧[[[π1=(π2/⌊)]∨[π1=(−π2⌊)]]∨[∃π3[[π3∈]∧[π1=(π2+π3⌊)]]]]]]]

[≤π1∈<]↔[[π1∈]∧[≤π1<]]

2.5 Metering

In section 2.1 congruity- and minority-relations were introduced for pairs of points. They allow for talking about distances and comparing them without actually **metering**[1)] them in the normal fashion.

What is metering? Metering produces a number for a given object so that one can compare it to objects of the same class and that by the number one has a proper ordering method of objects. Metering implies ordering, but one can ask the question the other way round, if there is a metering for a given ordering. In N-geometry there is an ordering for pairs of points by their distance with pair-congruity $\pi;\pi\approx\pi;\pi$ and pair-minority $\pi;\pi\prec\pi;\pi$, however, the abstract calcules of planar geometry contain **no numbers** for metering. Congruity of pairs of points establishes equivalence classes of pairs of points. One can choose certain **representatives** for each class as it was done by defining the proto-ray by $\pi 1\leq$. The points of the proto-ray replace the usual numbers for metering. The so-called proto-functions of section 2.4 demonstrate that the intuitive purpose of metering is taken care of. The proto-pair calibrates the metering and one can thus define:

cali-pair-metrition $(\pi'\pi)$

$(\pi 1'\pi 2)=(\pi o\oplus\pi e;\pi 1;\pi 2\llcorner)$

Cali-coordinates of a point with respect to proto-pair

cali-abscissation	$('\pi)$	**cali-ordination**	(π')	
cali-mabation	$(''\pi)$	**cali-mabotion**	(π'')	metered absolute coordination

$('\pi 1)=(\pi o;\pi e;\pi 1!\llcorner)$
$(\pi 1')=\pi o;\pi i;\pi 1!\llcorner)!$
$(''\pi 1)=(\pi o'(\pi o;\pi e;\pi 1!\llcorner)\llcorner)$
$(\pi 1'')=(\pi o'(\pi o;\pi i;\pi 1!\llcorner)!\llcorner)$

This is the best one can do for metering of distances within abstract calcule pi . But there are other objects in planar geometry that one would like to meter: angles and areas of triangles and of higher polygons. Metering for a class of objects necessitates proper congruity- and minority-relations. They may be obtained by mapping the objects to pairs of points so that the objects inherit their comparison values from congruity and minority of the pairs of points.

This can be done in the simple way that one defines object-congruity and object-minority with the use of pair-congruity and pair-minority. But one can also use the representatives of pairs for a number-like metering, in our case by the points of the proto-ray.

It will turn out that one can define angle-metering that can be done within the abstract calcule of planar N-geometry, as it will be developed in section 2.6 . It will become clear that one e.g. talk about zero, right, halfright, straight and full angle in the usual sense in N-geometry.

For pair-metering the functions proto-addition $(\pi{+}\pi\llcorner)$ and proto-negativation $(-\pi\llcorner)$ (establishing the ordered unlimited so-called lination group along the proto-line) were introduced. For angle-metering proto-angle-addition $(\pi 1^\circ{+}\pi 2\llcorner)$ and proto-angle-negativation $(^\circ{-}\pi\llcorner)$ will be introduced to establish an ordered limited so-called cyclation group on the full-segment from πo to πf , as given by $\leq\pi<$.

Area metering, however, cannot be done in N-geometry. This will become clear in the next two chapters: There are two entirely different concepts of area in E-geometry and L-geometry.

1) the expression 'metering' is chosen rather than 'measuring' as this expression is used in mathematics mostly with reference to subsets, whereas in geometry various kind of animals are getting metered.

2.6 Angle extrafuncta without metering

So far nothing has been said about angles. There is no entity 'angle' in N-geometry. However, one can define functa that refer to two angles to be equal or to be smaller and to classify angles heuristically e.g. as zero, acute, right, obtuse, straight, reflex. There is no full angle, it is the same as the zero angle - you cannot define a full angle by means of a triangle. In planar N-geometry the entities that allow for talking ins some way about angles are triples π1;π2;π3 of points, where the first one denotes the vertex in question, which can be classified with their heuristic angle meaning:

angularity	∠π;π;π——	greater equal zero, less full
tri-angularity	∠π;π;π	not zero, not straight
zero-angularity	π——π;π	zero
straight-angularity	π;π—— ——π	straight
line-angularity	—— ——π;π;π	zero or straight
protivity	(π;π;π	greater zero, less straight
contrivity	)π;π;π	greater straight, less full
lini-protivity	(——π;π;π	greater equal zero, less straight
lini-contrivity	)——π;π;π	greater equal straight, less full
right-angularity	π;π⊥π	right, greater equal zero, less full

The missing definitions will be given in section 2.7 .

absolute-angle-congruity says that two triangles have the 'same angle' at a vertex, **absolute-angle-minority** says that one triangle has a 'smaller angle' at a vertex than another triangle at one of its vertices. The comparison of angles is straightforward by comparison of distances. Notice that when talking about absolute angles no sense of angle is considered, meaning that π1;π2;π3≈π1;π3;π2 .

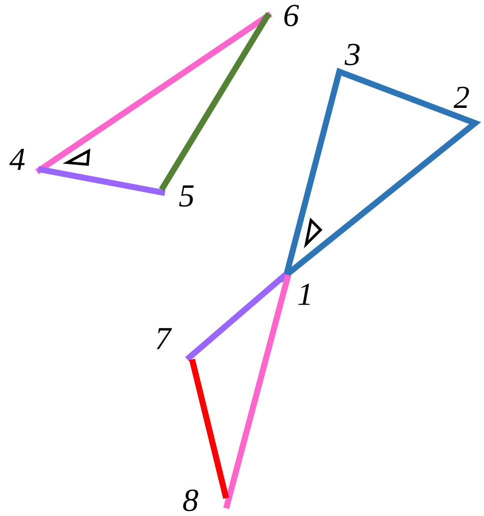
absolute-angle-congruity **absolute-angle-minority**

π;π;π≈π;π;π   π;π;π≺π;π;π

π7=(π2;π1⊕π4;π5⌊)
π8=(π3;π1⊕π6;π4⌊)

[π1;π2;π3≈π4;π5;π6]↔   *zero and straight angle inclusive*
[[[[[π1≠π2]∧[π1≠π3]]∧[π4≠π5]]∧[π4≠π6]]∧
[(π2;π1⊕π4;π5⌊);(π3;π1⊕π6;π4⌊)≈π5;π6]]

[π1;π2;π3≺π4;π5;π6]↔
[[[[[π1≠π2]∧[π1≠π3]]∧[π4≠π5]]∧[π4≠π6]]∧
[(π2;π1⊕π4;π5⌊);(π3;π1⊕π6;π4⌊)≺π5;π6]]



There are triples of points that are neither absolute-angle-congruent nor absolute-angle-minor.

¬[π1;π1;π3≈π4;π5;π6]   ¬[π1;π1;π3≺π4;π5;π6]   ¬[π4;π5;π6≺π1;π1;π3]
¬[π1;π2;π1≈π4;π5;π6]   ¬[π1;π2;π1≺π4;π5;π6]   ¬[π4;π5;π6≺π1;π2;π1]

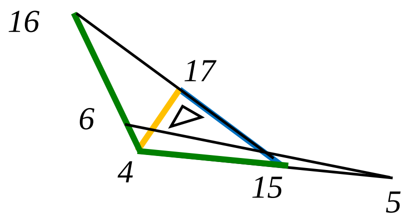
For **absolute-triangle-dichotomition** (⌿π;π;π⌊) one constructs an isosceles triangle and bisects the opposite side and constructs a new isosceles triangle of same sensitivity as the original one.

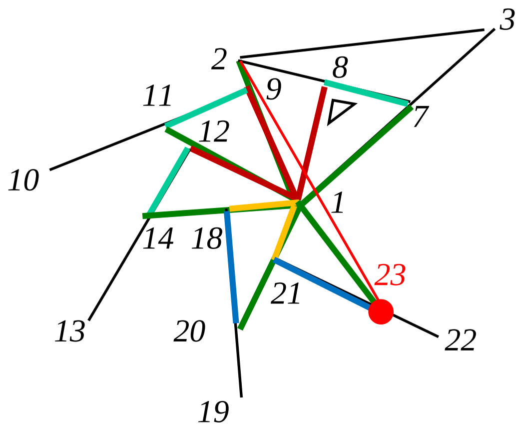
(⌿π1;π2;π3⌊)=(π1⊕(π2⌿(π1⊕π3;π1;π2⌊)⌊);π1;π2⌊)

absolute-triangle-combination defines the construction of a triangle from two triples $\pi_1;\pi_2;\pi_3$ and $\pi_4;\pi_5;\pi_6$ each with a selected vertex π_1 and π_4 resp., so that new triple $\pi_1;\pi_2;\pi_{23}$ has an angle at π_1, that is the sum of the two angles. The condition is tri-angularity. So far the heuristic justification. Notice that the new triangle $\pi_1;\pi_2;\pi_{17}$ is protive for acute angles, but can become contrive for obtuse angles. So far no sense of angles (protrive or contrive) has been introduced. The definition is done using ex- or and implicitly only isoscition, not circulition.

absolute-triangle-combination
$(\pi;\pi;\pi\oplus\pi;\pi;\pi\llcorner[[{}_1\pi\neq{}_2]\pi\wedge[{}_1\pi\neq{}_3\pi]]\wedge[[{}_4\pi\neq{}_5\pi]\wedge[{}_4\pi\neq{}_6\pi]])$

$\pi_7=(\pi_1\oplus\pi_3;\pi_1;\pi_2\llcorner)$
$\pi_8=(\pi_2\not-\pi_7\llcorner)$
$\pi_9=(\pi_1\oplus\pi_2;\pi_1;\pi_8\llcorner)$
$\pi_{10}=(\pi_1;\pi_9!\llcorner)$
$\pi_{11}=(\pi_9\oplus\pi_{10};\pi_7;\pi_8\llcorner)$
$\pi_{12}=(\pi_1\oplus\pi_{11};\pi_1;\pi_8\llcorner)$
$\pi_{13}=(\pi_1;\pi_{12}!\llcorner)$
$\pi_{14}=(\pi_{12}\oplus\pi_{14};\pi_7;\pi_8\llcorner)$ $\pi_1;\pi_2;\pi_{14}$ for first angle
$\pi_{15}=(\pi_4\oplus\pi_5;\pi_1;\pi_2\llcorner)$
$\pi_{16}=(\pi_4\oplus\pi_5;\pi_1;\pi_2\llcorner)$
$\pi_{17}=(\pi_{15}\not-\pi_{16}\llcorner)$
$\pi_{18}=(\pi_1\oplus\pi_{14};\pi_4;\pi_{17}\llcorner)$
$\pi_{19}=(\pi_1;\pi_{18}!\llcorner)$
$\pi_{20}=(\pi_{18}\oplus\pi_{19};\pi_{15};\pi_{17}\llcorner)$
$\pi_{21}=(\pi_1\oplus\pi_{20};\pi_4;\pi_{17}\llcorner)$
$\pi_{22}=(\pi_1;\pi_{21}!\llcorner)$ $\pi_1;\pi_{14};\pi_{23}$ for second angle
$(\pi_1;\pi_2;\pi_3\oplus\pi_4;\pi_5;\pi_6\llcorner)=\pi_{23}=(\pi_{21}\oplus\pi_{22};\pi_{15};\pi_{17}\llcorner)$





π_1 π_2 π_{23} is an isosceles triangle with the combined angle at π_1 . Zero and straight angles are included. The combination of two acute angles gives the same sensitivity as triangle $\pi_1;\pi_2;\pi_3$, the combination of two obtuse angles changes the sensitivity. The somewhat strange construction is chosen so that one starts from a pair of points $\pi_1;\pi_2$ and takes care that all angles are appended in the protive sense.

With the use of circulition absolute-triangle-combination could be expressed much simpler (but the above definition was chosen in foresight of E-geometry where circulition is not basic). The following ***THEOREM*** shows that one does not need the dichotomition trick either as there is a simple way of constructing protive triangles by circulition:

$(\pi_1;\pi_2;\pi_3\oplus\pi_4;\pi_5;\pi_6\llcorner)=$
$(\pi_1;(\pi_1;\pi_2\nabla\pi_3\llcorner)\nabla\pi_2;(\pi_1\oplus(\pi_1;\pi_2\nabla\pi_3\llcorner);(\pi_4\oplus\pi_5;\pi_1;(\pi_1;\pi_2\nabla\pi_3\llcorner)\llcorner);(\pi_4\oplus\pi_6;\pi_1;\pi_2\llcorner)\llcorner)\llcorner)$

as derived by:

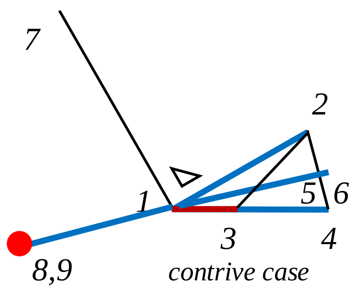
$\pi_7=(\pi_1;\pi_2\nabla\pi_3\llcorner)$
$\pi_8=(\pi_4\oplus\pi_5;\pi_1;\pi_7\llcorner)$
$\pi_9=(\pi_4\oplus\pi_6;\pi_1;\pi_2\llcorner)$
$\pi_{10}=(\pi_1\oplus\pi_7;\pi_8;\pi_9\llcorner)$
$\pi_{11}=(\pi_1;\pi_7\nabla\pi_2;\pi_{10}\llcorner)=(\pi_1;\pi_2;\pi_3\oplus\pi_4;\pi_5;\pi_6\llcorner)$

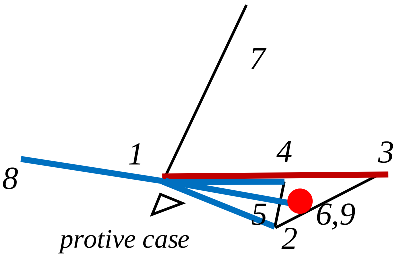
In the following part of this section **sensitivity** of angles and triangles is taken into account. **Triangle-dichotomition** ($\pi/\pi;\pi\llcorner$) is defined in a way so that reflex angles are bisected properly. This is the **first** functum where the sense of a triangle is crucial for the result.

triangle-dichotomition($\pi/\pi;\pi\llcorner[{}_1\pi\neq{}_2\pi]\wedge[{}_1\pi\neq{}_3\pi]$)

$\pi_4=(\pi_1\oplus\pi_3;\pi_1;\pi_2\llcorner)$
$\pi_5=(\pi_2/\pi_4\llcorner)$
$\pi_6=(\pi_1\oplus\pi_5;\pi_1;\pi_2\llcorner)$
$\pi_7=(!\pi_1;\pi_2\llcorner)$
$\pi_8=(\pi_6;\pi_1\oplus\llcorner)$
$\pi_9=(\pi_7\cap\pi_6;\pi_8)$

it is also valid for fringe cases
e.g. π_3 *on* π_4 π_5 *or on* π_1 π_2





Notice that the triangle-dichotomition point does not lie between the two legs of a contrive triangle, but opposite.

If one takes into account the sense of angles one gets **angle-congruity** $\pi;\pi;\pi\approx\angle\pi;\pi;\pi$ and **angle-minority** $\pi;\pi;\pi\prec\angle\pi;\pi;\pi$. One can do this very easily by comparing the bisected angle rather than the angle itself. **Proflex** angles between zero (included) and straight (excluded) belong to protrive are mapped to acute angles (greater equal zero and less right), **reflex** angles between straight (included) and full (excluded) are mapped to obtuse angles (greater equal right and less straight). One then compares the bisected angles and gets:

$[\pi_1;\pi_2;\pi_3\approx\angle\pi_4;\pi_5;\pi_6]\leftrightarrow$
$[[[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]\wedge[[\pi_4\neq\pi_5\wedge[\pi_4\neq\pi_6]]]\wedge[\pi_1;\pi_2;(\pi_1/\pi_2;\pi_3\llcorner)\approx\pi_4;\pi_5;(\pi_4/\pi_5;\pi_6\llcorner)]]$

$[\pi_1;\pi_2;\pi_3\prec\angle\pi_4;\pi_5;\pi_6]\leftrightarrow[[[[\pi_1\neq\pi_2]\wedge[\pi_1\neq\pi_3]]\wedge[[\pi_4\neq\pi_5\wedge[\pi_4\neq\pi_6]]]\wedge[\pi_1;\pi_2;(\pi_1/\pi_2;\pi_3\llcorner)\prec\pi_4;\pi_5;(\pi_4/\pi_5;\pi_6\llcorner)]]$

Now everything is prepared to introduce **segment-rotation**. Segment rotation rotates a pair of points $\pi_1;\pi_2$ by a given angle around π_1, where the sense of rotation is accounted for. In order to take care of sensitivity properly the rotation is constructed by a quadruple rotation with the quarter angle (obtained by angle-quadrisection. i.e. double triangle-dichotomition) , so that only protive triangles are involved, even if one starts with a contrive triangle $\pi_3;\pi_4;\pi_5$ or ends up with a contrive triangle $\pi_1;\pi_2;\pi_{17}$

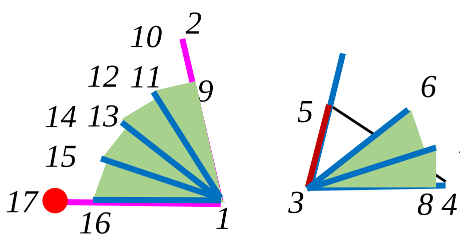
segment-rotation ($\pi;\pi\oplus\Diamond\pi;\pi;\pi\llcorner[[{}_1\pi\neq{}_2\pi]\wedge[{}_3\pi\neq{}_4\pi]]\wedge[{}_3\pi\neq{}_5\pi]\llcorner$)

$\pi_6=(\pi_3/\pi_4;\pi_5\llcorner)$ triangle-dichotomition
$\pi_7=(\pi_3/\pi_4;\pi_6\llcorner)$ angle-quadrisection
$\pi_8=(\pi_3;\pi_4!\pi_7\llcorner)$ protive right-triangle $\pi_3;\pi_8;\pi_7$
ready for 4 applications

4 rotations at $\pi_1;\pi_2$

$\pi_9=(\pi_1\oplus\pi_2;\pi_3;\pi_8\llcorner)$ $\pi_{10}=(\pi_1!\pi_9;\pi_7;\pi_8\llcorner)$
$\pi_{11}=(\pi_1\oplus\pi_{10};\pi_3;\pi_8\llcorner)$ $\pi_{12}=(\pi_1!\pi_{11};\pi_7;\pi_8\llcorner)$
$\pi_{13}=(\pi_1\oplus\pi_{12};\pi_3;\pi_8\llcorner)$ $\pi_{14}=(\pi_1!\pi_{13};\pi_7;\pi_8\llcorner)$
$\pi_{15}=(\pi_1\oplus\pi_{14};\pi_3;\pi_8\llcorner)$ $\pi_{16}=(\pi_1!\pi_{15};\pi_7;\pi_8\llcorner)$

$(\pi_1;\pi_2\oplus\Diamond\pi_3;\pi_4;\pi_5\llcorner)=$ $\pi_{17}=(\pi_1\oplus\pi_{16};\pi_1;\pi_2\llcorner)$



Center-rotation(π⊕◊π;π;π⌊[[1π≠2π]∧[2π≠3π]]∧[2π≠4π]) is the special case where the start π2 of the pair π2;π1 coincides with the rotation point of the triangle π2;π3;π4 . By rotating π1 around π2; is a bijective mapping of points π1 . Center-rotation is a rigid motion (besides translation and mirroring) [1) .

(π1⊕◊π2;π3;π4⌊)=(π2;π1⊕◊π2;π3;π4⌊)

angle-congruity π;π;π≈∠π;π;π
[π1;π2;π3≈∠π4;π5;π6]↔[π1;(π1⊬π2;π3⌊)≈π6;((π4⊕π5;π1;π3⌊)⊬(π4⊕π6;π1;π3⌊)⌊)]

angle-minority π;π;π⊰∠π;π;π
[π1;π2;π3⊰∠π4;π5;π6]↔[π1;(π1⊬π2;π3⌊)⊰π6;((π4⊕π5;π1;π3⌊)⊬(π4⊕π6;π1;π3⌊)⌊)

One can define **triangle-combination** (π;π;π⊕◊π;π;π⌊[[1π≠2π]∧[1π≠3π]]∧[[4π≠5π]∧[4π≠6π]]) that takes into account the sensitivity of triangles. One does a rotation of π1;π3 by the angle at π4 (automatically in the proper sense), followed by adjusting the line so that an isosceles triangle is formed with π1;π2 .

(π1;π2;π3⊕◊π4;π5;π6⌊)=(π1⊕(π1;π3⊕◊π4;π5;π6⌊);π1;π2⌊)

This point and the base line π1;π2 are a triangle with the combined angles, there are 8 possibilities:

π1;π2;π3	π4;π5;π6	π1;π2;(π1;π2;π3⊕◊4;π5;π6⌊)	
- protive	protive	protive or contrive	e.g. two obtuse protive triangles give a contrive e.g. two acute protive triangles give a protive
- protive	contrive	protive or contrive	
- contrive	protive	protive or contrive	
- contrive	contrive	protive or contrive	

One has **triangle-doublition** (π;π;π⊕◊⌊[1π≠2π]∧[1π≠3π]) as

(π1;π2;π3⊕◊⌊)=(π1;π2;π3⊕◊π1;π2;π3⌊) which conserves the orientation

One could define **angle-metrition** (π;π;π⊕◊π;π⌊[[1π≠2π]∧[1π≠3π]]∧[4π≠5π]) of an angle at 1π for triangle 1π; 2π; 3π with respect to a pair 4π;5π that would correspond to the straight angle but in section 2.7 it is done with respect to the proto-pair πo;πe that is a reasonable choice to represent the straight angle, so angle-metrition is skipped.

In section 2.7 functions are introduced that refer to the proto-pair πo , πe . Having a unit length and a right angle one immediately thinks of *sine* and *cosine* functions. Therefore the two functions **cali-sinition** (◊′π;π;π⌊[1π≠2π]∧[1π≠3π]) and **cali-cosinition** (′◊π;π;π⌊) are defined:

(′◊π1;π2;π3⌊)=(π1;π3;(π1⊕π2;πo;πe⌊)!⌊) gives distance *cosine* from πo
(◊′π1;π2;π3⌊)=(πo⊕π2;πe(π1;π3;(π1⊕π2;πo;πe⌊)!⌊) gives distance *sine* from πo

Cali-sinition and cali-cosinition corresponds to *sine* and *cosine* , one could compare angles by them. However, one has to be careful when angles between straight and full appear. Therefore one chooses another function.

[1) One has to check for the fringe case of the fixed center, i.e. π2=π1 , but this is not essential.

2.7 Calibrated-angle and proto-angle extrafuncta

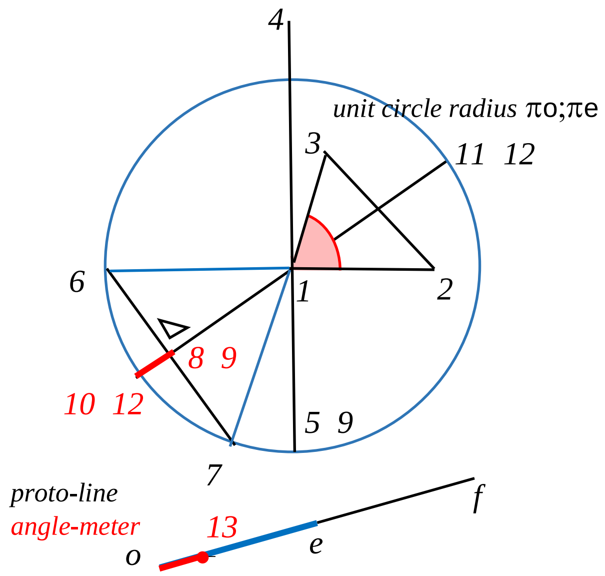
So far angle-congruity and -minority, triangle-combination and -dichotomition was based on the use of certain pairs of points that were constructed in every case with reference to the given points. One can do better in N-geometry by what was called representatives (of equivalence classes) in section 2.5 by so-called cali-pair-metrition (π′π⌊) that uses the proto-pair πo and πe along the proto-line.

A ternary function **cali-angle-metrition** (◊π;π;π′⌊) is introduced, that corresponds to the use of *cosine* of the half angle by reference to the unit distance. Don't worry, this does not mean that anything new is involved, no numbers are necessary and even less, no so-called 'real functions' and 'real numbers'.

(◊π;π;π′⌊[1π≠2π]∧[1π≠3π])

π4=(!π1;π2⌊)
π5=(π4;π1⊕πo;πe⌊)
π6=(π2;π1⊕πo;πe⌊)
π7=(π3;π1⊕πo;πe⌊)
π8=(π6/π7)
π9=(π8;π1↓π5↓(π1⊕π8;πo;πe⌊))
π10=(π1⊕π9;πo;πe⌊)
π11=(π9;π1⊕πo;πe⌊)
π12=(π10∪π11;π4)   chooses the greater distance

(◊π1;π2;π3′⌊)=π13=(πo⊕πe;π8;π12⌊)



Given three points π1 π2 π3 one can geometrically construct a point π13, whose distance to point πo can be taken as a convenient meter for the angle at π1 in a triangle π1 π2 π3 relative to the proto-pair of points. All the steps can be performed by successive applications of functions of abstract calcule pi . But be careful and don't let yourself confuse by the sketch that looks very Euclidean but is only done for for plausibility: the circle looks like a circle, dichotomition of angle looks familiar too and pair-congruity looks
like meaning the same Euclid distance. It is just a heuristic way to describe what is precisely expressed in the abstract calcule pi as construction of cali-angle-metrition. Cali-angle-metrition can be used both in E- and L-geometry. Cali-angle-metrition meters all angles from zero, along right, straight, up to but not including the full angle (what is called the **proto-cycle-segment**) If one had chosen the *sine* of the quarter-angle for metering angles as was done implicitly in triangle-combination the corresponding distances between πo;πo for angle *zero* and πo;πe for angle *one* would be less intuitive.

With cali-angle-metrition one can define the missing relations of angularity already in N-geometry:

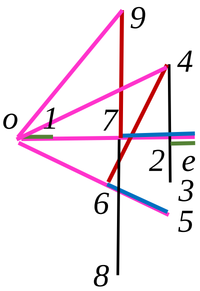
lini-protivity	[(—π1;π2;π3)↔[(π1′π2◊π3⌊)<πe]
lini-contrivity	[⌡—π1;π2;π3)↔[[πe≤(π1′π2◊π3⌊)]∧[(π1′π2◊π3⌊)<πf]]
halfright-angularity	[Hra(π1;π2;π3)]↔[(π1′π2◊π3⌊)=πw]
acute-angularity	[Acaπ1;π2;π3)]↔[(π1′π2◊π3⌊)<πr]
obtuse-angularity	[Ota(π1;π2;π3)]↔[[πr≤(π1′π2◊π3⌊)]∧[(π1′π2◊π3⌊)<πe]]
oblique-angularity	[Ola(π1;π2;π3)]↔[[[(π1′π2◊π3⌊)≠πo]∧[(π1′π2◊π3⌊)≠πr]]∧[(π1′π2◊π3⌊)<πe]]

With cali-angle-metrition one can express **angle-congruity** and **angle-minority** by comparing the angle-meters given as distances of the metrition point from the origin point, i.e. by equality and proto-minority

THEOREM strings:	angle-congruity	[π1;π2;π3≈∠π4;π5;π6]↔[(◊π1;π2;π3′⎿)=(◊π4;π5;π6′⎿)]
	angle-minority	[π1;π2;π3⦃∠π4;π5;π6]↔[(◊π1;π2;π3′⎿)<(◊π4;π5;π6′⎿)]

Metrition determines a metering distance for an angle given by a triangle π1;π2;π3 . One can do it the other was round as well and construct a **proto-triangle** (isosceles with proto-legs) πo;πe;π8 for point π1 on the proto-segment. The function (◊π′⎿≤1π<) is called **cali-meter-angulation** : It is constricted from **cali-pro-angulation** (′◊π⎿≤1π<) or **cali-contra-angulation** (◊′π⎿≤1π<) .

π2=(πe⊕πo;πo;π1⎿)	
π3=(πe;π2↓πa↓(πe;π2!⎿)	*above for zero*
π4=(πo;πe®⊕π2;π3⎿)=(′◊π1⎿)	
π5=(π4;π2⊕)=(◊′π1⎿)	*4 5 same direction as 2 3*
π6=(πo;π5;π4!)	
π7=(πe⊕πo;π5;π6⎿)	
π8=(π7;πo↓πl↓(!π7;πo⎿))	*low for o = 7*
π9=(π8;π7⊕π6;π4⎿))	



There is the lovely result for cali-meter-angulation, that holds for angles greater than straight ones and zero as well:

(◊π1′⎿≤1π<)=π9=

(((πe⊕πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿)!)⎿);πo↓πl↓(!(πe⊕πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿)!)⎿);πo⎿));(πe⊕πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿)!)⎿)⊕(πo;((πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿);(πe⊕πo;πo;π1⎿)⊕);(πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿)!);(πo;πe®⊕(πe⊕πo;πo;π1⎿);(πe;(πe⊕πo;πo;π1⎿)↓πa↓(πe;(πe⊕πo;πo;π1⎿)!⎿)⎿)⎿)

Another idea is to investigate the manipulation of angles with respect to their angle-meters, obtaining some proto-functions. It is getting a bit tricky due to the cycle-character of angles, restricted to the cycle-segment by ≤π< . One has to start with **proto-angle-bisection** (π°/⎿≤1π<)

π2=(′◊π1⎿)	angle by cali-pro-angulation: proto-triangle πo;πe;π2 with half angle
π3=(πo∤π2)	dichotomition for half-angle proto-triangle πo;πe;π3
(π1°/⎿)=π4=(◊πo;πe;π3′⎿)	cali-angle-metrition thereof

The next is **proto-angle-duplication** (π°+⎿≤1π<)

π2=(◊π1′⎿)	angle by cali-meter-angulation
π3=(πo;πe;π2⊕◊⎿)	triangle-doublition
(π1/+⎿)=(◊πo;πe;π3′⎿)	cali-angle-metrition thereof

One defines **proto-angle-addition** $(\pi_1{}^\circ{+}\pi_2\llcorner[\leq_1\pi<]\wedge[\leq_2\pi<])$ using the following trick, dichotomize the angles first and double the result

$\pi_3=(\prime\Diamond\pi_1\llcorner)$	first half angle by cali-pro-angulation: proto-triangle
$\pi_4=(\prime\Diamond\pi_2\llcorner)$	second half angle by cali-meter-angulation: proto-triangle
$\pi_5=(\pi o;\pi e;\pi_3\oplus\Diamond\pi o;\pi e;\pi_4\llcorner)$	triangle-combination
$\pi_6=(\Diamond\pi o;\pi e;\pi_5\prime\llcorner)$	cali-angle-metrition thereof
$\pi_7=(\pi_6\oplus\llcorner)$	proto-angle-duplication
$(\pi_1{}^\circ{+}\pi_2\llcorner)=(\Diamond\pi o;\pi e;\pi_7\prime\llcorner)$	cali-angle-metrition thereof

proto-angle-negativation $({}^\circ{-}\pi_1\llcorner\leq_1\pi<)$ makes use of negativation

$\pi_2=(\prime\Diamond\pi_1\llcorner)$	angle by cali-pro-angulation: proto-triangle $\pi o;\pi e;\pi_2$ with half angle
$\pi_3=(\pi o\div\pi e;\pi_2\llcorner)$	reflection
$({}^\circ{-}\pi_1\llcorner)=(\Diamond\pi o;\pi e;\pi_3\prime\llcorner)$	cali-angle-metrition thereof

Finally **proto-angle-subtraction** $(\pi^\circ{-}\pi\llcorner[\leq_1\pi<]\wedge[\leq_2\pi<])$ is given by $(\pi_1{}^\circ{-}\pi_2\llcorner)=(\pi_1{}^\circ{+}({}^\circ{-}\pi_2\llcorner)\llcorner)$

THEOREM proto-angle-negativation and proto-angle-negativation $({}^\circ{-}\pi_1\llcorner)=(\pi o;\pi_1\downarrow\pi o\downarrow(\pi f-\pi_1\llcorner))$

Proto-angle-functions are defined on the proto-cycle-segment, with their values on the proto-cycle-segment too The following cyclation ***THEOREM*** strings justify the naming of the above proto-angle-functa as there is a locally ordered group:

	<u>cyclative group</u>	
TPC1	conclusity	$\forall\pi_1[\forall\pi_2[[[\leq\pi_1<]\wedge[\leq\pi_2<]]\rightarrow[\leq(\pi_1{}^\circ{+}\pi_2\llcorner)<]]]$
TPC2	neutrivity	$\forall\pi_1[[\leq\pi_1<]\rightarrow[(\pi_1{}^\circ{+}\pi o\llcorner)=\pi_1]]$
TPC3	commutativity	$\forall\pi_1[\forall\pi_2[[[\leq\pi_1<]\wedge[\leq\pi_2<]]\rightarrow[(\pi_1{}^\circ{+}\pi_2\llcorner)=(\pi_2{}^\circ{+}\pi_1\llcorner)]]]$
TPC4	associativity	$\forall\pi_1[\forall\pi_2[\forall\pi_3[[[[\leq\pi_1<]\wedge[\leq\pi_2<]]\wedge[\leq\pi_3<]]\rightarrow[(\pi_1{}^\circ{+}(\pi_2{}^\circ{+}\pi_3\llcorner)\llcorner)=((\pi_1{}^\circ{+}\pi_2\llcorner){}^\circ{+}\pi_3\llcorner)]]]$
TPL5	invertivity	$\forall\pi_1[[\leq\pi_1<]\rightarrow[(\pi_1{}^\circ{+}({}^\circ{-}\pi_1\llcorner)=\pi o]]$
TPC6	low monotony	$\forall\pi_1[\forall\pi_2[[[[[\leq\pi_1]\wedge[\pi_1<\pi e]]\wedge[<\pi_2]]\wedge[\pi_2<\pi e]]\rightarrow[\pi_1<(\pi_1{}^\circ{+}\pi_2\llcorner)]]]$

The important fact is that one can do everything with respect to angles formally in N-geometry, there is congruity and minority of angles, there are acute, right, obtuse and straight angles, one can combine angles and one can bisect angles, one can meter them by distances with respect to a proto-distance. And all is possible without referring to a parallel Axiom and without any kind of **numbers** in calcule <u>pi</u> .

One can introduce calibrated versions of opposition $(\pi\div\pi;\pi)$ and center-rotation $(\pi\oplus\Diamond\pi;\pi;\pi)$.

cali-rotation	$(\prime\pi\oplus\Diamond\pi)$	$(\pi_1\oplus\Diamond\pi_2)=(\pi_1\oplus\Diamond\pi o;\pi e;\pi_2\llcorner)$	rotation of π_1 around πo by angle of triangle $\pi o;\pi e;\pi_2$ at πo (essentially one parameter as only direction of π_2 is relevant)
cali-opposition	$(\prime\pi\div\pi)$	$(\prime\pi_1\div\pi_2)=(\pi o\div\pi_1;\pi_2\llcorner)$	sort of translation of π_1 by $\pi o;\pi_2$ (essentially two parameters)
cali-reflection	$(\prime\div\pi)$	$(\prime\div\pi_1)=(\pi o;\pi e\div\pi_1\llcorner)$	mirroring with respect to proto-line (no parameter involved)

Cali-reflection of a triangle flips between protive and contrive triangles.

2.8 Triangle construction, congruity and other ***THEOREM*** strings

Many ***THEOREM*** strings of E-geometry already can be proven in N-geometry. Among them are triangle constructions that lead to congruity ***THEOREM*** strings and special pairs of points for triangles that lead to unique intersection ***THEOREM*** strings. Let us start with ***Syniom*** strings for triangulation functions:

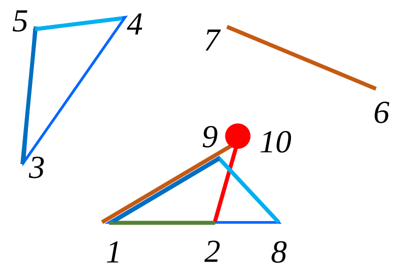
side-side-side-triangulation $(\pi;\pi\nabla\pi;\pi\nabla\pi;\pi\llcorner 1\pi;2\pi\therefore(2\pi;1\pi\oplus 3\pi;4\pi\llcorner);(1\pi;2\pi\oplus 5\pi;6\pi\llcorner))$

$(\pi1;\pi2\nabla\pi3;\pi4\nabla 5;\pi6\llcorner)=(\pi1;\pi2\nabla(\pi2;\pi1\oplus\pi3;\pi4\llcorner);(\pi1;\pi2\oplus\pi5;\pi6\llcorner)\llcorner)$

This is the only instance in this chapter where isoscition is not sufficient and circulition is necessary.

triangulation with two sides and one angle

side-angle-side-triangulation
$(\pi;\pi\nabla\pi;\pi;\pi\nabla\pi;\pi\llcorner[[1\pi\neq 2\pi]\wedge[\angle 3\pi;4\pi;5\pi]]\wedge[6\pi\neq 7\pi])$



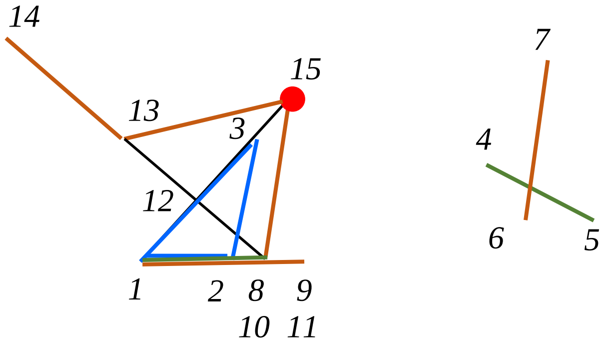
$\pi8=(\pi1\oplus\pi2;\pi3;\pi4\llcorner)$
$\pi9=(\pi1;\pi8\nabla\pi3;\pi5\nabla\pi4;\pi5\llcorner)$
$\pi10=(\pi1\oplus\pi9;\pi6;\pi7\llcorner)=(\pi1;\pi2\nabla\pi3;\pi4;\pi5\nabla\pi6;\pi7\llcorner)$

$(\pi1;\pi2\nabla\pi3;\pi4;\pi5\nabla\pi6;\pi7\llcorner\llcorner)=$
$(\pi1\oplus(\pi1;(\pi1\oplus\pi2;\pi3;\pi4\llcorner)\nabla\pi3;\pi5\nabla\pi4;\pi5\llcorner);\pi6;\pi7\llcorner)$

angle-side-side-triangulation

$(\pi;\pi;\pi\nabla\pi;\pi\nabla\pi;\pi\llcorner[[\angle 1\pi;2\pi;3\pi]\wedge[4\pi\neq 5\pi]]\wedge[6\pi\neq 7\pi]$

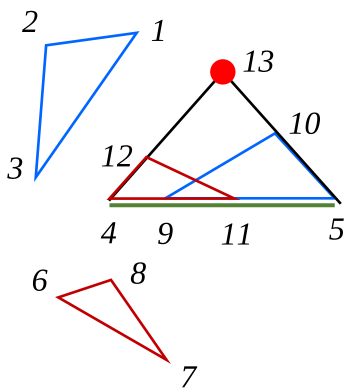
triangle $\pi1$ $\pi10$ $\pi15$



$\pi8=(\pi1\oplus\pi2;\pi4;\pi5\llcorner)$
$\pi9=(\pi1\oplus\pi2;\pi6;\pi7\llcorner)$
$\pi10=(\pi\cap\pi8;\pi9)$
$\pi11=(\pi\cup\pi8;\pi9)$
$\pi12=(\pi1;\pi3!\pi10\llcorner)$
$\pi13=(\pi10;\pi12\oplus)$
$\pi14=(\pi10;\pi13\oplus\pi1;\pi9\llcorner)$
$\pi15=(\pi13;\pi10\nabla\pi14\llcorner)$

triangulation with one side and two angles

angle-side-angle-triangulation
$(\pi;\pi;\pi\nabla\pi;\pi\nabla\pi;\pi;\pi\llcorner[[[[\angle 1\pi;2\pi;3\pi]\wedge[4\pi\neq 5\pi]]\wedge[\angle 6\pi;7\pi;8\pi]]$
$\wedge[$ *intersection condition*$])$



$\pi9=(\pi5\oplus\pi4;\pi1;\pi3\llcorner)$
$\pi10=(\pi9;\pi5\nabla\pi2;\pi3\nabla\pi2;\pi2\llcorner)$
$\pi11=(\pi4\oplus\pi5;\pi6;\pi7\llcorner)$
$\pi12=(\pi4;\pi11\nabla\pi6;\pi8\nabla\pi7;\pi8\llcorner)$
$\pi13=(\pi4;\pi12\otimes\pi5;\pi10\llcorner\pi4;\pi12\nmid\pi5;\pi10\llcorner)$
$=(\pi1;\pi2;\pi3\nabla\pi4;\pi5\nabla\pi6;\pi7;\pi8\llcorner)$

result: point 13 forms triangle with line 4 5

$(\pi1;\pi2;\pi3\nabla\pi4;\pi5\nabla\pi6;\pi7;\pi8\llcorner)=$
$(\pi4;(\pi4;(\pi4\oplus\pi5;\pi6;\pi7\llcorner)\nabla\pi6;\pi8\nabla\pi7;\pi8\llcorner)\otimes\pi5;((\pi5\oplus\pi4;\pi1;\pi3\llcorner);\pi5\nabla\pi2;\pi3\nabla\pi2;\pi2\llcorner)\llcorner)$

side-angle-angle-triangulation
~~(π;π;π∇π;π;π∇π;π⎿[[[1π≠2π]∧[∠3π;4π;5π]]∧[∠6π;7π;8π]]∧[*angle-sum less straight*])~~

It cannot be constructed in N-geometry, however it can be done in both E- and L-geometry. It is simple in E-geometry, as it is replaceable by angle side-angle-triangulation due to straight angle sum. In L-geometry it is more complicated, including determinations of horo-angle and horo-distance that have no counterpart in E-geometry.

Triangulation with three angles
angle-angle-angle-angulation
~~(π;π;π∇π;π;π∇π;π;π⎿[[[∠1π;2π3π]∧[∠4π;5π;6π]]∧[∠7π;8π;9π]]∧[*angle-sum less straight*])~~

It cannot be constructed in N-geometry and E-geometry, however, it will turn out to be possible in L-geometry.

Triangle-congruity ***THEOREM*** strings :

∀π1[∀π2[∀π3[[[π1;π2;π3≅π2;π1;π3]∧[π1;π2;π3≅π2;π1;π3]]]]] *independent of sense and ordering*

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[∀π7[∀π8[[[[*circulity holds for congruent sides*
[π1;π2∴π3;π4]∧[π1;π2≈π5;π6]]∧[π1;π3≈π5;π7]]∧[π2;π4≈π6;π8]]→[π5;π6∴π7;π8]]]]]]]]]

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[∀π7[∀π8[[[[
[π1;π2∴π3;π4]∧[π1;π2≈π5;π6]]∧[π1;π3≈π5;π7]]∧[π2;π4≈π6;π8]]→ *circulity and congruent sides produce*
[π1;π2;(π1;π2∇π3;π4⎿)≅π5;π6;(π5;π6∇π7;π8⎿)]]]]]]]]] *congruent triangles*

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[π1;π2∴ *circulition produces triangle*
[(π2;π1⊕π3;π4⎿);(π1;π2⊕π5;π6⎿)]]→[∠π1;π2;(π1;π2∇(π2;π1⊕π3;π4⎿);(π1;π2⊕π5;π6⎿)⎿)]]]]]]]

∀π1[∀π2[∀π3[∀π4[[π1;π2∴π3;π4]→[∠π1;π2;(π1;π2∇π3;π4⎿)]]]]] *circulity implies circulition tri-angularity*

congruity ***sentence*** 2a sas (*two congruent sides, enclosed congruent angles)*

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[[[[[[[∠π1;π2;π3]∧[∠π4;π5;π6]]∧
[π1;π2≈π4;π5]]∧[π1;π3≈π4;π6]]∧[π1;π2;π3≈π4;π5;π6]]→[π1;π2;π3≅π4;π5;π6]]]]]]

congruity ***sentence*** 2b ssa *(two congruent sides, opposite congruent angles, pro case)*

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[[[[[[[∠π1;π2;π3]∧[∠π4;π5;π6]]∧
[π1;π2≈π4;π5]]∧[π1;π3≈π4;π6]]∧[π2;π3;π1≈π5;π6;π4]]→[π1;π2;π3≅π4;π5;π6]]]]]]

congruity ***sentence*** 3a asa (*one congruent side, two adjacent congruent angles)*

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[[[[[[[∠π1;π2;π3]∧[∠π4;π5;π6]]∧
[π1;π2≈π4;π5]]∧[π1;π2;π3≈π4;π5;π6]]∧[π3;π1;π2≈π6;π4;π5]]→[π1;π2;π3≅π4;π5;π6]]]]]]

congruity ***sentence*** 3b saa (*one congruent side, one adjacent and one opposite congruent angles)*

∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[[[[[[[∠π1;π2;π3]∧[∠π4;π5;π6]]∧
[π1;π2≈π4;π5]]∧[π1;π2;π3≈π4;π5;π6]]∧[π2;π3;π1≈π5;π6;π4]]→[π1;π2;π3≅π4;π5;π6]]]]]

Same orientation of triangles
∀π1[∀π2[∀π3[∀π4[∀π5[∀π6[[[∠π1;π2;π3]∧[∠π4;π5;π6]]→
[[[[⎛π1;π2;π3]∧[⎛π4;π5;π6]]∨[[⎠π1;π2;π3]∧[⎠π4;π5;π6]]]↔
[π3 —(π1;π2∇π3;((π1;π2⊕π4;π5⎿);π2∇π4;π6∇π5;π6⎿)⎿)—((π1;π2⊕π4;π5⎿);π2∇π4;π6∇π5;π6⎿)]]]]]]]]

Triangular intersection ***THEOREM*** strings :

angle-bisector-intersection *center inscribed circle*

∀π1[∀π2[∀π3[[∠π1;π2;π3]→[(π1;(π1⁄π2;π3└)⊗π2;(π2⁄π3;π1└)└)=(π1;(π1⁄π2;π3└)⊗π3;(π3⁄π1;π2└)└)]]]]

In a corresponding fashion, *just too lazy to do it*:
perpendicular-bisector-intersection *center circumscribed circle*
altitude-intersection
median-intersection *center centroid*

'1st Legendre' ***THEOREM*** : absolute-triangle-combination of triangle angles less than or equal straight:
∀π1[∀π2[∀π3[[∠π1;π2;π3]→[⎛π1;π2;(π1;π2;(π1;π2;π3⊕π2;π3;π1└)⊕π3;π2;π1└)]
∨[π2— π1—(π1;π2;(π1;π2;π3⊕π2;π3;π1└)⊕π3;π2;π1└)]]]]

'2nd Legendre' ***THEOREM*** : straight angle-sum in one triangle implies straightness in all triangles

A final observation: **point sets** are given as relations with one argument exposed to the other ones: e.g.
perpendicular-equidistancy π;π;π≈≈π *defines equidistant-line relative* 4π . Some more:

straight-line-particity	π3	STL(π;π;π)	[π1≠π2]∧[—π1;π2;π3]
circle-line-particity	π3	CIL(π;π;π)	[π1≠π2]∧[π1;π3≈π1;π2]
ellipse-line-particity	π4	ELL(π;π;π;π)	[π1 —π2—π3]∧[π1;(π1;π4⊕π2;π4└)≈π1;(π1;π3⊕π2;π3└)]
hyperbola-line-particity	π4	HYL(π;π;π;π)	[π1 —π2—π3]∧[π1;(π4⊕π1;π2;π4└)≈π1;(π3⊕π1;π2;π3└)]
circle-area-particity	π4	CIA(π;π;π)	[π1≠π2]∧[π1;π3⦔ π1;π2]
ellipse-area-particity	π4	ELA(π;π;π;π)	[π1 —π2—π3]∧[π1;(π1;π3⊕π2;π3└)⦔ π1;(π1;π4⊕π2;π4└)]
hyperbola-area-particity	π4	HYA(π;π;π;π)	[π1 —π2—π3]∧[π1;(π4⊕π1;π2;π4└)⦔ π1;(π3⊕π1;π2;π3└)]

What **one cannot do in abstract calcule** pi is talking about **areas** in any way (no shear-mapping, no cathetus theorem, no Pythagoras theorem, no area-equality, no area-minority, no relative area-metering, no area-combination of triangles and so on).

2.9 Symmetry transformations

Given cali-rotation (′π⊕◊π) , cali-opposition (′π÷π) and cali-reflection (′÷π) of section 2.7 one can ask if one can combine them to produce cali-motions. Certainly one can do it, it just remains questionable to what effort. It would be nice to obtain a group of symmetry transformations. In N-geometry only cali-rotation and cali-reflection preserve distance-meters, meaning that they induce symmetry transformations.

It will turn out that cali-rotation (′πε⊕◊πε) , cali-opposition (′πε÷πε) and cali-reflection (′÷πε) in calcule piepsilon of E-geometry preserve the distance of segments and angles as well. So by combination one can define function **eu-motion** (′πε⊕◊πε;πε÷πε) . It induces symmetry transformation of E-geometry forming the group of two-dimensional Euclidean-motion.

It will turn out that in calcule pilambda of L-geometry a similar procedure allows for defining **lo-motion** (″πλ⊕◊πλ;πλ÷πλ) corresponding to the group of two-dimensional projective motions.

Deep meaning lies in the fact that calcule piepsilon of E-geometry has the additional **gauge-symmetry** given by **cali-tension** (′πε×πε└<2πε) that has no counterpart in L-geometry. See section 3.5 for its definition and meaning.

3. Planar E-geometry

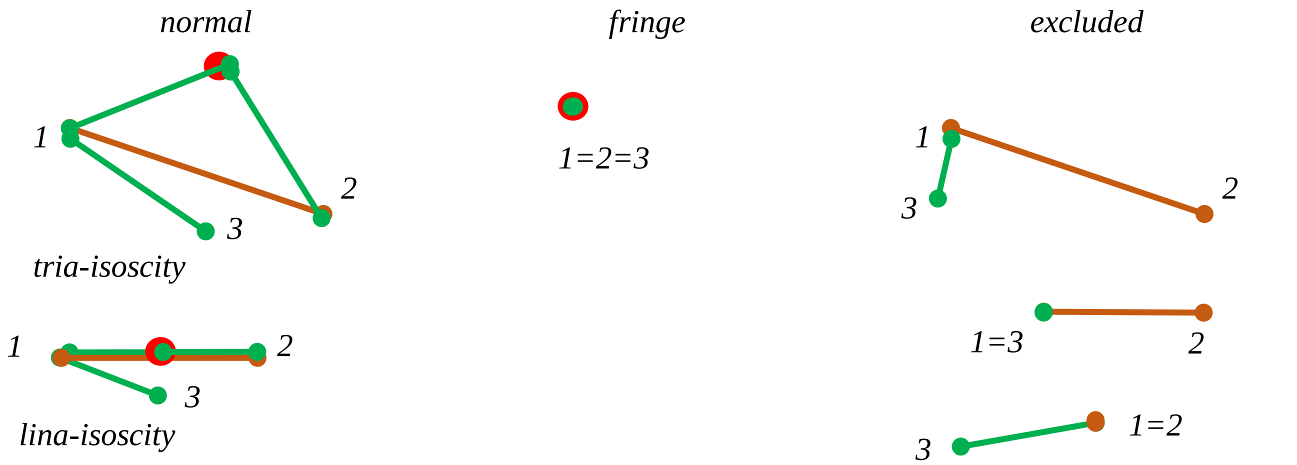
3.1 Ontological basis of calcule piepsilon of planar Euclid E-geometry

Drop circulition (circle-circle) (π;π∇π;π⎿) and circulity π;π… ∴π;π from calcule pi and replace it by **isoscition** which produces isosceles triangles from intersection of circles with equal radii, its condition is **isoscity** π;π… ∴π . It will turn out that one then can define circulition as an ***extra-function-constant***, if one makes use of the unique parallels that is new in abstract calcule pi of planar Euclid E-geometry.

sort	πε	Euclid-point, Евклид-точка
basis-function-constant		*condition*
appension	(πε;πε⊕πε;πε⎿	[1πε≠2πε]∨[3πε=4πε])
isoscition	(πε;πε∇πε⎿	[1πε;2πε… ∴3πε]∨[[1πε=2πε]∧[1πε=3πε]])
linisection	(πε;πε⊗πε;πε⎿	1πε;2πε∤3πε;4πε)

New necessary and convenient ternary ***extra-relation-constant*** strings for ***Axiom*** strings

isoscity	πε;πε… ∴πε
[πε1;πε2… ∴πε3]↔	[[πε1≠πε2]∧[(πε1;((πε2;πε1⊕πε1;πε2⎿);πε1⊕πε1;πε3⎿)⊕πε1;πε3⎿)⊕ (πε1;((πε2;πε1⊕πε1;πε2⎿);πε1⊕πε1;πε3⎿)⊕πε1;πε3⎿);πε2⎿)=πε2]]
lina-isoscity	πε;πε…πε
[πε1;πε2…πε3]↔	[[πε1≠πε2]∧[πε2=(πε1;((πε2;πε1⊕πε1;πε2⎿);πε1⊕πε1;πε3⎿)⊕πε1;πε3⎿)]]
tria-isoscity	πε;πε∴πε
[πε1;πε2∴πε3]↔	[[πε1;πε2… ∴πε3]∧[¬[πε1;πε2…πε3]]]



Possible arrangements for isoscition

extra-relation-constant strings for pair congruity πε;πε≈πε;πε , tri-angularity ∠πε;πε;πε , internity πε—πε—πε and parallelity πε;πε||πε;πε are taken without changes from abstract calcule pi .

Besides the modification of the ingredients for the abstract calcule piepsilon of planar Euclid E-geometry the inclusion of the ***Unique-parallel-axiom*** and the ***Axiom*** mater of inductivity changes the character of the abstract calcule totally. Whereas abstract calcule pi contains ***limbHOOD*** strings, i. e. ***sentence*** strings, that are neither ***TRUTH*** or ***FALSEHOOD*** strings, abstract calcule piepsilon is a **complete** calcule, which means that every ***sentence*** is either a ***TRUTH*** or a ***FALSEHOOD*** string. Another essential difference will be that one can express cissectivity (and thence parallelity) without the use of ***entitor*** character as a mere junctive ***formula*** string, see section 3.3 . By the way, the same will be true in Lobachevsky L-geometry, although with different junctive ***formula*** strings.

3.2 Axioms of planar Euclid E-geometry and inductivity

Firstly planar N-geometry is modified, as circulition can be replaced by weaker isoscition. Therefore ***Axiom*** strings ***A9*** to ***A14*** are replaced by ***A9e*** to ***A13e*** ; a numbering gap ***A14*** occurs on purpose. As mentioned at the end of section 3.1 one could replace nonjunctive cissectivity π;π╫π;π of calcule <u>pi</u> in the ontological basis of calcule <u>piepsilon</u> right from the beginning by junctive eu-cissectivity[1)]┼πε;πε;πε;πε┼ (defined in section 3.3). Secondly the ***Unique-parallel-axiom A21e*** (with character ***e*** for Euclid) is supplemented to N-geometry (and closes the numbering gap of N-geometry).

<u>isoscition</u>

A9e auto-isoscition ∀πε1[(πε1;πε1∇πε1⎿)=πε1]

A10e tria-isoscition congruities ∀πε1[∀πε2[∀πε3[[πε1;πε2∴πε3]→ [[πε1;(πε1;πε2∇πε3⎿)≈πε1;πε3]∧[πε2;(πε1;πε2∇πε3⎿)≈πε1;πε3]]]]]

A11e tria-isoscition uniqueness ∀πε1[∀πε2[∀πε3[∀πε4[[[π1;π2∴π3]∧[π1;π4≈π1;π3]]→ [(π1;π2∇π4⎿)=(π1;π2∇π3⎿)]]]]]

A12e tria-isoscition duplicity of orientation ∀πε1[∀πε2[∀πε3[[πε1;πε2∴πε3]→ [(πε1;πε2∇πε3⎿)≠(πε2;πε1∇(π1;π2⊕π1;π3⎿)⎿)]]]]

A13e lina-isoscition dichotomition ∀πε1[∀πε2[∀πε3[[πε1;πε2…πε3]→ [(πε1;πε2∇πε3⎿)=((πε2;πε1⊕πε1;πε2⎿);πε1⊕πε1;πε3⎿)]]]]

<u>parallelity</u>

A21e unique parallel ∀πε1[∀πε2[∀πε3[∀πε4[[[[π1≠π2]∧[π3≠π4]]∧[πε1;πε2||πε3;πε4]]→ [¬[∃πε5[[πε1;πε2||πε3;πε5]∧[∠πε3;πε4;πε5]]]]]]]]

Furthermore a mater (usually called 'Axiom-scheme' but the expression 'scheme' is preferred to be used otherwise) of ***Axiom*** strings is added for inductivity. Without **mater of inductivity** calcule <u>piepsilon</u> would not be complete. E.g. one could add ***Axiom*** strings ***A26*** that would allow for some or all dividings of angles.

An ***Axiom*** mater is expressed in metalanguage Mencish (see section1.2). Metacalcule ***<u>piepsilon</u>*** (relating to object-calcule <u>piepsilon</u>) contain metastrings that are metaobjects: e.g. the metavariables are written as ***πε1*** ; ***πε2*** etc. . The metafunction ***(πε ; πε∫πε)*** (see section1.2) replaces strings. The metaproperty ***sentence*(πε)** takes care of proper syntax of the sentences of the object-language. But no further details are given; it is treated reference *(10)* .

Axiom <u>mater of inductivity</u>

∀πε1[[[[[sentence(∀πε1[πε1])] ∧ [¬ [πε1 ⊃ πε2]] ∧ [¬ [πε1 ⊃ πε3]]] ∧ [¬ [πε1 ⊃ πε4]]] → [TRUTH([[[(πε1;πε1∫πε0)]∧[(πε1;πε1∫πεe)]]∧[∀πε1[∀πε2[∀πε3[∀πε4[[[[[[[πε1]∧[(πε1;πε1∫πε2)]]∧[(πε1;πε1∫πε3)]]∧[(πε1;πε1∫πε4)]]∧[πε1≠πε2]]→ [(πε1;πε1∫(πε1;πε2⊕πε3;πε4⎿))]]∧ [[[[πε1]∧[(πε1;πε1∫πε2)]]∧[(πε1;πε1∫πε3)]]∧[(πε1;πε1∫πε4)]]∧[[[┼πε1;πε2┼πε3;πε4┼]→ [(πε1;πε1∫(πε1;πε2⊗πε3;πε4⎿))]]]]∧[[[[[πε1]∧[(πε1;πε1∫πε2)]]∧[(πε1;πε1∫πε3)]]∧[πε1;πε2∴πε3]]→ [(πε1;πε1∫(πε1;πε2∇πε3⎿))]]]]]]]→[∀πε1[πε1]])]] (also implying two-dimensionality)

[1)] the prefix 'eu' is chosen for 'Euclid' as 'lo' will be chosen for 'Lobachevsky' - by coincidence Greek ευ means pretty!.

3.3 Fundamental extrafuncta

Parallelity πε;πε||πε;πε and the syn- , anti-, pro- and contra- relations are taken from N-geometry (section 2.3) , remember that they contain an ***entitor*** . One can define corresponding relations in E-geometry **without *entitor***, using parallelograms, e.g. **mono-parallelity** |πε;πε;πε;πε|

mono-parallelity	[|πε1;πε2;πε3;πε4|]↔[[[[πε1≠πε2]∧[πε3≠πε4]]∧[¬[πε1;πε2—πε3;πε4]]]∨ [[πε2;πε3≈(πε1;πε2⊕ε3;πε4└);πε4└)]∧[πε1;πε4≈(πε2;πε1⊕ε3;πε4└);πε3└)]]]

and get ***THEOREM*** parallelity is mono-parallelity in E-geometry.

∀πε1[∀πε2[∀πε3[∀πε4[[πε1;πε2||πε3;πε4]↔[|πε1;πε2;πε3;πε4|]]]]]

The second parallelogram in the definition of mono-parallelity excludes the cases where the line πε3;πε4 crosses πε1;πε2 in a special fashion. Using mono-parallelity instead of regular-parallelity there is **eu-parallelity** |πε;πε—πε;πε| , **eu-cissectivity** ǂπε;πε;πε;πεǂ and **eu-linisectivity** ǂπε;πεǂπε;πεǂ **without *entitor***:

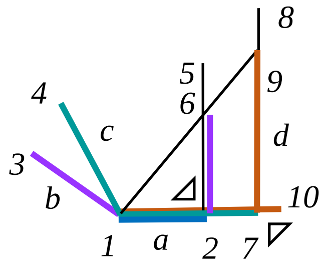
[|πε1;πε2—πε3;πε4|]↔[[|πε1;πε2;πε3;πε4|]∨[πε1;πε2—πε3;πε4]]

[ǂπε1;πε2;πε3;πε4ǂ]↔[¬[[[|πε1;πε2—πε3;πε4|]∨[πε1;πε2ǂπε3;πε4]]∨[πε1;πε2|πε3;πε4]]]

[ǂπε1;πε2ǂπε3;πε4ǂ]↔[[πε1;πε2ǂπε3;πε4]∨[ǂπε1;πε2;πε3;πε4ǂ]]

Notice that the two following constructions is done strictly in calcule piepsilon , syn-perculation is used and could be expanded:(πε1;πε2!└)=(πε1;(πε1;πε2⊕)∇(επ1;πε2⊕)└) . The constructions cannot be done in N-geometry or L-geometry as a linisection is included, that cannot be guaranteed in N-geometry or L-geometry.

proportion
(πε;πε∫πε;πε└1πε≠2πε)



πε5=(πε1;πε2!└)
πε6=(πε2⊕πε5;πε1;πε3└)
πε7=(πε1⊕πε2;πε1;πε4└)

the construction is done such that the fringe case πε1=πε4 is included

πε8=(πε1;πε7↓πε1↓(πε1;πε7!└))
πε9=(πε1;πε7↓πε1↓(πε7;(πε1;πε7!└)⊗πε1;πε6└))
here E-geometry relevant, linisection guaranteed

πε10=(πε1⊕πε2;(πε1⊕πε2;πε7;πε9└)

proportion a/b=c/d d=(bc)/a as distance to πε1

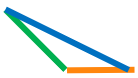
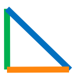
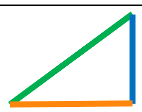
(πε1;πε2∫πε3;πε4└)=πε10=
(πε1⊕πε2;(πε1⊕πε2;(πε1⊕πε2;πε1;πε4└);(πε1;(πε1⊕πε2;πε1;πε4└)↓πε1↓
((πε1⊕πε2;πε1;πε4└);(πε1;(πε1⊕πε2;πε1;πε4└)!└)⊗πε1;(πε2⊕(πε1;πε2!└);πε1;πε3└)└))└)

square-division (πε;πε∫πε└1πε≠2πε)
(πε1;πε2∫πε3└)=(πε1;πε2∫πε2;πε3└)

square division $d=b^2/a$

In N-geometry little use was made of circulition, most functions depended only on isoscition, essentially circulition was only necessary for constructing a triangle from its three sides. Now that there is only isoscition in the ontological basis of abstract calcule piepsilon it is yet to be shown how to do **circulition**.

The idea of the construction comes from the semiconcrete calcule of planar Descartes geometry that is introduced further down in section 3.9, where one observes that for metering distances only so-called **hypothetion**, i.e. Pythagoras-addition-root $sr(x^2+y^2)$ and so-called **cathetetion**, i.e. Pythagoras-subtraction-square-root $sr(x^2-y^2)$ appear beside the four basic arithmetical operations, geometrically speaking everything can be reduced to the use rectangular triangles and Pythagoras ***THEOREM*** . It will be shown that general square roots can be constructed. Farther below in this section the preparation for square root is done. However, in calcule piepsilon there are no numbers for metering. But still one can do the necessary constructions. *Tricky, isn't it.* Use will be made of representing distances by pairs of points. For better understanding only ***basis-function-constant*** strings are used. As one has to take into account fringe cases like a right angle and degenerate collinic case one has to include jections. For treating acute and obtuse angles with one expression the emination and emaxation functions are necessary.

case	*heuristic sketch*	*a*	*b*	*c*	*d^2= a^2+b^2*	*e^2 : c^2-d^2 or d^2-e^2*	*= < >*	*f= $e^2/(2a)$*	*h= $sr(b^2-f^2)$*
1.1.1		*1*	*sr(5)*	*sr(2)*	*6*	*4*	*<*	*2*	*1*
1.1.2		*1*	*sr(2)*	*1*	*5*	*4*	*<*	*2*	*1*
2.1.1		*2*	*sr(2)*	*sr(2)*	*6*	*4*	*<*	*1*	*1*
2.1.2		*2*	*1.5*	*2.5*	*6.25*	*0*	*=*	*0*	*1.5*
2.1.3		*1*	*sr(2)*	*sr(5)*	*3*	*2*	*>*	*1*	*1*
1.2.1		*1*	*2*	*1*	*5*	*4*	*<*	*2*	*0*
1.2.2		*1*	*1*	*0*	*2*	*2*	*<*	*1*	*0*
2.2.1		*2*	*1*	*1*	*5*	*4*	*<*	*1*	*0*
2.2.2		*1*	*0*	*1*	*1*	*0*	*=*	*0*	*0*
2.2.3		*1*	*1*	*2*	*sr(2)*	*2*	*>*	*1*	*0*
2.3		*0*	*0*	*0*	*0*	*0*	*=*	*0*	*0*

Principal arrangements for circulition including collinic cases

An **isoscelator** (see section 1.5) allows only for the same radius of the two circles. Contrary to the usual requirement of freely using two separate roundedges for circle-circle-intersection (circulition) an isoscelator is sufficient for E-geometry: That is why the ***Axiom*** strings ***A9*** to ***A14*** of N-geometry got replaced above.

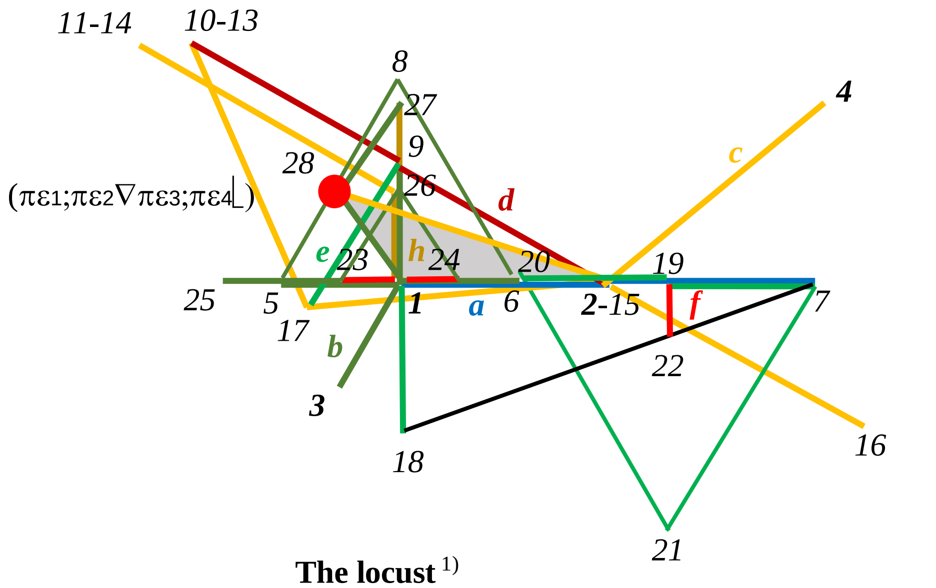
Locust *THEOREM :* circulition i.e. intersection of two circles with the use of an isoscelator (see section 1.5). The function locustition (πε;πε∇πε;πε⌊) for its construction fulfills circulition ***Axiom*** strings ***A9*** to ***A14*** of section 2.2. if circulity applies.

From Heron's triangle formula in E-geometry *(ref. 12)* one gets the $h=sr(b^2-((a^2+b^2-c^2)/(2a))^2)$ as height formula: relative to base line a : The radicand cannot be negative - this is guaranteed by circulity (as defined in section 2.1) . There is case 1 for $a^2+b^2<=c^2$ and case 2 for $a^2+b^2>c^2$. In the preceding table the two cases are specified in detail. The trivial possibility of πε1=πε2=πε3 is met also.

<u>construction of triangle</u>	(πε1;πε2;(πε1;πε2∇πε3;πε4⌊1πε;2πε… ∴3πε;4πε)			
with sides *a*, *b* and *c*	$d^2=a^2+b^2$	$e^2=d^2-c^2$	$f=e^2/2a$	$h^2=b^2-f^2$

one successively prepares for and performs pro-hypothetions, pro-cathetions (see section 2.3) and a square-division

illustrated for the triangular case 1.1.1 where $d^2<c^2$



The locust [1)]

<u>*8 auxiliary triangles*</u>

right for pro-hypothetion	πε1;πε2;πε9		
equilateral for perpendiculation	πε5;πε6;πε8	πε20;πε7;πε21	
isosceles for pro-cathetetion	πε13;πε15;πε17	πε23;πε24;πε6	πε1;πε27;πε28
right for square-division	πε1;πε18;πε7	πε19;πε22;πε7	

πε5=(πε2;πε1⊕πε1;πε3⌊)	*appension*	πε17=(πε13;πε15∇πε16)	*isoscition for e*
πε6=(πε5;πε1⊕)	*pair-doublition*	πε18=(πε8;πε1⊕πε9;πε15⌊)	*preparing for*
πε7=(πε1;πε2⊕)		πε19=(πε7⊕πε1;πε9;πε15⌊)	*square division f*
πε8=(πε5∇πε6)	*equitriangulation*	πε20=(πε7;πε19⊕)	
πε9=(πε1⊕πε8;πε1;πε3⌊)	*dispension for d*	πε21=(πε7∇πε20)	*equitriangulation*
πε10=(πε2;πε9⊕)		πε22=(πε7;πε18⊗πε19;πε21⌊)	*transsection for f*
πε11=(πε2;πε9⊕πε2;πε4⌊)		πε23=(πε2;πε1⊕πε19;πε22⌊)	
πε12=(πε10;πε9⊕)		πε24=(πε23;πε1⊕)	
πε13=(πε10;πε11∩πε9⌊)	*only emination*	πε25=(πε2;πε23⊕πε1;πε3⌊)	
πε14=(πε10;πε11∪πε9⌊)	*only emaxation*	πε26=(πε23;πε24↓πε1↓(πε23;πε24∇πε25⌊))	
πε15=(πε13;πε9⊕)		*isoscition for h , decision for fringe case*	
πε16=(πε9;πε15⊕πε9;πε14)		πε27=(πε1;πε25⊕)	

(πε1;πε2∇πε3;πε4⌊)= *finally with 3 decisions because of fringe cases:*
πε28=(πε1;πε27↓(πε3;πε11↓πε5↓πε6)↓(πε3;πε11↓(πε1;πε27∇πε3⌊))↓((πε1;πε27∇πε3⌊);πε26⊕))

[1)] according to Eva Hutzelmeyer

By successive insertion from πε5 up to πε30 into πε25 one obtains an expression for **locustition** (πε;πε∇πε;πε⎿) with condition ⎿1π;2π…∴3π;4π (it is **not** explicitly developed as it is a very lengthy expansion that fills many pages with many appensions (π;π⊕π;π⎿) , isoscitions (π;π∇π⎿) and decisions (π;π↓π↓π⎿) , but **only one** linisection (π;π ⊗π;π⎿) .

Circulation (π1;π2∇π3;π4⎿) is the **only** basic-function of N-geometry [1)] that is not transferred with the same axiomatic definition to E-geometry. However, there it can be replaced by locustition as an extra-function defined by composition.

*For definition of locustition Heron's triangle **THEOREM**. was used. As an exception, this does not fit into the top-down treatment, as areas only will appear in section 3.4 . Nevertheless it seemed appropriate to put it into this section right after proportition and square division as specific functions of E-geometry. Heron's triangle **THEOREM** could be proven in section 3.4 , but it is left to the reader.*

How about orientation? No problem: protivity and contrivity had been defined with the use of isoscition only in N-geometry, so the concept is valid in E-geometry and L-geometry as well. The same applies for the concept of angles with the angle-congruity, angle-minority, triangle-dichotomition, triangle-combination, rotation and so on. However, there is the one outstanding new feature, that is based on ***Axiom A21e*** 'unique parallels' . In Legendre's first ***THEOREM*** the possibility of angle-sums less than straight has to be dropped:

THEOREM : absolute-triangle-combination of triangle is straight
∀π1[∀π2[∀π3[[∠π1;π2;π3]→[π2—π1—(π1;π2;(π1;π2;π3⊕π2;π3;π1⎿)⊕π3;π2;π1⎿)]]]]

From this one gets an easy for the construction of a triangle of one side and two adjacent angles **side-angle-angle-triangulation** (π;π∇π;π;π∇π;π;π⎿⎛3π;4π;(3π;4π;5π⊕6π;7π;8π⎿)) . Notice that the condition is expressed by protivity.

Due to straight angle sum it can be reduced to angle-side-angle-triangulation (π;π;π∇π;π∇π;π;π⎿) . Do a combination of the two angles and take its difference to the straight angle as the second angle.

(π1;π2∇π3;π4;π5∇π6;π7;π8⎿)= (π3;π4;π5∇π1;π2∇π6;(π7;π6⊕);(π3;π4;π5⊕π6;π7;π8⎿)⎿)

Desargues *THEOREM* 'parallel triangles'

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[[∠πε1;πε2;πε3]∧[∠πε4;πε5;πε6]]→
[[[[πε1;πε2|πε4;πε5]∧[πε2;πε3|πε5;πε6]]∧[πε3;πε1|πε5;πε6]]↔
[[[πε1;πε4|πε2;πε5]∧[πε2;πε5|πε3;πε6]]∨[[[[[πε1;πε4∤πε2;πε5]∧[πε2;πε5∤πε3;πε6]]∧[πε3;πε6∤πε1;πε4]]∧
[(πε1;πε4⊗πε2;πε5⎿)=(πε2;πε5⊗πε3;πε6⎿)]]∧[(πε2;πε5⊗πε3;πε6⎿)=(πε3;πε6⊗πε1;πε4⎿)]]]]]]]]]

Pascal *THEOREM* 'transitivity of pairs of parallels'

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[[[[
[πε1—πε2—πε3]∧[πε4—πε5—πε6]]∧[πε3;πε5||πε6;πε2|]]∧[πε3;πε4||πε6;πε1|]]∧[πε1;πε2∤πε4;πε5]]∧
[[[[(πε1;πε2⊗πε4;πε3⎿)—πε1—πε2]∨[πε2—πε3—[(πε1;πε2⊗πε4;πε3⎿)]]∧
[(πε1;πε2⊗πε4;πε3⎿)—πε4—πε5]]∨[πε5—πε6—[(πε1;πε2⊗πε4;πε3⎿)]]]→[πε1;πε4||πε2;πε3]]]]]]]

<u>Important observation</u>: although this quaternary relation eu-cissectivity is somewhat lengthy when it is expanded in order to express it by appension, isoscition, linisection and decision it serves the desired purpose: there is no ***entitor*** involved, junctive logic only! This means one could start with eu-cissectivity right from the beginning in the ontological base and the ***Axiom*** strings.

[1)] contrary to L-geometry, where circulition is a basic-function as well (section 4.1)

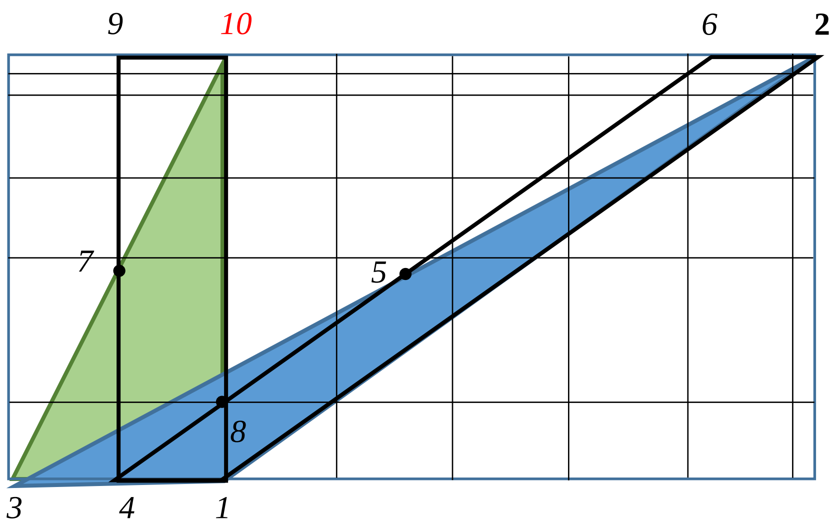
3.4 Aria on area extrafuncta

Infinite straight lines and rays, angles, circles, polygons like triangles or quadrangles and areas of polygons, none of these are part of the ontological basis of geometry of O. If one would enrich the ontological basis by some of them the theory would become even more complicated: straight lines and circles would be relative easy, but already angles would pose some difficulty and what the hack is an 'area' ? One has to realize that the expression 'area' is used in two meanings in everyday English:

- **areal set,** i.e. a set of points that lie within one or more closed boundary curves
- **areal meter**, i.e. number for the size of an areal set

The problem is that there are neither point sets nor numbers in geometry of O calcules, and thus not in piepsilon . The only objects that are available are polygons given as tuples of points. But still one can talk about two polygons have the same size of areas, or that one is smaller than the other one. And this one can do without numbers, but rather by comparing the parts of polygons that one cuts out with scissors, **scissor-congruence** is the expression that heuristically characterizes this method. As all polygons consist of triangles it is sufficient to treat triangles. Triangles can be built up by triangles too. The heuristic idea is that congruent triangles have equal areas and that areas are in some way 'additive'.

What one needs are definitions of area-eqivality of triangles πε;πε;πε≈≈πε;πε;πε and area-minority of triangles πε;πε;πε⌇⌇πε;πε;πε that meet our intuitive understanding of areas. The definitions for polygon-area-eqivality and polygon-area-minority can be given on this basis for all arities. On the way to find the triangle-area-relations one has to use what was called relative metering in section 2.5 where this method was used for angles. Absolute metering will be introduced in section 3.5 where the calcule pi is enriched by some sort of numbers. One starts with shear-mapping of a triangle to a right triangle of equal area by scissor-congruence: the proof for **triangle-rectification** (πε!πε;πε⌊) , however, necessitates Archimedes Axiom.



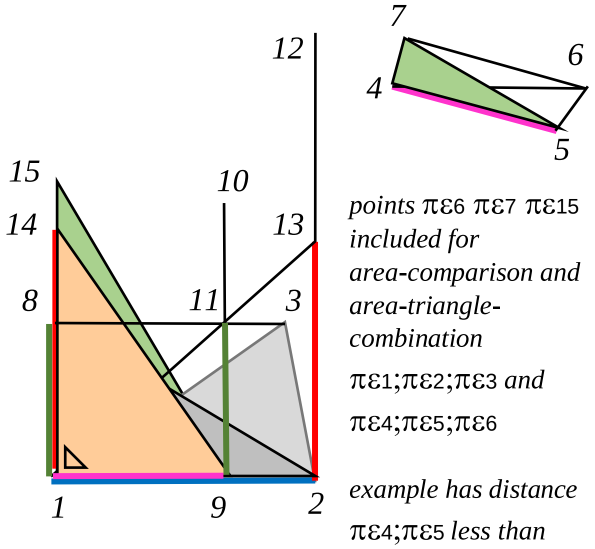
Scissor-congruence buildup of right triangle πε1 πε10 πε3 that is area-equivalent to the original triangle πε1 πε2 πε3 is obtained by constructing parallelogram πε4 πε2 πε2 πε6 and rectangle πε4 πε1 πε10 πε9 using bisection points πε5 and πε7 resp. . Starting from πε8 one constructs a ladder of parallelograms and rectangles that area-combinations of the same two triangles. The point πε10 is obtained by triangle-rectification πε10=(πε1!πε2;πε3⌊) that has already been defined in N-geometry; however, **height** of a triangle has no meaning with respect to area in N-geometry (and not in Lobachevsky geometry as will be seen). Of course the construction of parallelograms by attaching the mirrored triangle to a triangle is only possible in E-geometry. If one wants to avoid the Archimedes-reasoning for obtuse-angles as above, one can pick the longest sides of triangle πε1;πε2;πε3 and πε4;πε5;πε6 resp. as bases using emination and emaxation and follow the above procedure with the base line between (properly renumbered) point πε1 and point (πε1∪πε2;(πε1⊕πε2;πε2;πε3⌊)) . The reasoning for scissor-congruence then includes only parallelograms and rectangles and is thus straightforward.

area-exometrition
(πε;πε;πε♦πε;πε⌊[1πε≠2πε]∧[4πε≠5πε])

construction of right triangle πε1;πε9;πε14
with same area as πε1;πε2;πε3
and base line of length πε4;πε5

πε8=(!πε1;πε2;πε3⌊) *rectification*
πε9=(πε1⊕πε2;πε4;πε5⌊) *new base line*
πε10=(πε1;πε5!⌊)
πε11=(πε9⊕πε10;πε1;πε8⌊)
πε12=(πε1;πε2!⌊)
πε13=(πε1;πε11⊗πε2;πε12⌊) *linisection ok*
πε14=(πε1⊕πε8;πε2;πε13⌊) *triangle comparison*
=(πε1;πε2;πε3♦πε4;πε5⌊)

πε7=(πε4!πε5;πε6⌊)
πε15=(πε1⊕πε14;πε4;πε7⌊)



points πε6 πε7 πε15 *included for area-comparison and area-triangle-combination*
πε1;πε2;πε3 *and* πε4;πε5;πε6

example has distance πε4;πε5 *less than* πε1;πε2

Tri-angularity condition ∠1πε;2πε;3πε is not necessary, area-exometrition includes the fringe cases of linicity as long as [1πε≠2πε]∧[4πε≠5πε].

Use was made of proportionality πε1;πε9 to πε1;πε2 equaling πε9;πε11 to πε2;πε13 . If base distance πε4;πε5 is less than distance πε1;πε2 one could construct the points in N-geometry and thus in L-geometry as well. However, in the other case there is a linisection problem and thus the definition of area-exometrition depends on parallelity ***Axiom*** and cannot be meaningfully used in L-geometry.

Now everything is prepared for talking about the area of triangles without metering area by some numbers or similar concept (see section 3.5). One defines **area-equality** πε;πε;πε≈≈πε;πε;πε and **area-minority** πε;πε;πε⦉⦉πε;πε;πε (by comparing heights of triangles with base segments of equal length, neglecting sensitivity). Fringe cases of collinic triples are included.

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[πε1;πε2;πε3≈≈πε4;πε5;πε6]↔
[[[[πε1≠πε2]∧[πε4≠πε5]]∧[πε1;(πε1;πε2;πε3♦πε4;πε5⌊)≈πε4;(πε4!πε5;πε6⌊)]]]]]]]

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[πε1;πε2;πε3⦉⦉πε4;πε5;πε6]↔
[[[πε1≠πε2]∧[πε4≠πε5]]∧[πε1;(πε1;πε2;πε3♦πε4;πε5⌊)⦉πε4;(πε4!πε5;πε6⌊)]]]]]]]

THEOREM self-referring area-exometrition is rectification

∀πε1[∀πε2[∀πε3[[πε1≠πε2]→([πε1;πε2;πε3♦πε1;πε2⌊)=(!πε1;πε2;πε3⌊)]]]]

THEOREM of area-conserving rectification

∀πε1[∀πε2[∀πε3[[πε1≠πε2]→[πε1;πε2;πε3≈≈πε1;πε2;(!πε1;πε2;πε3⌊))]]]]

THEOREM of triangle area splitting

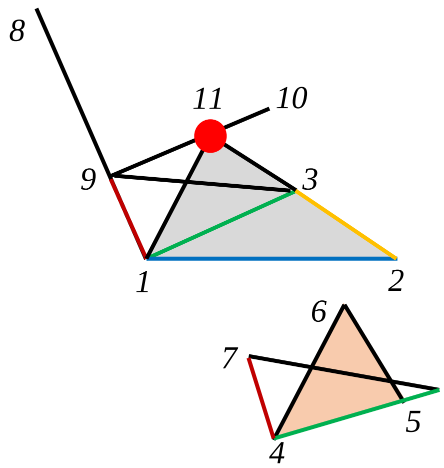
∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[[∠πε1;πε2;πε3]∧[πε1—πε4—πε3]]→
[(πε1;πε2;πε3♦πε5;πε6⌊)=(πε1;(πε1;πε4;πε3♦πε5;πε6⌊)⊕πε4;(πε4;πε2(πε4;πε2;πε3)♦πε5;πε6⌊)]]]]]]]

General **area-triangle-combination** (πε;πε;πε⊕♦πε;πε;πε⌊[∠1πε;2πε;3πε]∧[4πε≠5πε]) is introduced in the following sense: a triangle is combined with another triangle to produce a right triangle that has the same base line and is built from two triangles that have the same areas as the outset triangles resp. ; the first three points must not be collinic. This, of course, is heuristic language, let's define it it in a precise fashion:

(πε;πε;πε⊕♦πε;πε;πε⌊[∠1πε;2πε;3πε]∧[4πε≠5πε])

πε7=(πε4;πε5;πε6♦πε1;πε3⌊) *area-exometrition*
πε8==(!πε1;πε2⌊) *anti-perculation*
πε9=(πε1⊕πε8;πε4;πε7⌊)
πε10==(!πε9;πε1⌊)
πε11=(πε9;πε10⊕πε2;πε3⌊)

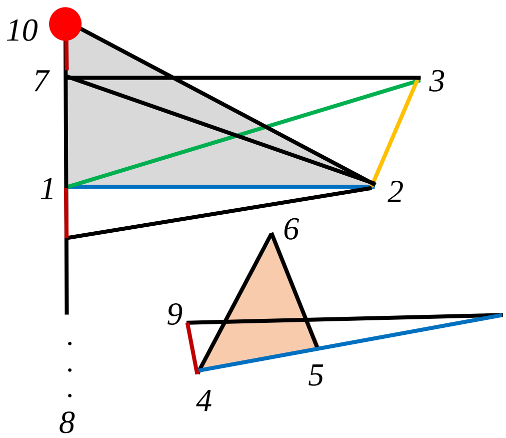
combined triangle π1;π2;π11



Alternatively can introduce **area-right-combination** (πε;πε;πε♦⊕πε;πε;πε⌊[1πε≠2πε]∧[4πε≠5πε]) of triangles in the following sense: a triangle is combined with another triangle to produce a right triangle. To this end the two triangles are transformed to right triangles with a common base that can be attached and sheared. This combination includes the case where the triples of points are on a line too.

(πε;πε;πε♦⊕πε;πε;πε⌊[1πε≠2πε]∧[4πε≠5πε])

πε7==(!πε1;πε2;πε3⌊) *rectification*
πε8==(πε2;πε1!⌊) *syn-perculation*
πε9=(πε4;πε5;πε6♦πε1;πε2⌊) *area-exometrition*
πε10=(πε8;πε7⊕ε4;πε9⌊)



THEOREM: same area of two combinations

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[[∠πε1;πε2;πε3]∧[πε4≠πε5]]→
[(πε1;πε2;(πε1;πε2;πε3⊕♦πε4;πε5;πε6⌊)♦πε1;πε2)=(πε1;πε2;πε3!♦πε4;πε5;πε6⌊)]]]]]]]

THEOREM: combination with line triple

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[[[∠πε1;πε2;πε3]∧[πε4≠πε5]]∧[—4πε;5πε;6πε]]→
[(πε1;πε2;πε3⊕♦πε4;πε5;πε6⌊)=πε3]]]]

This corresponds to sort of adding a zero-area, in a certain sense the combinations of triangles are associative and commutative as well, but it is a bit complicate to express it, as metering so far is relative to a distance. Constructions for area-equality and area-minority (based on for area-exometrition) are not symmetric with respect to the two triangles in question, they single out a cathetus of the second triangle for comparison. However one can do better: **area-endometrition** is an independent construction for every triangles itself. One makes use of the following ***THEOREM*** :

Cathetus *THEOREM* (*Kathetensatz*) is proven by two shear-mappings and one congruence. In the following the usual squares are not shown, but only the corresponding right-isoscelic triangles (obtained with diagonal).

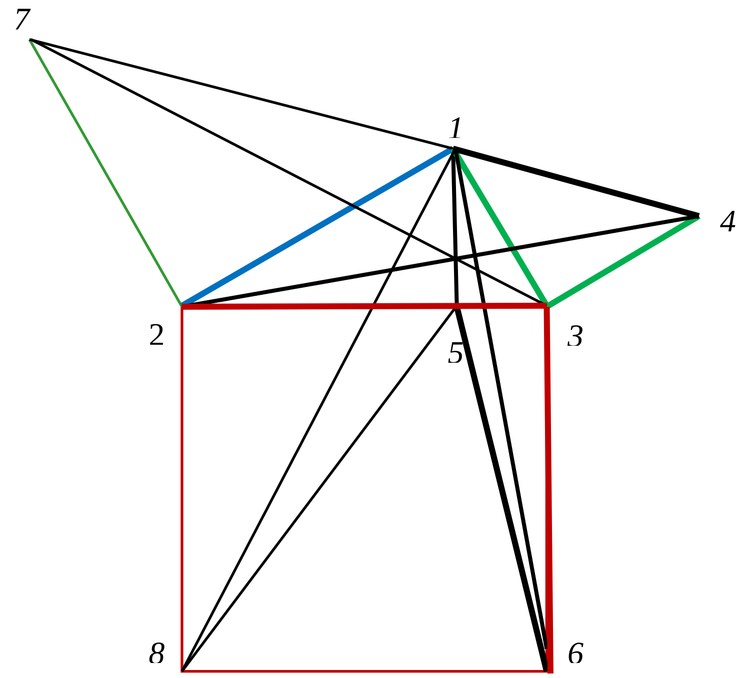
πε3;πε6≈πε2;πε3
πε3;πε6≈πε2;πε8
πε3;πε6≈πε6;πε8

πε3;πε4≈πε1;πε3
πε2;πε7≈πε1;πε2
πε2;πε4≈πε1;πε8
πε3;πε7≈πε1;πε8

πε3;πε5;πε6≈≈πε1;πε3;πε6
πε1;πε5;πε3≅πε2;πε3;πε4
πε2;πε3;πε4≈≈πε1;πε3;πε4

πε2;πε5;πε8≈≈πε1;πε2;πε8
πε1;πε2;πε8≅πε2;πε3;πε7
πε2;πε3;πε7≈≈πε1;πε2;πε7

πε3;πε5;πε6≈≈πε1;πε3;πε4
πε2;πε5;πε8≈≈πε1;πε2;πε7



cathetus *THEOREM*

∀πε1[∀πε2[∀πε3[[[[[πε1≠πε2]∧[πε1≠πε3]]∧[πε1⊥πε2;πε3]]∧[⎛ πε1;πε2;πε3]]→
[πε1;πε3;(πε2;πε1⊕πε1;πε3└)≈≈πε3;(πε2;πε3;πε1!└) ;(πε1;(πε2;πε3;πε1!└)⊕πε2;πε3└))]]]]

πε4=(πε2;πε1⊕πε1;πε3└)
πε5=(πε2;πε3;πε1!└) perpendiculation
πε6=(πε1;πε5⊕πε2;πε3└)

πε1;πε3;πε4≈≈πε3;πε5;πε6

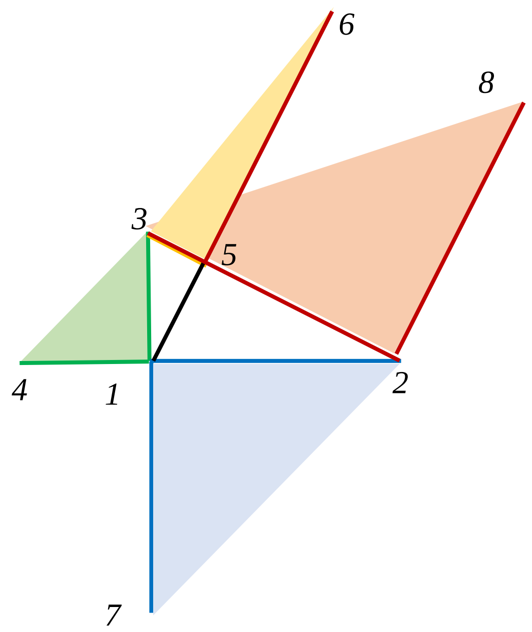
preparing for Pythagoras

πε7=(πε2;πε1!!└) syn-riscolation
πε8=(πε3;πε2!!└) syn-riscolation

corollary **Pythagoras *THEOREM***

πε2;πε3;πε8≈≈(πε1;πε2;πε7⊕♦πε1;πε3;πε4└)

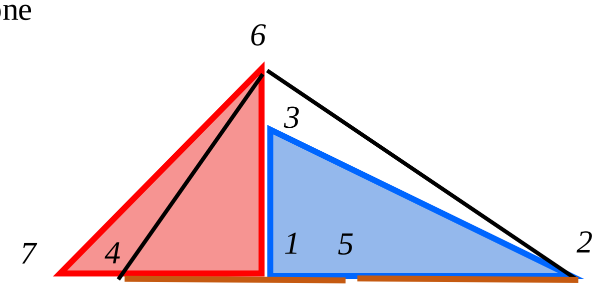
∀πε1[∀πε2[∀πε3[[[[[πε1≠πε2]∧[πε1≠πε3]]∧
[πε1⊥πε2;πε3]]∧[⎛ πε1;πε2;πε3]]→[πε2;πε3;(!πε3;πε2!└)≈≈
(πε1;πε2;(!πε2;πε1!└)⊕♦πε1;πε3;πε4└)]]]]



Geometric mean *THEOREM* (*Höhensatz*) is proven by three shear-mappings. It has the advantage that one does not have to know which leg is the smaller one

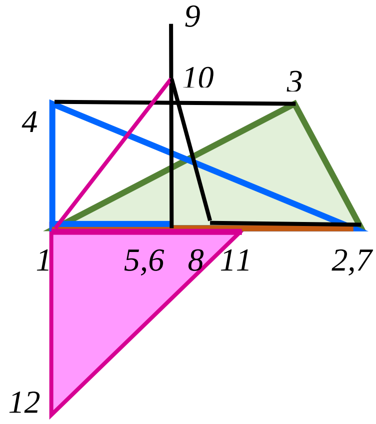
πε4=(πε3!!πε1)
πε5=(πε2⁄πε4)
πε6=(πε5;πε2®⊕πε1;πε4)
πε7=(πε1!!πε6)

πε1;πε2;πε3≈≈πε1;πε6;πε7 right isoscelic triangle



From cathetus ***THEOREM*** one can determine a right isoscelic triangle for a given triangle that has the same are, which means that one can meter the area of a triangle by a distance, i.e. the cathetus of a right isoscelic triangle. This is used for the introduction of **area-endometrition** (♦πε;πε;πε⌊1πε≠2πε) .

area-endometrition



πε4=(πε1!πε2;πε3⌊)	rectification
πε5=(πε1⊕πε2;πε1;πε4⌊)	
πε6=(πε2∩πε5;πε1⌊)	
πε7=(πε2∪πε5;πε1⌊)	new hypotenuse
πε8=(πε1⁄πε7⌊)	Thales
dichotomition	
πε9=(!πε6;πε7⌊)	
πε10=(πε8;πε1®⊕πε6;πε9⌊)	syn-resection
πε11=(πε1⊕πε2;πε1;πε10⌊)	
πε12=(πε11;πε1!⌊)	
=(♦πε1;πε2;πε3⌊)	
πε1;πε2;πε3≈≈πε1;πε11;πε12	right isoscelic triangle

Now one can also express area-equality and area-minority using ***THEOREM*** strings

[πε1;πε2;πε3≈≈πε4;πε5;πε6]↔[[[[πε1≠πε2]∧[πε4≠πε5]]∧[πε1;(♦πε1;πε2;πε3⌊)≈πε4;(♦πε4;πε5;πε6⌊)]]∨
[[¬[∠πε1;πε2;πε3]]∧[¬[∠πε4;πε5;πε6]]]]

[πε1;πε2;πε3≺≺πε4;πε5;πε6]↔[[[[πε1≠πε2]∧[πε4≠πε5]]∧[πε1;(♦πε1;πε2;πε3⌊)≺πε4;(♦πε4;πε5;πε6⌊)]]∨
[[¬[∠πε1;πε2;πε3]]∧[∠πε4;πε5;πε6]]]

One has to pay a price for independent area-endometrition: the metering for the combination of triangles is not done by appending two distances, but rather by a slightly more complicated procedure: the diagonal of a right angle with the two meters.

But such an alternative way of metering has already occurred in section 2.6 for triangle-combination and shown to be totally acceptable. The deeper reason is that monotonous functions generate new meterings from a given metering. In the present case the procedure is based on 'Pythagoras' ***THEOREM***

It means that one takes the square root of area for metering, which implies that the corresponding combination function **area-combo-endometrition** (πε;πε;πε♦♦πε;πε;πε⌊[1πε≠2πε]∧[4πε≠5πε]) is essentially the square root of the sum of the two squares (as will become more clear in the section 3.6 that allows for talking about addition, squaring etc.)

area-endometrition (with respect to πε1)	πε7=(♦πε1;πε2;πε3⌊)
area-endometrition (with respect to πε4)	πε8=(♦πε4;πε5;πε6⌊)
	πε9=(πε1⊕πε2;πε4;πε8⌊)
	πε10=(πε1⊕πε2;πε7;πε9⌊)
	πε11=(πε2;πε1!⌊)
so one can define area-combo-endometrition:	πε12=(πε1⊕πε11;πε7;πε9⌊)
with respect to πε1 and line πε1;πε2	=(πε1;πε2;πε3♦♦πε4;πε5;πε6⌊)

and has a ***THEOREM*** (that precedes the 'normal' addition of areas as given in the section 3.6) stating that the area of combined triangles can be also determined by a Pythagoras kind of procedure.

∀πε1[∀πε2[∀πε3[∀πε4[∀πε5[∀πε6[[[1πε≠2πε]∧[4πε≠5πε]]→
[(♦πε1;πε2;(πε1;πε2;πε3⊕♦πε4;πε5;πε6⌊)⌊)=(πε1;πε2;πε3♦♦πε4;πε5;πε6⌊)]]]]]]]

3.5 Cali-extrafuncta and metering

Lets add some functions that are based on proto-pair πεo;πεe. *heuristic comment*

cali-multix-portion	(′πε;πε∫⌊)	(′πε1;πε2∫⌊)=(πεo;πεe∫πε1;πε2⌊)	o1 . o2
cali-quadrix-portion	(∫πε′⌊)	(∫πε1′⌊)=(πεo;πεe∫πε1;πε1⌊)	$o1^2$
cali-divix-portion	(∫πε;πε′⌊2πε≠πεo)	(∫πε1;πε2′⌊)=(πεo;πε1∫πεe;πε2⌊)	o2 / o1
cali-recix-portion	(∫πε′⌊1πε≠πεo)	(∫πε1;πε2′⌊)=(πεo;πε1∫πεe;πεe⌊)	1 / o1

area-equality and area-minority fulfill the requirements for an class-ordo-relation. With reference to the proto-pair πεo πεe one can specify a representative for every class of area-equal triangles:

Cali-area-exometrition (′πε;πε;πε♦⌊1πε≠2πε)
(′πε1;πε2;πε3♦⌊)=(πε1;πε2;πε3♦πεo;πεe⌊)

Now one has ***THEOREM*** strings that express area-equality and area-minority with reference to cali-area-exometrition

[πε1;πε2;πε3≈≈πε4;πε5;πε6]↔[[[[πε1≠πε2]∧[πε4≠πε5]]∧[(′πε1;πε2;πε3♦⌊)=(′πε4;πε5;πε6♦⌊)]]∨[[¬[∠πε1;πε2;πε3]]∧[¬[∠πε4;πε5;πε6]]]]

[πε1;πε2;πε3⊰⊰πε4;πε5;πε6]↔[[[[πε1≠πε2]∧[πε4≠πε5]]∧[πεo;(π′ε1;πε2;πε3♦⌊)⊰πεo;(′πε4;πε5;πε6♦⌊)]]∨[[¬[∠πε1;πε2;πε3]]∧[∠πε4;πε5;πε6]]]

Alternative area-square-metering with application of 'mean' ***THEOREM*** . The meter is the side of a right isoscelic triangle that has the same area. For the start with any triangle, one needs shear-mapping and application of 'mean theorem' . It means that one takes the square root of area for metering.

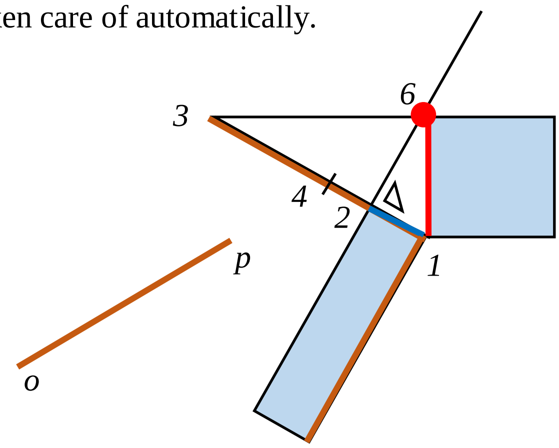
By the way: Hilbert is in error in his definition of biradical numbers in §9 of reference (3) when he introduces a fifth (total) function $sr(1+x^2)$ besides addition, subtraction, multiplication and division, he forgot the sixth function $sr(1-x^2)$ that is partial, or he should have simply chosen $sr(x)$ as fifth (partial) function.

Cali-biradication (′πε;πε′⌊1πε≠2πε⌊) is the inverse to cali-quadrix-portion: (πε1∫′(′πε1;πε2′⌊)⌊)=πε2 , it can be programmed by a geometrical construction for the expression: $sr(abs(x))=(sr(1+x^2)/sr(2)) sr(abs(1-((1-x)/sr(1+x^2))^2))$ using cali-multix-portion and cali-syn-resection and as in neutral geometry cali-adipension i.e. absolute value of $(1-x)$, cali-hypothetion and cali-cathetetion that have been identified in E-geometry as hypotenuse and cathetus Pythagoras square roots $sr(1+x^2)$ and $sr(1-x^2)$. The critical condition that the radicand of a square root must be nonnegative means for $sr(1-x^2)$ that x^2 must not be greater 1 . In the following construction this is taken care of automatically.

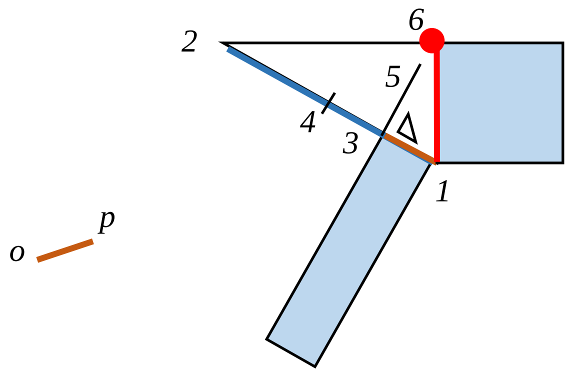
cali-endo-biradication
(′πε;πε⌊[1πε≠2πε]∧[1πε;2πε⊰πεo;πεe])
in E-geometry for sr(x) with x<=1

πε3=(πε1⊕πε2;πεo;πεe⌊)
πε4=(πε1∤πε3⌊)
πε5=(!πε2;πε1⌊)
πε6=(πε4;πε1®⊕πε2;πε5⌊)

(′πε1;πε2⌊)=πε6=
((πε1∤(πε1⊕πε2;πεo;πεe⌊)⌊);πε1®⊕πε2;(!πε2;πε1⌊)⌊)



cali-exo-biradication (πε;πε′⎿πεo;πεe⎨1πε;2πε)
in E-geometry for sr(x) with 1<= x



πε3=(πε1⊕πε2;πεo;πεe⎿)
πε4=(πε1⌿πε2⎿)
πε5=(!πε3;πε1⎿)
πε6=(πε4;πε1®⊕πε3;πε5⎿)

(πε1;πε2′⎿)=
((πε1⌿πε2⎿);πε1®⊕πε3;(!(πε1⊕πε2;πεo;πεe⎿);πε1⎿)⎿)

cali-biradication (′πε;πε′) *in E-geometry for sr(abs(x))*

πε3=(πε1⊕πε2;πεo;πεe⎿)
πε4=(πε1∩πε2;πε3) *picking smaller distance*
πε5=(πε1∪πε2;πε3) *picking greater distance*
πε6=(πε1⌿πε3⎿)
πε7=(!πε4;πε1⎿)
πε8=(πε6;πε1®⊕πε4;πε7⎿)
(′πε1;πε2′)=
((πε1⌿(πε1∪πε2;(πε1⊕πε2;πεo;πεe⎿))⎿);πε1®⊕(πε1∩πε2;(πε1⊕⊕πε2;πεo;πεe⎿));
(!(πε1∩πε2;(πε1⊕πε2;πεo;πεe⎿));πε1⎿)⎿)

cali-area-endometrition (♦πε;πε;πε′⎿1πε≠2πε) is based on the result (♦πε1;πε2;πε3⎿) of area-endometrition where the result is located on the proto-line

(♦πε1;πε2;πε3′⎿)=(πεo⊕πεe;πε1;(♦πε1;πε2;πε3⎿)⎿)

3.6 Proto- and cali-extrafuncta

Now one can do one step further and relate all metering to the proto-line. In section 2.4 there were already defined: proto-linity, proto-positivity, proto-addition, proto-negativation etc. . All of them based on appension. They turned the proto-line into an ordered unlimited group. Based on unique ***Unique-parallel-axiom A21e*** and linisection one has proportition and this leads to the definition of proto-multiplication, proto-reciprocation and thus extending the proto-line to an ordered unlimited corpus[1] (in German *Körper*, in English usually called *field* which, however, is a very bad name, as so many different systems or entities are called *field* in mathematics). Based on ***Unique-parallel-axiom A21e*** and isoscition one can introduce proto-biradication etc. and thus one extends the proto-line even further to an ordered unlimited corpus of biradical numbers, a simple extension of the rational-number corpus to an algebraic corpus. By the way: in proto-functions of E-geometry one sees that appension of points is related to addition of numbers, linisection to multiplication and isoscition (and circulation) to biradication.

<u>based on linisection</u>

proto-multiplication (πε×πε⌊[⊆1πε]∧[⊆2πε])
(πε1×πε2⌊)=(πεo;πεe∫πε1;πε2⌊)

proto-quadration (πε×⌊⊆1πε)
(πε1×⌊)=(πε1×πε1⌊)

proto-reciprocation (/πε⌊[⊆1πε]∧[1πε≠πo])
(/πε1⌊)=(πεo;πε1∫πεe;πε e ⌊)

proto-division (πε/πε⌊[[⊆1πε]∧[⊆2πε]]∧[2πε≠πo])
(πε1/πε2⌊)=(πε1× (/πε2⌊)⌊)

proto-reduction (πε–/ +⌊1πε≠πεm)
(πε1–/+π ⌊)=((πε1–πεe⌊)/(πε1+πεe⌊)) *(x-1)/(x+1)*

proto-upduction (πε+/–⌊1πε≠πεe)
(πε1–/+π ⌊)=((πε1+πεe⌊)/(πε1–πεe⌊)) *(x+1)/(x-1)*

cali-tension (′πε×πε⌊<2πε)
(′πε1×πε2⌊)=(πεo⊕πε1;πεo;((πεo⊕πεe;πεo;πε1⌊)×πε2⌊)) *stretching with center* πεo

<u>based on isoscition</u>

proto-biradication (√πε⌊≤1πε) inverse of proto-quadration
is obtained by applying cali-biradication to πεo;πε1 and putting the result on the positive protoray:
(√πε1⌊)=(πεo⊕πεe;πεo;(′πεo;πε1⌊)⌊)

Now one can express ***individual-constant*** strings based on proto-pair πεo;πεe (representing *0* and *1*)

proto-full	πεf=(πεo;πεe⊕)	*2*	*cos(pi)=-1*
proto-straight	πεe	*1*	*cos(pi/2)=0*
proto-right	πεr=(πεe–((√πεf⌊)/⌊)⌊)	*1-(sr(2))/2)=0,292…*	*cos(pi/2)=(sr(2))/2*

THEOREM (πε1/)=(πε1/πεf⌊)

THEOREM proto-biradication is non-negative inverse of proto-quadration
∀πε1[[≤1πε]→[((√πε1⌊)×⌊)=πε1]] ∀πε1[[⊆1πε]→[(√(πε1×⌊)⌊)=(+πε1⌊)]]

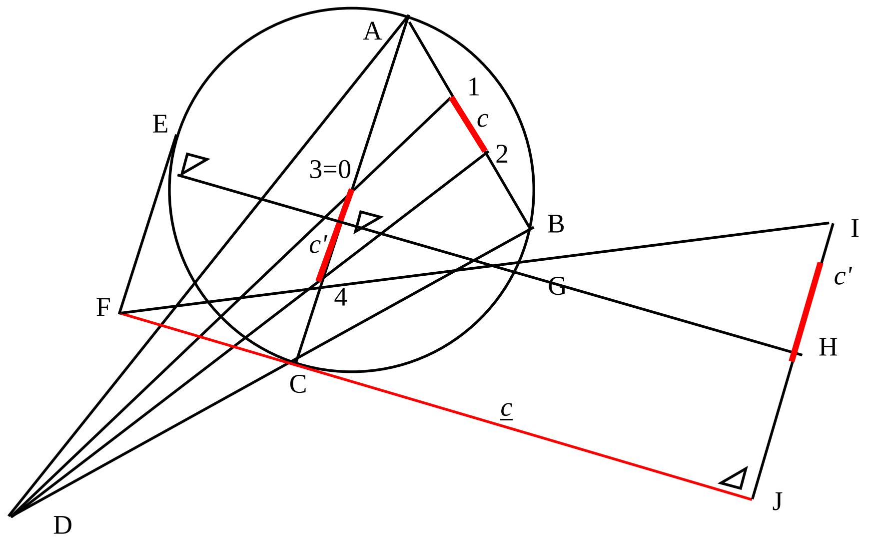
[1] One could call proto-functions on the proto-line **linic Euclid-functions** as opposed to **planar Euclid-functions**

cross-ratios will be useful for Lobachevsky geometry in chapter 4 . For Euclid distances between four points A 1 2 B (on a line in this order) it is defined: $\underline{c} = (d_{A2}\, d_{B1}) / (d_{2B}\, d_{1A}) = (d_{A2} / d_{2B}) / (d_{A1} / d_{1B})$ with $\underline{c}$ values greater or equal to *1* , value *1* if points 1 and 2 are the same. For an ordering A 2 1 B the function gives values between *0* (excluded) and *1* .

proto-cross-ration (πε;πε×/×πε;πε⎣[[3πε<1πε]∧[1πε<4πε]]∧[[3πε<2πε]∧[2πε<4πε]])
(πε1;πε2×/×πε3;πε4⎣)=(((πε3–πε2⎣)×(πε4–πε1⎣)⎣)/((πε4–πε2⎣)×(πε3–πε1⎣)⎣)⎣) *((u-y)(v-x))/((v-y)(u-x))*

<u>A little extra for the friends of projective geometry:</u>

For constructing cross-ratio in general one has to map pair 1and 2 so that one point 1' is at the center 0:



		construction central distance
	1 2	start pair with distance *c*
	A B	chord
13		ray
A3C	C	chord thru center
	3	*=0*
BC		ray
13xBC=	D	linisection of rays
DA		ray (nice for understanding)
D2		ray
D2xAC=	4	linisection of ray and chord
		gives distance *c'*

	construction cross-ratio
E	thru 3, orthogonal to AC
F	distance *1*, orthogonal to 3E
G	linisection F4xE3
H	equal distance 3G and GH
I	orthogonal 3G in distance *c'*
J	linisection IH with FC
	gives cross-ratio $\underline{c}$ as FJ

cali-syn-resection (′πε;πε®⊕⎣[[⊂1πε⊃]∧[⊂2πε⊃]]∧[1πε≠2πε]) intersects cali-circle with line of pair inside, in same direction, **cali-anti-resectio**n (′πε;πε⊕®⎣[[⊂1πε⊃]∧[⊂2πε⊃]]∧[1πε≠2πε]) in opposite direction

(′πε1;πε2®⊕⎣)=(πεo;πεe®⊕πε1;πε2 ⎣) (′πε1;πε2⊕®⎣)=(πεe;πεo⊕®πε1;πε2 ⎣)

The cross-ratio of a pair inside the cali-circle and its intersections with the cali-circle is called **cali-cross-ration** (πε;πε/′⎣[⊂1πε⊃]∧[⊂2πε⊃]) it has result on positive proto-ray outside the cali-circle

′πε1)⎣)⎣)
(πε1;πε2/′⎣)=(((((®⊕πε1;πε2′⎣))′πε2)×((⊕®πε1;πε2⎣)′πε1)⎣)/(((⊕®πε1;πε2⎣)′πε2)×(((®⊕πε1;πε2′⎣)⎣)

cali-cross-metrition (πε;πε′/⎣[⊂1πε⊃]∧[⊂2πε⊃]]) is obtained from cali-cross-ration by proto-reduction, it has result on proto-radius ≤πε# (πε1;πε2′/⎣)=((πε1;πε2/′⎣)–/+⎣)

One can define (either with the use of proto-biradication or directly) some functions that are meaningful in connection with triangle construction:

THEOREM proto-hypothetion (πε√+πε└), proto-cathetetion (πε√–πε└), proto-hypocation (√πε+└) and proto-cathecation (√–πε└) can be expressed by proto-biradication in E-geometry representing *$sr(x^2+y^2)$* , *$sr(x^2- y^2)$* , *$sr(1+x^2)$* and *$sr(1-x^2)$* resp.

∀πε1[[≤1πε]→[((√πε1└)×└)=πε1]]   ∀πε1[[⊆1πε]→[(√(πε1×└)└)=(+πε1└)]]

Special cases on the proto-diameter lead to four more proto-functions with condition [⊂1πε⊃]∧[⊂2πε⊃] :

proto-ortho-metrition
(πε1√–/πε2└)=(πε2/(√–πε1└)└)   conventional *$y/sr(1-x^2)$*

and its inverse **proto-ortho-immetrition**
(πε1√–×πε2└)=(πε2×(√–πε1└)└)   conventional *$y.sr(1-x^2)$*

proto-para-metrition
(πε1–+/πε2└)=(πε2/(πεe–(πε1×(πε1+πε2└)└)└)└)   conventional *y/(1-x(x+y))*

and its inverse **proto-para-immetrition**
(πε1–/+πε2└)=((πε2×(πεe–(πε1×└)└)└)/(πεe+(πε1×πε2└)└)└)   conventional *$y(1-x^2)/(1+xy)$*

THEOREM **proto-hypothetion of areas**, stating that the cali-area-endometer of a combined triangle is given by proto-hypothetion of the cali-area-meters (♦πε;πε;πε′└) of the two triangles; located on the proto-line. (♦πε1;πε2;(πε1;πε2;πε3⊕♦πε4;πε5;πε6└)′└)=((♦πε1;πε2;πε3′└)√+(♦πε1;πε2;πε3′└)└)

cali-area-quad-metrition (πε♦πε′πε└1πε≠2πε) is obtained by proto-quadration of cali-area-endometrition. It corresponds to the conventional area metering. The result is located on the proto-line Notice that cali-area-protometers have to be **bisected** with (πε/└) to get a result that corresponds to the fact that the area of a triangle is **half** the product of base and height lengths.

(πε1♦πε2′πε3└)=((πεo,(♦πε1;πε2;πε3′└)×└)/└)

THEOREM **proto-addition of areas**, stating that the cali-area-meter of a combined triangle is the sum with 'proto-addition (πε+πε└) of the cali-area-meters of the two triangles (observe that cali-area meters (♦πε;πε;πε′└) are located on the proto-line.

(♦πε1;πε2;(πε1;πε2;πε3⊕♦πε4;πε5;πε6└)′└)=((♦πε1;πε2;πε3′└)+(♦πε1;πε2;πε3′└)└)

There is a constructive way to produce the full plane of E-geometry by the so-called coordinate method. There is the ***THEOREM*** of **unique existence of coordinates** (πε2 and πε3 for πε1):

∀πε1[∃πε2[∃πε3[[[–πε2]∧[–πε3]]∧[[πε1=(πε3;πεo↓πε2↓(πε2⊕(πε1;(πε3;(πε1⊕πε2;πε1;πε3└)∇πε1└)↓(πε2;πεo↓πεi↓(πεo;πε2!└)↓(πεo;πεe÷(πε2;πεo↓πεi↓(πεo;πε2!└)└));πεo;πε3└))]∧   *existence*

[∀πε4[∀πε5[[[[–πε4]∧[–πε5]]∧[[πε1=(πε5;πεo↓πε4↓(πε4⊕(πε1;(πε5;(πε1⊕πε4;πε1;πε5└)∇πε1└)↓(πε4;πεo↓πεi↓(πεo;πε4!└)↓(πεo;πεe÷(πε4;πεo↓πεi↓(πεo;πε4!└)└));πεo;πε5└))]→[[πε2=πε4]∧[πε3=πε5]]]]]]]   *uniqueness*

<u>Proof:</u> Coordinates are obtained by **perpendiculations** to the two axis, the proto-line and the ortho-line:
πε2=(′πε1)=(πεo;πεe;πε1!└)   πε3=(πε1′)=(πεo;πεi;πε1!└)

proto-angle-addition (πε°+πε└[≤1πε<]∧[≤2πε<]) is revisited:

THEOREM trigonometric expression [1)] for proto-angle-addition. For angles less than the straight one that is angles of the radity-segment it holds:

1) this leads to proto-dyadic-angle-metrition (°πε°└≤1πε∈) recursive definition, only for **dyadic** points in cycle-segment, values between πεo and πεf as it will be introduced in section 3.7 .

[[≤π1#]∧[≤π2#]]→
[(πε1°+πε2⎿)=((((πε1+πε2⎿)−(πε1×πε2⎿)⎿)+(√(((πε1+πε1⎿)−(πε1×πε1⎿)⎿)×((πε2+πε2⎿)−(πε2×πε2⎿)⎿)⎿)⎿))

corresponding to *f(x+y)=1-cos((x+y)/2)=1-cos(x/2)cos(y/2)+sin(x/2)sin(y/2)=*
f(x)+f(y)-f(x)f(y)+sr((2f(x)-f(x)^2*)(2f(y)-f(y)*^{2}*))* for *0<=x,y<1* where *f(x)=1-cos(pi.x/2)*

from N-geometry. For **proto-angle-bisection** (πε°/⎿) one has

THEOREM (πε1°/⎿)=(πεe−(√(πεe−(πε1/)⎿)⎿)⎿) corresp. to *f(x/2)=1-cos(x/4)=1-sr(1-cos(x/2)/2)*

Now one can express as further ***individual-constant*** strings based on πεo;πεe (representing *0* and *1*)
proto-halfright πεw=(πεr°/⎿)=(πεe−(√(πεe−((πεe−((√πεf⎿)/)⎿)/)⎿)⎿)⎿) corresp. to*1-cos(pi/4)=*
=(πεe−(√((πεe+((√πεf⎿)/)⎿)/)⎿)⎿) *1-sr(((1+sr(2)/2)/2)=0,073…*

3.7 Abstract calcule <u>deltaalpha</u> of biradical numbers

The proto-line with its proto-functions in abstract calcule <u>piepsilon</u> correspond perfectly to the abstract calcule <u>deltaalpha</u> of ordered corpus of biradical numbers that has the following ontological basis:

sort	δα	
basis-individual-constant	δαn	δαu
basis-function-constant		
addition	(δα+δα)	
negativation	(−δα)	
multiplication	(δα×δα)	
reciprocation	(/δα⎿1δα≠δαn)	
biradication	(√δα⎿δαn≤1δα)	
basis-relation-constant		
positivity	<δα	

basis-functum strings of <u>deltaalpha</u> are defined via the ***Axiom*** strings. Only the angle-related ***extra-functum*** strings are written down explicitly, the other ones are straightforward as proto-functa of <u>pi</u> .

extra-individual-constant	δαb δαmu	δαb=(δαu+δαu) δαmu=(−δαu)
extra-relation-constant		
nonpositivity	δα≤	
negativity, nonnegativity	≤δα δα<	
minority, equal-minority	δα<δα δα≤δα	
radity	≤δα#	[≤δα1#↔[[δαn ≤δα1]∧[δα1<δαu]
diametrity	#δα#	[#δα1#↔[[δαmu<α1]∧[δα1<δαu]
cyclity	≤δα<	[≤δα1<]↔[[δαn ≤δα1]∧[δα1<δαb]]
dyadicity	δα∈	recursive[1)] definition
dyadic-cyclity	≤δα∈<	recursive definition
rationality	∈δα	recursive definition
extra-function-constant		
duplication	(δα+)	
subtraction	(δα−δα)	
quadration	(δα×)	

[1)] in E-geometry recursion occurred only in connection with extensions to dyadic numbers and only for relations, not for functions. For calcule <u>delta-alpha</u> it may occur for relations as well and function-definition by cases is not excluded

absolution	(+δα)	
division	(δα/δα⌊2δα≠δαn)	
bisection	(δα/)	
hypothetion	(δα√+δα)	*$sr(x^2+y^2)$*
cathetetion	(δα√−δα⌊(2δα×)≤(1δα×))	*$sr(x^2-y^2)$*
hypocation	(√δα+)	*$sr(1+x^2)$*
cathecation	(√δα−⌊1δα≤δαu)	*$sr(1-x^2)$*
angle-bisection	(δα°/⌊≤1δα<)	angle-functions are partial in cycle-segment
angle-duplication	(δα°+⌊≤1δα<)	
angle-addition, cyclation	(δα°+δα⌊[≤1δα<]∧[≤2δα<])	
angle-negativation	(°−δα⌊≤1δα<)	
angle-subtraction	(δα°−δα⌊[≤1δα<]∧[≤2δα<])	
dyadic-angle-metrition	(°δα°⌊≤1δα∈<)	partial in Klein-numbers in cycle-segment

With the following 17 ***Axiom*** strings, that are not written down explicitly:

A1	neutrivity of addition	***A11***	distributivity addition multiplication
A2	associativity of addition	***A12***	distributivity negativation multiplication
A3	commutativity of addition	***A13***	unus positivity
A4	addition of negativation	***A14***	exclusivity of positivity
A5	double negativation	***A15***	quadration of biradication [δαn≤δα1]→[((√δα1⌊)×(√δα1⌊))=δα1]
A6	neutrivity of multiplication		
A7	associativity of multiplication	***A16***	distributivity biradication multiplication [[δαn≤δα1]∧[δαn≤δα2]]→ [(√(δα1×δα2)⌊)=((√δα1⌊)×(√δα2⌊)]
A8	commutativity of multiplication		
A9	multiplication by reciprocation		
A9	multiplication by reciprocation	***A17***	distributivity biradication reciprocation [[δαn<δα1]→[(√(/δα1⌊)⌊)=[(/(√δα1⌊)⌊)]
A10	double reciprocation		

and an ***Axiom*** <u>mater of inductivity</u> ***∀δα1[[[sentence(*** ∀δα1[***δα1***]***)] ∧ [¬ [δα1 ⊃*** δα2 ***]]] →***
[[[TRUTH([***(δα1;***δα1∫δαn ***)] ∧ [TRUTH(*** ∀δα1[[***δα1***]→[[[[***(δα1;***δα1∫−δα1***)***]∧[[δα1≠δαn)]→
[***(δα1;***δα1∫(/δα1⌊)***)***]]]∧[[δαn≤δα1)]→[***(δα1;***δα1∫(√δα1⌊)***)***]]]∧[∀δα2[[***(δα1;***δα1∫δα2***)***]→
[[***(δα1;***δα1∫(δα1+δα2)***)***]∧[***(δα1;***δα1∫(δα1×δα2)***)***]]]]]]***)]] →[TRUTH(*** ∀δα1[***δα1***]***)]]]***

<u>Metatheorem</u> for the two abstract calcules <u>piepsilon</u> of planar Euclid geometry and <u>deltaalpha</u> of ordered corpus of biradical numbers: biradical numbers can be constructed geometrically in an abstract sense

<u>Metaproof idea</u>: the ***Axiom*** strings of abstract calcule <u>deltaalpha</u> of the ordered corpus of biradical numbers with ***nullum-constant*** δαn and ***unus-constant*** δαu, functions addition (δα+δα), negativation (−δα) , multiplication (δα+δα) , reciprocation (/δα ⌊1δα≠δαn) and biradication (√δα ⌊≤δα)and relation positivity correspond one-to-one to ***THEOREM*** strings for points of the proto-line of abstract calcule <u>piepsilon</u> with proto-origin and proto-unit, functions proto-addition, proto-negativation, proto-multiplication, proto-reciprocation and proto-biradication and relation proto-positivity.

Angle-functions of <u>deltaalpha</u> get definitions where **one definition by cases** is necessary:

angle-bisection [≤δα1<]→[(δα1°/⌊)=(δαu°−(√(δαu°−(δα1/)⌊)⌊)⌊)]

angle-duplication [[≤δα1]∧[δα1<δαu]]→[(δα1°+⌊)=(((δα1+)+)−((δα1×)+))] *$(4x-2x^2)$ mod 2*
2 cases! [[δαu≤δα1]∧[δα1<δαb]]→[(δα1°+⌊)=(δαb−(((δα1+)+)−((δα1×)+)))]

angle-addition [[≤δα1]∧[≤δα2<]]→[(δα1°+δα2⌊)= *$(x+y-xy+sr((2x-x^2).(2y-y^2)))$ mod 2*
((((δα1/)+(δα2/))−((δα1/)×(δα2/)))+(√((δα1−((δα1/)×))×(δα2−((δα2/)×)))⌊));δαb↓δαn↓
((((δα1/)+(δα2/))−((δα1/)×(δα2/)))+(√((δα1−((δα1/)×))×(δα2−((δα2/)×)))⌊))°+⌊)]

angle-negativation [≤δα1<]→[(°−δα1└)=(δαn;δα1↓δαo↓(δαb−δα1))]
angle-subtraction [[≤δα1<]∧[≤δα2<]]→[(δα1°−δα2└)=(δα1°+(°−δα1└)└)]

The following functa are defined recursively, which is beyond our self-imposed rule that allows only for junctive logic for relations and composition of functions. However, this is not a problem as they are not really necessary, but just **nice to have for understanding**. **Dyadic-cyclity** determines dyadic points between δαn and δαb. **Dyadic-angle-metrition** corresponds to the conventional trigonometric function *1-cos(pi.x/2)* for dyadic input between *0* and *2* , as one can see from:

*1-cos(pi.**0**/2)=0* , *1-cos(pi.(**1/4**)/2)=1-sr((1+sr(1/2))/2)* , *1-cos(pi.(**1/2**)/2)=1-sr(1/2)* , *1-cos(pi.**1**/2)=1*
*1-cos(pi.(**3/2**)/2)=1+sr(1/2* and *1-cos(pi.**2**/2)=0*

dyadic-cyclity ≤δα∈< [≤δα1∈<]↔[[δα1=δαu]∨[∃δα2[[≤δα2∈<]∧[[δα1=(/δα2└)]∨
[∃δα3[[≤δα3∈<]∧[δα1=(δα2°+δα3└)]]]]]]]

dyadic-angle-metrition (°δα°└≤1δα∈<)*start recursion* *at anchor* (°δαu°└)=δαu
bisection recursion [≤(δα1°/└)∈<]→[(°(δα1°/└)°└)=((°δα1°└)°/└)]
addition recursion [[≤δα1∈<]∧[≤δα2∈<]]→[(°(δα1°+δα2└)°└)=((°δα1°└)°+(°δα2°└)└)]

rationality ∈δα [∈δα1]↔[[δα1=δαu]∨[∃δα2[[∈δα2]∧[[[δα1=(−δα2)]∨
[[δα2≠δαn]∧[δα1=(/δα2)]]]∨[∃δα3[[∈δα3]∧[δα1=(δα2+δα3)]]]]]]]

3.8 Semiconcrete calcule DELTAalpha of biradical numbers

The following introduction to calcules in general is just a sketch, actually it deserves a much more thorough treatment, as it goes to the very roots of mathematics and logics. This is done in the forthcoming publication *(ref. 11)* ***Opus logico-mathematicus - With the Calculation Criterion of Truth*** .

So far only **abstract calcules** were treated: abstract calcule pi of planar N-geometry, abstract calcule piepsilon of planar E-geometry and abstract calcule deltaalpha of biradical numbers. An abstract calcules has ***basis-individual-constant*** , ***basis-function-constant*** and ***basis-relation-constant*** strings as its ontological basis. Furthermore there is a set of ***Axiom*** strings (that may be infinite using metalingual so-called ***Axiom*** maters). The mathematics of abstract calcules rests on the 'if-then-principle': if there exist certain individuals that fulfill the ***Axiom*** strings, there are more ***sentence*** strings that are true as well; they are called ***THEOREM*** strings. The existence itself is not claimed, it is simply 'if-then'. Abstract calcules are given names that are underlined small Latin words that quote the Greek letters that are used for the sort of individuals, e.g. piepsilon for ***sort*** πε that is chosen for the points of the geometric Euclidean plane. An abstract calcules does not contain ***individual*** strings, it just talks **about** individuals by their names (may they exist or not).

Besides abstract calcules there are calcules where the individuals actually are given as ***individual*** strings of characters, they are called **concrete calcules** . If it is decidable[1] if a string is an ***individual*** the calcule is called **omniconcrete** , if not **semiconcrete** . An omniconcrete calcule has a ***sort*** with capital Greek letters, and a correspondent name with underlined all-capital-Latin words. They can be either finite or infinite. Examples of finite omniconcrete calcules are finite groups. Simple examples of infinite omniconcrete calcules are calcule ALPHAALPHA of power-Robinson decimal natural numbers with ***sort*** AA and calcule RHO of decimal rational numbers with ***sort*** P .

[1] Decidability cannot be treated here in detail. Decidability means that there is a procedure that can be performed by a machine for answering a question. This implies that decidability is closely connected with calculable functions. There are various aspects of decidability, depending on which kind of question it is applied to. Some examples appear in the following.

Semiconcrete calcules have sorts with starting capital Greek letters followed by small Greek letters, they are given correspondent names that start with underlined all-capital-Latin words followed by all-small-Latin-words. Example of a semiconcrete calcules are introduced in the following: calcule DELTAalpha with with ***sort*** Δα and calcule PIdelta with with ***sort*** Πδ .

For a concrete calcule it also must be decidable if a given string is an ***individual*** string [1] or not, e.g. 12345 is a ***decimal-natural-number*** string, 123 :3 is a ***decimal-rational-number*** string. If equality of individuals means string-equality the calcule is called **uniconcrete** . The calcule RHO of rational numbers only has reduced ratios (that are unique) and is therefore uniconcrete. If one allows for unreduced ratios to represent rational number one has the calcule RHORHO of decimal ratio numbers; although the ***individual*** strings are no longer unique, it is decidable if e.g. the strings 122 :2 and 366 :6 represent the same number (multiply numerator and denominator crosswise and check for string-equality). Such a calcule is called **multiconcrete**.

Concrete calcules are not introduced via ***Axiom*** strings, but rather some rules that allow for the calculation of functions and relations (via some basic true sentences). Of course one can define an abstract calcule alpha of natural numbers and an abstract calcule rho of rational numbers that have ***Axiom*** strings. And then one can show that e.g. the concrete calcules ALPHA of decimal natural numbers and e.g. calcule RHO of decimal rational numbers fulfill the ***Axiom*** strings of the related abstract calcules, usually one says that they are 'concrete models[1]' of the abstract calcule.

Now one can turn to the question if there are omniconcrete calcules that are 'models'[2] of abstract geometry calcules. It will turn out that the answer is 'no'. This is closely connected with the question of decidability. It is not surprising that there are no models for the abstract calcule pi of planar neutral N-geometry, that is **incomplete** , allowing for supplementing it with two kind of parallel ***Axiom*** strings. But the abstract calcules piepsilon of planar Euclid E-geometry has no omniconcrete models either.

The best one can do is to take refuge to **semiconcrete calcules**. A semiconcrete calcule has ***individual*** strings, but it is not decidable in general if a certain string represents an ***individual*** or not. In the case of planar geometries one can construct ***individual*** strings that represent points, however, the admissibility of these strings and the equality of points are **not decidable**.

Semiconcrete calcule DELTAalpha of decimal biradical numbers of ***sort*** Δα comprises the numbers that can be constructed starting from Δαu=1 by addition (Δα+Δα) , negativation (–Δα) , multiplication (Δα×Δα) , reciprocation (/Δα⌊1Δα≠0) and biradication (√Δα⌊0≤1Δα) . It is a **semiconcrete model** of the abstract calcule deltaalpha of biradical numbers.

Decimal-biradical-number strings (notice the capital initial) are certain decimal numbers built from 0 1 2 3 4 5 6 8 9 . Cipher 7 is left out for good reasons as will become clear in section 3.9 . ***Decimal-biradical-number*** strings are defined by metalingual recursion that includes ***TRUTH*** expressions for reciprocation (not to be zero) and biradication (not to be negative). The ***decimal-digit*** characters 2 3 4 5 6 denote addition, multiplication, negativation, reciprocation and biradication resp. , the characters 8 the 9 play the roles parentheses. One starts with a metadefinition for an auxiliary entity ***decimal-biradical*** (notice the small initial) that is purely syntactical.

∀Δα1[[***decimal-biradical(Δα1)*** *] ↔ [[* ***dual-natural-number(Δα1)*** *] ∨ [∃ Δα2[[* ***decimal-biradical(Δα2)*** *] ∧ [[[[Δα1=* 84*Δα2*9 *] ∨ [Δα1 =* 85*Δα2*9 *]] ∨ [Δα1 =* 86*Δα2*9 *]] ∨ [∃ Δα3[[* ***Decimal-biradical(Δα3)*** *] ∧ [[Δα1 =Δ*8*α2*2*Δα3*9 *] ∨ [Δα1 =*8*Δα2*3*Δα3*9 *]]]]]]]]]*

1) ***individual*** strings in Bavarian notation (section 1.3) contain only Arial 12 fonts 0,1,2,3,4,5,6,7,8,9,A,B,C, …

2) a calcule or sub calcule is a model of another calcule if there is an isomorphy of the ontological bases and the axioms of an abstract calcule or the basic true sentences of a concrete calcule; however this has yet to be expressed precisely.

The ontological basis of DELTAalpha can be written as follows:

sort :: Δα

individual-constant :: Δαn ¦ Δαu where ¦ is a metalingual short-notation 'or'

Δαn=0 Δαu=1

basis-function-constant :: (Δα+Δα) ¦ (–Δα) ¦ (Δα×Δα) ¦ (/Δα⌊1Δα≠0) ¦ (√Δα⌊≤1Δα)
basis-relation-constant :: ≤Δα

The defining features are not given in detail: they are basically replacing + × – / √ ( ) by 2 3 4 5 6 8 9 respectively and using the rules for arithmetics of ***dual-natural-number*** strings , reduction of ratios and biradication of square roots.

The metaproperty ***Decimal-biradical-number*** [1)]that is metadefined in a more complicated fashion:

∀$\Delta\alpha_1$[[***Decimal-biradical-number(**$\Delta\alpha_1$**)** *] ↔ [[* ***dual-natural-number(**$\Delta\alpha_1$**)** *] ∨*
[∃ $\Delta\alpha_2$[[***Decimal-biradical-number(**$\Delta\alpha_2$**)** *] ∧ [[[[$\Delta\alpha_1$ =*84*$\Delta\alpha_2$*9 *] ∨ [[$\Delta\alpha_1$ =* 85*$\Delta\alpha_2$*9 *] ∧*
[***TRUTH(*** *$\Delta\alpha_1$≠*0*)]]] ∨ [[$\Delta\alpha_1$ =* 86*$\Delta\alpha_2$*9 *] ∧ [* ***TRUTH(*** 0≤*$\Delta\alpha_1$)]]] ∨ [∃ $\Delta\alpha_3$[*
[***Decimal-biradical-number(**$\Delta\alpha_3$**)** *] ∧ [[$\Delta\alpha_1$ =Δ*8*α*2*2*Δα*3*9 *] ∨ [$\Delta\alpha_1$ =*8*Δα*2*3*Δα*3*9 *]]]]]]]]]*

The appearance of ***TRUTH*** expressions shows that it is not decidable for a given string if is a ***Decimal-biradical-number*** (which is an **alethic** [2)] and not a syntactic requirement). Notice that the ***dual-natural-number*** strings are the usual 0 1 10 11 100 101 110 111 1000 and so on.

Decimal-biradical-number strings are ***decimal-biradical*** strings that fulfill additional ***TRUTH*** conditions. In a semiconcrete calcule one has an alethic metaproperty for individuals, therefore it is written with a capital initial:

Individual:: ***Decimal-biradical-number***

This is not the place to discuss in detail. Some examples of what is a simple ***TRUTH*** should be enough:

	conventional
84119=(–11)	*-3*
828610919=((√10)+1)	*sr(2)+1*
81013858611999=(101×(/(√11)))	*5/sr(3)*

Arithmetic without biradication can be performed within ***dual-natural-number*** strings, this includes reduction of ratios of ***dual-natural-number*** strings.

(((1001+(–11))×10)+10)=1010 *(((5+(-3)).2)+2)=6*

It is not claimed that the representation of ***Decimal-biradical-number*** strings is unique. Two strings can represent the same ***Individual*** . That is the least problem anyhow, considering the undecidability problems. As stated above: undecidability is the essential problem with semiconcrete calcules. But one needs reciprocation and biradication for planar geometry and has to accept the unsatisfactory situation.

[1)] For better reading one could synonymously replace characters 2 3 4 5 6 8 9 by A M N R S { } with mnemonic for Addition, Multiplication, Negativation, Reciprocation and Square- or biradication..

[2)] 'alethic' refers to Greek αληϑεια for truth

3.9 Semiconcrete calcule PIdelta of planar Descartes E-geometry

Analytic geometry is the study of geometry using a coordinate system. **Mathematical analysis** is the branch of mathematics dealing with limits and related theories. It was shown above that Euclid geometry at first hand has nothing to do with limits. On the other hand proto-line and ortho-line have been introduced in section 2.3 already for absolute geometry. And in section 3.6 the path to coordinates for Euclid geometry has been cleared. For an E-geometry with coordinates the name **Descartes E-geometry** is chosen.

Semiconcrete calcule PIdelta of decimal planar E-geometry of Descartes-numbers is obtained from abstract calcule piepsilon of planar Euclid E-geometry by noticing that the coordinates of section 3.6 are biradical numbers. This is where the semiconcrete calcule DELTAalpha of decimal biradical numbers of section 3.8 comes into play: one takes ***Decimal-biradical-number*** strings to represent the abscissa (proto-axis or *x-axis)*, ***Decimal-biradical-number*** strings without the origin preceded by 7 to represent the ordinate (ortho-axis or *y-axis*) and pairs of ***Decimal-biradical-number*** strings without the origin separated by 7 to represent the point between the two axis.

This somewhat weird definition is chosen such that one can talk within the calcule about coordinates: by means of functions one can map every coordinate to a point on the x-axis and stay within the calcule and can produce every pair-number from two ***Decimal-biradical-number*** strings. By including the biradical numbers within the calcule PIdelta as abscissa one avoids to talk about two calcules.

Abscissa-number :: Decimal-biradical-number

$$\forall\Pi\delta_1[\,[\,\textbf{Ordinate-number}(\Pi\delta_1)\,] \leftrightarrow$$
$$[\,\exists\Pi\delta_2[\,[\,[\,\textbf{Decimal-biradical-number}(\Pi\delta_2)\,] \wedge [\,\neg\,[\;\textbf{Truth}\;(\Pi\delta_1 = 0]\,]\,]\,] \wedge [\;\Pi\delta_1 = 7\Pi\delta_2]\,]\,]\,]$$

$$\forall\Pi\delta_1[\,[\,\textbf{Pair-number}(\Pi\delta_1)\,] \leftrightarrow$$
$$[\,\exists\Pi\delta_2[\,\exists\Pi\delta_3[\,[\,[\,[\,\textbf{Decimal-biradical-number}(\Pi\delta_2)\,] \wedge [\,\neg\,[\;\textbf{Truth}\;(\Pi\delta_1 = 0]\,]\,]\,] \wedge$$
$$[\,[\,\textbf{Decimal-biradical-number}(\Pi\delta_3)\,] \wedge [\,\neg\,[\;\textbf{Truth}\;(\Pi\delta_3 = 0]\,]\,]\,]\,] \wedge\;[\;\Pi\delta_1 = \Pi\delta_2\,7\,\Pi\delta_3]\,]\,]\,]\,]$$

Descartes-number :: Individual :: Abscissa-number ¦ Ordinate-number ¦ Pair-number

The ontological basis of semiconcrete calcule PIdelta immediately shows the connection to abstract calcule piepsilon (PIdelta is a **semiconcrete model** of piepsilon) :

sort :: Πδ

individual-constant :: Πδo ¦ Πδe ¦ Πδm ¦ Πδi ¦ Πδa ¦ Πδl ¦ Πδc ¦ Πδr ¦ Πδw

basis-function-constant ::

geometry functions

appension $(\Pi\delta;\Pi\delta\oplus\Pi\delta;\Pi\delta\lfloor[{}_1\Pi\delta\neq{}_2\Pi\delta]\vee[{}_3\Pi\delta={}_4\Pi\delta])$ ¦

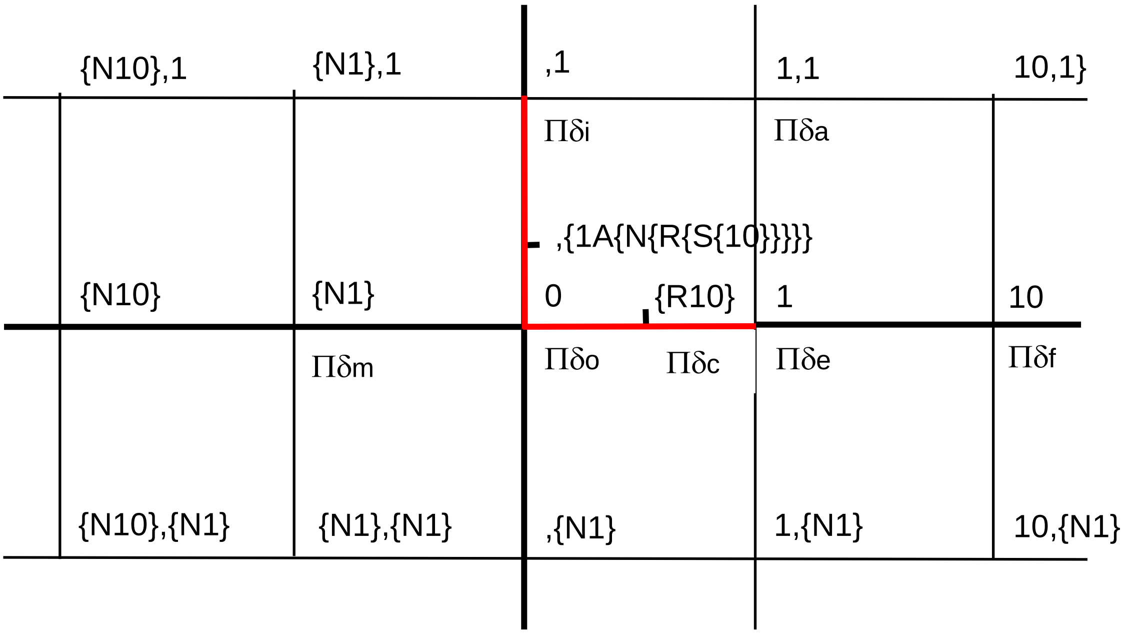
linisection $(\Pi\delta;\Pi\delta\otimes\Pi\delta;\Pi\delta\lfloor\dagger{}_1\Pi\delta;{}_2\Pi\delta;{}_3\Pi\delta;{}_4\Pi\delta)$ ¦

isoscition $(\Pi\delta;\Pi\delta\nabla\Pi\delta\lfloor[{}_1\Pi\delta;{}_2\Pi\delta\ldots\therefore{}_3\Pi\delta]\vee[[{}_1\Pi\delta={}_2\Pi\delta]\wedge[{}_1\Pi\delta={}_3\Pi\delta]])$

coordinate functions

abscission value on proto-line $(\downarrow\Pi\delta)$

ordination value on proto-line $(\Pi\delta\downarrow)$



The coordinate-functions are easy, they are defined by:

$$\forall\Pi\delta_1[[\ \textbf{Decimal-biradical-number}(\Pi\delta_1)\ ]\rightarrow$$
$$[[\textbf{TRUTH}(\ [[(\downarrow\Pi\delta_1)=0]\wedge[(\Pi\delta_1\downarrow)=\Pi\delta_1]]\wedge[[(\downarrow\Pi\delta_1)=\Pi\delta_1]\wedge[(\Pi\delta_1\downarrow)=0]]\ )]\wedge$$
$$[\ \forall\Pi\delta_1[[\textbf{Ordinate-number}(\Pi\delta_2\ ]\rightarrow[\textbf{TRUTH}(\ [(\downarrow\Pi\delta_1,\Pi\delta_2)=\Pi\delta_1]\wedge[(\Pi\delta_1,\Pi\delta_2\downarrow)=\Pi\delta_2]\ )]]]]]$$

$$\forall\Pi\delta_1[[\ \textbf{Decimal-biradical-number}(\Pi\delta_1)\ ]\rightarrow$$
$$[[\textbf{TRUTH}(\ [[[0*0=0]\wedge[[\Pi\delta_1\neq 0]\rightarrow[(\Pi\delta_1*0)=\Pi\delta_1]]]\wedge[(0*\Pi\delta_1)=7\Pi\delta_1]]\ )]\wedge$$
$$[\ \forall\Pi\delta_2[[\textbf{Ordinate-number}(\Pi\delta_2)\ ]\rightarrow[\textbf{TRUTH}(\ (\Pi\delta_1*\Pi\delta_2)=\Pi\delta_1,\Pi\delta_2)\ ]]]]]$$

Proto-linity $\subseteq\Pi\delta$ is defined $\forall\Pi\delta_1[[\textbf{TRUTH}(\subseteq\Pi\delta_1)\ ]\leftrightarrow[\textbf{Decimal-biradical-number}(\ \Pi\delta_1)\ ]]$

Notice that $\forall\Pi\delta_1[\ \ldots$ means 'for all ***Descartes-number*** strings' and $\exists\Pi\delta_1[\ \ldots$ means 'there exist a ***Descartes-number*** string' as the ***individual*** strings of calcule PIdelta are ***Descartes-number*** strings. There are yet to be defined the functions appension, linisection and isoscition in semiconcrete calcule PIdelta . It is obvious how to do it: reduce it to manipulation of biradical numbers. To this end use is made of the fact that along the proto-line points are given exactly by ***Decimal-biradical-number*** strings .

Using coordinates e.g. $(\downarrow\Pi\delta_1)$ and $(\Pi\delta_1\downarrow)$ of $\Pi\delta_1$ and doing some geometric and algebraic manipulations one develops the calculation of the ***basis-functum*** strings. Its tedious.

appension $\quad (\Pi\delta;\Pi\delta\oplus\Pi\delta;\Pi\delta\llcorner)$
notice condition $\quad [\Pi\delta_1)\neq\Pi\delta_2]\leftrightarrow[(\downarrow\Pi\delta_1)\neq(\downarrow\Pi\delta_2)]\vee[(\Pi\delta_1\downarrow)\neq(\Pi\delta_2\downarrow)]$

$$\forall\Pi\delta_1[\forall\Pi\delta_2[\forall\Pi\delta_3[\forall\Pi\delta_4[[(\downarrow\Pi\delta_1)\neq(\downarrow\Pi\delta_2)]\vee[(\Pi\delta_1\downarrow)\neq(\Pi\delta_2\downarrow)]]\rightarrow[[[$$
$$[[[(\downarrow\Pi\delta_1)<(\downarrow\Pi\delta_2)]\wedge[(\Pi\delta_1\downarrow)\leq(\Pi\delta_2\downarrow)]]\rightarrow[(\Pi\delta_1;\Pi\delta_2\oplus\Pi\delta_3;\Pi\delta_4\llcorner)= \qquad \textit{case 1}$$
$$((\downarrow\Pi\delta_2)+((((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1))/((\Pi\delta_2\downarrow)-(\downarrow\Pi\delta_2)))\times$$
$$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))+(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow)))))))),$$
$$((\Pi\delta_2\downarrow)+((((\Pi\delta_2\downarrow)-(\Pi\delta_1\downarrow))/((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1)))\times$$
$$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))+(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))))))))]]\wedge$$
$$[[[(\downarrow\Pi\delta_1)\leq(\downarrow\Pi\delta_2)]\wedge[(\Pi\delta_2\downarrow)<(\Pi\delta_1\downarrow)]]\rightarrow[(\Pi\delta_1;\Pi\delta_2\oplus\Pi\delta_3;\Pi\delta_4\llcorner)= \qquad \textit{case 2}$$
$$((\downarrow\Pi\delta_2)+((((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1))/((\Pi\delta_2\downarrow)-(\downarrow\Pi\delta_2)))\times$$
$$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))+(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow)))))))),$$
$$((\Pi\delta_2\downarrow)+((((\Pi\delta_2\downarrow)-(\Pi\delta_1\downarrow))/((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1)))\times$$
$$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))-(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow)))))))))]]]\wedge$$
$$[[[(\downarrow\Pi\delta_2)<(\downarrow\Pi\delta_1)]\wedge[(\Pi\delta_1\downarrow\leq)(\Pi\delta_2\downarrow)]]\rightarrow[(\Pi\delta_1;\Pi\delta_2\oplus\Pi\delta_3;\Pi\delta_4\llcorner)= \qquad \textit{case 3}$$

$((\downarrow\Pi\delta_2)+((((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1))/((\Pi\delta_2\downarrow)-(\downarrow\Pi\delta_2)))\times$
$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))-(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow)))))))),$
$((\Pi\delta_2\downarrow)+((((\Pi\delta_2\downarrow)-(\Pi\delta_1\downarrow))/((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1)))\times$
$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))+(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))))))))]]]\wedge$
$[[[(\downarrow\Pi\delta_2)\leq(\downarrow\Pi\delta_1)]\wedge[(\Pi\delta_2\downarrow)<(\Pi\delta_1\downarrow)]]\rightarrow[(\Pi\delta_1;\Pi\delta_2\oplus\Pi\delta_3;\Pi\delta_4\lfloor)=$ *case 4*
$((\downarrow\Pi\delta_2)+((((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1))/((\Pi\delta_2\downarrow)-(\downarrow\Pi\delta_2)))\times$
$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))-(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow)))))))),$
$((\Pi\delta_2\downarrow)+((((\Pi\delta_2\downarrow)-(\Pi\delta_1\downarrow))/((\downarrow\Pi\delta_2)-(\downarrow\Pi\delta_1)))\times$
$(\surd((((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4))\times((\downarrow\Pi\delta_3)-(\downarrow\Pi\delta_4)))-(((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))\times((\Pi\delta_3\downarrow)-(\Pi\delta_4\downarrow))))))))]]]]]]$

Linisectivity and isoscity are needed for the conditions of linisection and isoscition resp. . They are defined by similar procedure as appension by use of coordinates, geometric and algebraic manipulations. But it is boring.

linisectivity $\dashv\Pi\delta;\Pi\delta;\Pi\delta;\Pi\delta$
$\forall\Pi\delta_1[\forall\Pi\delta_2[\forall\Pi\delta_3[\forall\Pi\delta_4[[\dashv\Pi\delta_1;\Pi\delta_2;\Pi\delta_3;\Pi\delta_4]\leftrightarrow$ *get busy*

isoscity $\Pi\delta;\Pi\delta\ldots\therefore\Pi\delta$
$\forall\Pi\delta_1[\forall\Pi\delta_2[\forall\Pi\delta_3[[\Pi\delta_1;\Pi\delta_2\ldots\therefore\Pi\delta_3]\leftrightarrow$ *get busy*

linisection $(\Pi\delta;\Pi\delta\otimes\Pi\delta;\Pi\delta\lfloor\dashv_1\Pi\delta;_2\Pi\delta;_3\Pi\delta;_4\Pi\delta)$
$\forall\Pi\delta_1[\forall\Pi\delta_2[\forall\Pi\delta_3[\forall\Pi\delta_4[[\dashv\Pi\delta_1;\Pi\delta_2;\Pi\delta_3;\Pi\delta_4]\rightarrow[\ \ldots\ (\Pi\delta_1;\Pi\delta_2\otimes\Pi\delta_3;\Pi\delta_4\lfloor)=\ldots$ *get busy*

isoscition $(\Pi\delta;\Pi\delta\nabla\Pi\delta \quad \lfloor_1\Pi\delta;_2\Pi\delta\ldots\therefore_3\Pi\delta)$
$\forall\Pi\delta_1[\forall\Pi\delta_2[\forall\Pi\delta_3[[\Pi\delta_1;\Pi\delta_2\ldots\therefore\Pi\delta_3]\rightarrow[\ \ldots\ (\Pi\delta_1;\Pi\delta_2\nabla\Pi\delta_3\lfloor)=\ldots$ *get busy*

However, one must not forget, that one cannot in general effectively do the calculations. The necessary conditions cannot be checked:

- number zero equality (for reciprocation and division)
- number equality
- nonnegativity (for square root)
- point equality
- point equality of pairs of points
- sectivity of two segments
- isoscity for intersecting circles of same radius.

The numbers that are used in the calculations belong to the semiconcrete calcule DELTAalpha and that is where the undecidability and incalculability in general enters.

Now one can modify the metatheorem of section 3.7 for the two semiconcrete calcules PIdelta of planar Descartes geometry and DELTAalpha of ordered corpus of decimal biradical numbers:

biradical numbers can be semiconstructed geometrically
There is no decision procedure for equality of biradical numbers!
There is no decision procedure for vanity (equal zero) of biradical numbers!
There is no decision procedure for positivity of biradical numbers!
There is no decision procedure for equality of points in planar Descartes E-geometry.!

4. Planar L-geometry

4.1 Ontological basis of calcule pilambda of planar Lobachevsky L-geometry

The abstract calcule pilambda of **Двухмерная Л-Геометрия Лобачевского** :is obtained from abstract calcule pi with the same ontological basis, just replace π by πλ .

sort	πλ	Lobachevsky-point, Лобачевский-точка

However, the introduction of the ***Multiple-parallel-axiom*** changes the character of the abstract calcule completely. Whereas abstract calcule pi contains ***limbHOOD*** strings, i. e. ***sentence*** strings, that are neither ***TRUTH*** or ***FALSEHOOD*** strings, abstract calcule pilambda is a **complete** calcule like piepsilon., which means that every ***sentence*** is either a ***TRUTH*** or a ***FALSEHOOD*** string (for more section 4.9). It will turn out that one has to add one more ***Axiom*** . Another essential difference will be the way that one expresses cissectivity (and thence parallelity) without the use of ***entitor*** character as a mere **junctive *formula*** string, see section 4.3 . This was true in Euclid E-geometry too, although with different junctive ***formula*** strings.

4.2 Axioms of planar Lobachevsky L-geometry and inductivity

From calcule pi of N-geometry one takes ***Axiom*** strings ***A1*** to ***A20*** and ***A22*** to ***A25*** One ***Multiple-parallel-axiom A21l*** and - surprisingly - another one ***Proto-length-axiom A26*** . The existence of a parallel through a point with respect to a pair of points is a ***THEOREM*** of N-geometry. In E-geometry ***Axiom A21e*** requests that it is unique. There is more than one parallel in planar L-geometry. Negating ***A21e*** gives the ***Multiple-parallel-axiom A21l*** (character ***l*** for Lobachevsky) :

A21l multiple parallel ∀πλ1[∀πλ2[∀πλ3[∀πλ4[
[[[πλ1≠πλ2]∧[πλ3≠πλ4]]∧[πλ1;πλ2||πλ3;πλ4]]→[∃πλ5[[πλ1;πλ2||πλ3;πλ5]∧[∠πλ3;πλ4;πλ5]]]]]]]

One has yet to prove the ***THEOREM*** of L-geometry that there are many, actually infinitely many, parallels through a point with respect to a pair of points.

In addition to the nontriviality ***Axiom A25*** one has to put forward another ***Axiom*** hat requires special features for the proto-pair πλo;πλe, the so-called ***Proto-octomidial-axiom*** This procedure is characteristic for Lobachevsky planar geometry, it cannot be applied in Euclid planar geometry. The simplest choice is the requirement that the proto-length produces the octomidial equilateral triangle. The relevance and the consequence of ***A26*** is discussed in section 4.9 .

A26 proto-octomidial equilateral triangle
(πλo∇(πλo∇(πλo∇(πλo∇(πλo∇(πλo∇(πλo∇(πλo;πλe∇πλe⌊)⌊)⌊)⌊)⌊)⌊)⌊)⌊)=πλe

Just as in E-geometry an ***Axiom*** **mater** is needed to guarantee completeness:

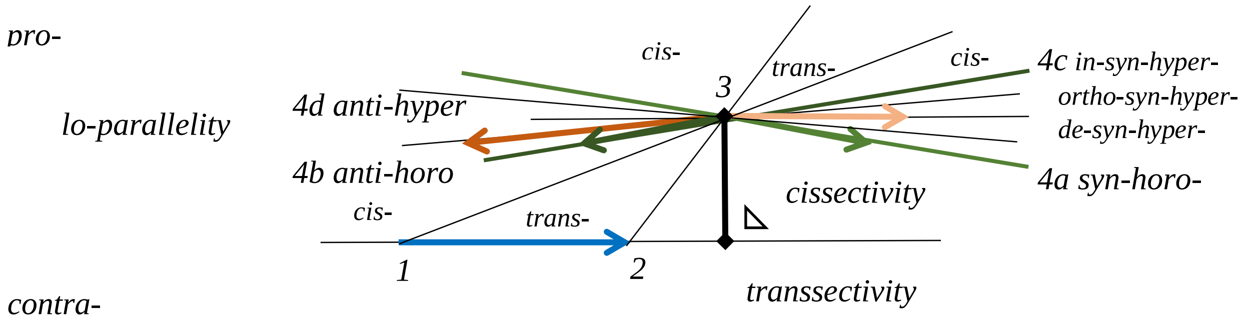
Axiom mater of inductivity ***∀πλ1[[[[***

[sentence(∀πλ1[***πλ1***]***)] ∧ [¬ [πλ1 ⊃*** πλ2 ***]] ∧ [¬ [πλ1 ⊃*** πλ3 ***]]] ∧ [¬ [πλ1 ⊃*** πλ3 ***]]] →***
[TRUTH([[[***(πλ1;***πλ1∫πλo***)***]∧[***(πλ1;***πλ1∫πλe***)***]]∧[∀πλ1[∀πλ2[∀πλ3[∀πλ4[
[[[[[[***πλ1***]∧[***(πλ1;***πλ1∫πλ2***)***]∧[***(πλ1;***πλ1∫πλ3***)***]]∧[***(πλ1;***πλ1∫πλ4***)***]]∧[πλ1≠πλ2]]→
[***(πλ1;***πλ1∫(πλ1;πλ2⊕πλ3;πλ4⌊)***)***]]∧
[[[[***πλ1***]∧[***(πλ1;***πλ1∫πλ2***)***]]∧[***(πλ1;***πλ1∫πλ3***)***]]∧[***(πλ1;***πλ1∫πλ4***)***]]∧[[[⫲πλ1;πλ2⫲πλ3;πλ4⫲]→
[***(πλ1;***πλ1∫(πλ1;πλ2⊗πλ3;πλ4⌊)***)***]]]]∧[[[[[***πλ1***]∧[***(πλ1;***πλ1∫πλ2***)***]∧[***(πλ1;***πλ1∫πλ3***)***]]∧
[***(πλ1;***πλ1∫πλ4***)***]]∧[πλ1;πλ2∴πλ3;πλ4]]→[***(πλ1;***πλ1∫(πλ1;πλ2∇πλ3;πλ4⌊)***)***]]]]]]]→[∀πλ1[***πλ1***]]***)]]***

It means: start with proto-pair πλo;πλe and generate **all** points from it by appension, linisection and circulition.

4.3 Fundamental extrafuncta

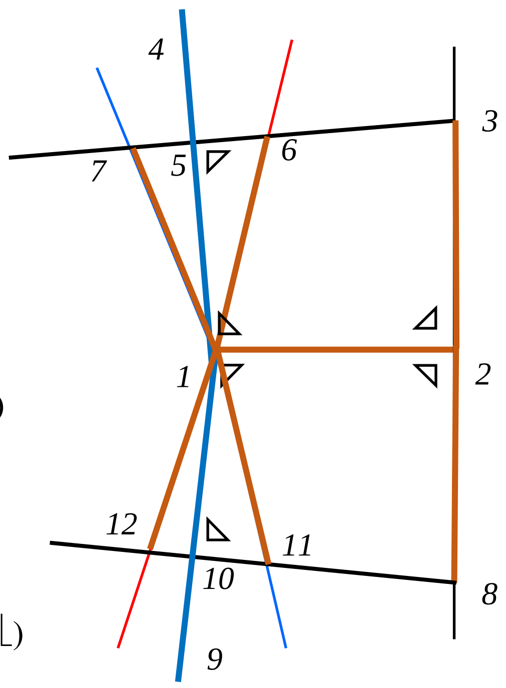
As opposed to E-geometry in L-geometry there is a whole spectrum of parallel lines thru a point πλ3 relative to a line given by πλ1;πλ2 . The point πλ3 can be either on the protive or the contrive side of the πλ1;πλ2 . The points, defining the parallel lines, have the same or the opposite direction, syn- or anti- .Between the two limiting syn- and anti-horo- cases there are the syn- and anti-hyper- cases resp. . Relative to point πλ3 one can identify ortho-hyper- (orthogonal) in-hyper- (increasing) and de-hyper- (decreasing); however, these classifications are only valid with respect to the point πλ3 as every hyper-parallel has a point with respect to which it is an ortho-hyper-parallel. For a thorough classification of two pairs of points πλ1;πλ2 an πλ3;πλ4 with respect to parallelity of the second one to the first one there are 20 cases of parallelity (besides syn- and anti-line-equality).



extra-function-constant strings in connection with the special kind of parallelity in L-geometry,

pro-horo-parallel-angulation	(⇑πλ;πλ⌊)	1πλ≠2πλ
contra-horo-parallel-angulation	(⇓πλ;πλ)	⌊1πλ≠2πλ
syn-horo-parallelation	(πλ;πλ⇑πλ⌊	∠1πλ;2πλ;3πλ
anti-horo-parallelation	(πλ;πλ⇓πλ⌊⌊)	∠1πλ;2πλ;3πλ
syn-angle-horo-parallelation	(πλ;πλ;πλ⇑⌊	∠1πλ;2πλ;3πλ
anti-angle-horo-parallelation	(πλ;πλ;πλ⇓⌊)	∠1πλ;2πλ;3πλ

<u>Construction of horo-parallel-angle</u>[1)] , determine an angle from a distance, using **Legendre-quadrates** πλ1;πλ2;πλ3;πλ5 and πλ1;πλ2;πλ8;πλ10



pro-horo-parallel-angulation

(⇑πλ;πλ⌊1πλ≠2πλ) triangle πλ1 πλ2 πλ6

πλ3=(πλ1!!πλ2⌊)
πλ4=(!πλ1;πλ2⌊)
πλ5=(πλ1;πλ4;πλ3!⌊)
πλ6=(πλ1;πλ2®⊕πλ3;πλ5⌊)

(⇑πλ1;πλ2⌊⌊)=πλ6=
(πλ1;πλ2®⊕(πλ1;πλ2!⌊);(πλ1;(!πλ1;πλ2⌊);(πλ1!!πλ2⌊)!⌊)⌊)

contra-horo-parallel-angulation

(⇓πλ;πλ⌊1πλ≠2πλ) triangle πλ1 πλ2 πλ11

(⇓πλ1;πλ2⌊)=πλ11=
(πλ1;πλ2⊕®(!πλ1;πλ2!⌊);(πλ1;(πλ2;πλ1!⌊);(!πλ1;πλ2!⌊)!⌊)⌊)

1) the usual 'parallel-angle' neglects the fact that there are two directions for horo-parallels

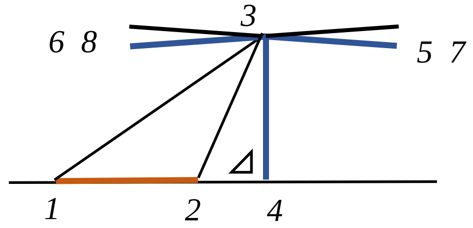
The horo-parallel-angles at πλ1 between πλ2 and πλ6=(⇑πλ1;πλ2⌊) and πλ11=(⇓πλ1;πλ2⌊) are used as limits for lo-cissectivity[1)] πλ∤πλ;πλ∤πλ and lo-parallelity π|||π—π|||π conditions. The important fact is that there is no ***entitor*** as compared to cissectivity π;π⧺π;π and parallelity π;π|—|π;π of N-geometry.

The inverse problem is **syn-angle-horo-parallelation** (πλ;πλ;πλ⇑⌊∠1πλ;2πλ;3πλ) and **anti-angle-horo-parallelation** (πλ;πλ;πλ⇓⌊) : given an angle by a triangle, find the point πλ4 on line πλ1 πλ2 with distance πλ1 πλ4 that has this angle as horo-parallel-angle.

Construction of horo-parallel through a point relative to a line

syn-horo-parallelation
(πλ;πλ⇑πλ⌊∠1πλ;2πλ;3πλ)

πλ4=(πλ1;πλ2!πλ3⌊)
πλ5=(⇑πλ3;πλ4⌊)
πλ6=(⇓πλ3;πλ4⌊)
πλ7=(πλ3;(πλ1;πλ2∇πλ3;πλ3⌊)↓πλ5↓πλ6)
πλ8=(πλ3;(πλ2;πλ1∇πλ3;πλ3⌊)↓πλ5↓πλ6)



example case pro-

(πλ1;πλ2⇑πλ3⌊)=πλ7=
(πλ3;(πλ1;πλ2∇πλ3;πλ3⌊)↓(⇑πλ3;(πλ1;πλ2!πλ3⌊)⌊)↓(⇓πλ3;(πλ1;πλ2!πλ3⌊)⌊))

anti-horo-parallelation
(πλ;πλ⇓πλ⌊∠1πλ;2πλ;3πλ)

(πλ1;πλ2⇓πλ3⌊)=πλ8=
(πλ3;(πλ2;πλ1∇πλ3;πλ3⌊)↓(⇑πλ3;(πλ1;πλ2!πλ3⌊)⌊)↓(⇓πλ3;(πλ1;πλ2!πλ3⌊)⌊))

where care is taken of both cases, pro- and contra- .

extra-relation-constant strings in connection with parallelity and linisectivity in pilambda are introduced. As it was the case in E-geometry no ***entitor*** character appears; 'lo' from '**Lo**bachevsky' .

lo-cissectivity	πλ∤πλ;πλ∤πλ		
lo-linisectivity	πλ∤πλ∤πλ∤πλ	lo-cissectivity or transsectivity	
lo-parallelity	πλ|||πλ—πλ|||πλ	poly-parallelity or line-equality	
poly-parallelity	πλ|||πλ;πλ|||πλ	horo- or hyper-parallelity	
hyper-parallelity	πλ||πλ,πλ||πλ	horo-parallelity	πλ|πλ,πλ|πλ
syn-hyper-parallelity	πλ{{πλ;πλ{{πλ	syn-horo-parallelity	πλ{πλ;πλ{πλ
anti-hyper-parallelity	πλ}}πλ;πλ}}πλ	anti-horo-parallelity	πλ}πλ;πλ}πλ
pro-hyper-parallelity	πλ((πλ;πλ((πλ	pro-horo-parallelity	πλ(πλ;πλ(πλ
contra-hyper-parallelity	πλ))πλπ;λ))πλ	contra-horo-parallelity	πλ)πλ;πλ)πλ
syn-pro-hyper-parallelity	πλ{{πλ;πλ((πλ	syn-pro-horo-parallelity	πλ{πλ;πλ(πλ
anti-pro-hyper-parallelity	πλ}}πλ;πλ((πλ	anti-pro-horo-parallelity	πλ}πλ;πλ(πλ
syn-contra-hyper-parallelity	πλ{{πλ;πλ))πλ	syn-contra-horo-parallelity	πλ{πλ;πλ)πλ
anti-contra-hyper-parallelity	πλ}}πλ;πλ))πλ	anti-contra-horo-parallelity	πλ}πλ;πλ)πλ

syn-horo-parallelity	[πλ1{πλ2;πλ3{πλ4]↔	[—πλ3;πλ4;(πλ1;πλ2⇑πλ3⌊)]
anti-horo-parallelity	[πλ1}πλ2;πλ3}πλ4]↔	[—πλ3;πλ4;(πλ1;πλ2⇓πλ3⌊)]
horo-parallelity	[πλ1|πλ2;πλ3|πλ4]↔	[[πλ1{πλ2;πλ3{πλ4]∨[πλ1}πλ2;πλ3}πλ4]

[1)] the prefix lo- is attached to avoid confusions with functions of E-geometry where necessary

syn-hyper-parallelity	[πλ1≀≀πλ2;πλ3≀≀πλ4]↔	*between the horo-parallels, same directions*
	[[[⎛πλ3;(πλ1;πλ2⇑πλ3└);πλ4]∧[⎠πλ3;(πλ1;πλ2⇓πλ3└);πλ4]∨	
	[[⎠πλ3;(πλ1;πλ2⇑πλ3└);πλ4]∧[⎛πλ3;(πλ1;πλ2⇓πλ3└);πλ4]]]	
anti-hyper-parallelity	[πλ1ʃʃπλ2;πλ3ʃʃπλ4]↔	*between the horo-parallels, opposing directions*
	[[[⎛πλ3;(πλ1;πλ2⇑πλ3└);πλ4]∧[⎠πλ3;(πλ1;πλ2⇓πλ3└);πλ4]∨	
	[[⎠πλ3;(πλ1;πλ2⇑πλ3└);πλ4]∧[⎛πλ3;(πλ1;πλ2⇓πλ3└);πλ4]]]	
hyper-parallelity	[πλ1‖πλ2;πλ3‖πλ4]↔ [[πλ1≀≀πλ2;πλ3≀≀πλ4]∨[πλ1ʃʃπλ2;πλ3ʃʃπλ4]]	
poly-parallelity	[πλ1⦀πλ2;πλ3⦀πλ4]↔[[πλ1‖πλ2;πλ3‖πλ4]∨[πλ1|πλ2;πλ3|πλ4]]	
lo-parallelity	[πλ1⦀πλ2—πλ3⦀πλ4]↔[[πλ1⦀πλ2;πλ3⦀πλ4]∨[πλ1;πλ2—πλ3;πλ4]]	

lo-cissectivity	[πλ1┤πλ2;πλ3┤πλ4]↔
	[¬[[[|πλ1;πλ2—πλ3;πλ4|]∨[πλ1;πλ2┤πλ3;πλ4]]∨[πλ1;πλ2|πλ3;πλ4]]]
lo-linisectivity	[πλ1┤πλ2┤πλ3┤πλ4]↔
	[[πλ1┤πλ2;πλ3┤πλ4]∨[πλ1,πλ2┤πλ3;πλ4]]

Although this quaternary relation is somewhat lengthy when it is expanded in order to express it by appension, circulition, linisection and decision it serves the desired purpose: there is no ***entitor*** involved, junctive logic only! This means one could start with lo-cissectivity right from the beginning in the ontological base and the ***Axiom*** strings. Just as one could replace cissectivity by junctive eu-cissectivity in E-geometry. This again shows another strong distinction between the two geometries.

4.4 Right and light triangles, polygons, triangulations

A triangle is called **light** if the sum of two angles equals the third angle (the light angle), it has light-angularity πλ≈≈πλ;πλ . In E-geometry light is right, in L-geometry light and right is never the same. The two equal angles of a liso-triangle are each less than a halfright angle.

[πλ1≈≈πλ2;πλ3]↔ [[∠πλ1;πλ2;πλ3]∧[πλ1;πλ2;πλ3≈
πλ2;(πλ3;πλ2∇(πλ1;πλ2⊕πλ1;πλ3└);(πλ1;πλ3⊕πλ1;πλ2└)└);πλ1└)]]

A light isoscelic triangle is called **liso-triangle**, it has liso-angularity πλ≈≈πλ≈πλ .

[πλ1≈≈πλ2≈πλ3]↔ [[πλ1≈≈πλ2;πλ3]∧[πλ1;πλ2≈πλ3]]

The following classifications of quadrangles are meaningful in L-geometry:

A **quadrilateral** is a quadrangle with at least two right angles

Lambert-quadrangularity	3 right angles (Spitzeck)	πλ⊥πλ⊥πλ⊥πλ
Saccheri- quadrangularity	2 right angles adjacent	πλ⊥πλ⊥πλ;πλ
Bolyai- quadrangularity	2 right angles opposite	πλ⊥πλ;πλ⊥πλ

A **quadrate** is a quadrisymmetrical with at least 2 equal sides

Lambert-quadrity	3 right angles with 2 pairs of equal sides	πλ≈πλ⊥πλ≈πλ
Saccheri-quadrity	2 right angles adjacent with 3 equal sides	πλ⊥πλ⊥πλ≈πλ
Bolyai-quadrity	2 right angles opposite with 4 equal sides	πλ⊥πλ≈πλ⊥πλ
Gauss-quadrity	4 equal angles, 4 equal sides	πλ≈πλ≈πλ≈πλ

A Gauss-quadrate is obtained by a liso-triangle combined with itself along the great side.

polygons specified by angles only:
three angles in triangularity
base angle and apex angle in isoscelic-triangularity
angle in equilateral-triangularity
angle in liso-triangularity
angle in Lambert-quadrity
angle in Saccheri-quadrity
angle in Bolyai-quadrity
angle in Gauss-quadrity

In section 4.5 the concept of Lobachevsky area will be introduced for triangles and quadrangles.

ordering equivalence relations

lo-triangle-area-equality	πλ;πλ;πλ≈~πλ;πλ;πλ
lo-triangle-area-minority	πλ;πλ;πλ⎨~πλ;πλ;πλ
lo-tri-quadrangle-area-equality	πλ;πλ;πλ≈~πλ;πλ;πλ;πλ
lo-tri-quadrangle-area-minority	πλ;πλ;πλ⎨~πλ;πλ;πλ;πλ
lo-quadrangle-area-equality	πλ;πλ;πλ;πλ≈~πλ;πλ;πλ;πλ
lo-quadrangle-area-minority	πλ;πλ;πλ;πλ⎨~πλ;πλ;πλ;πλ

Special constructions of triangles from sides and angles:

side-angle-angle-triangulation (πλ;πλ∇πλ;πλ;πλ∇πλ;πλ;πλ⌊) which if trivial in E-geometry
See 'Aufgabe 14' Perron, page 47. *Too lazy to translate into my language.*

angle-right-triangulation, riso-triangulation (∇πλ;πλ;πλ⌊)

πλ4=(!πλ1;πλ2⌊)
πλ5=(πλ1;πλ2⇈πλ3⌊)
πλ6=(πλ1;πλ4⇈πλ3⌊)
πλ7=(πλ6;πλ1⊕)
πλ8=(πλ1;πλ6⊕πλ1;πλ5⌊)
πλ9=(πλ1;πλ7⊕πλ1;πλ5⌊)
πλ10=(πλ6;πλ7∇πλ8;πλ9⌊)
πλ11=(πλ1;πλ10!⌊)
πλ12=(πλ10;πλ11⊗πλ1;πλ3⌊)=(!!πλ1;πλ2;πλ3⌊)

triangle πλ10;πλ1;πλ12 is right isoscelic , the auxiliary triangle πλ1;πλ5;πλ6 is not isoscelic.

Riso-triangulation is a special case of
angle-angle-right-triangulation (πλ;πλ;πλ∇πλ;πλ;πλ⌊)

See 'Aufgabe 11' Perron, page 46. *Too lazy to translate into my language.*

Angle-angle-right-triangulation is a special case of the Liebmann-functions
angle-angle-angle-base-triangulation (πλ;πλ;πλ∇πλ;πλ;πλ∇πλ;πλ;πλ⌊)
angle-angle-angle-tip-triangulation (πλ;πλ;πλ∇∇πλ;πλ;πλ∇πλ;πλ;πλ⌊)

which were put forward for the first time by Liebmann - it is a bit complicated. See 'Aufgabe 16' Perron, page 47. *Again too lazy to translate into my language*. Together with πλ1 the tip-function and the base-function produce a protive triangle with the three given angles in the protive sense:

⎛πλ1;(πλ1;πλ2;πλ3∇∇πλ4;πλ5;πλ6∇πλ7;πλ8;πλ9⌊);(πλ1;πλ2;πλ3∇πλ4;πλ5;πλ6∇πλ7;πλ8;πλ9⌊)

4.5 Metering, angles, areas

Angles could be treated perfectly in N-geometry with the outstanding features of angle-congruity, angle-minority, triangle-combination, and cali-meter-angulation. Angles reach from zero-angle to less than full-angle. Right, halfright, straight, acute, obtuse and oblique angles can be identified. Therefore nothing really new happens with respect to angle metering in calcule pilambda of planar L-geometry.

Of course, the proto-pair πλo;πλe that is essential in cali-angle-metrition (◊πλ;πλ;πλ′└) as obtained from N-geometry (◊π;π;π′└) , as reference is taken as πλo;πλe . It is only different in so far from the proto-pair πεo;πεe of calcule piepsilon of planar E-geometry as it cannot be taken as a unit for proto-multiplication or proto-reciprocation of E-geometry: there is no proto-multiplication or -reciprocation in L-geometry that combines with proto-addition (as in the case of rational numbers) . Proto-addition (πλ+πλ└) and proto-angle-addition (πλ°+πλ└) , however, can be directly transferred from N-geometry.

In N-geometry it is shown that the triangle-combination of three angles is less or equal to the straight angle. In E-geometry the equality holds, whereas in L-geometry the minority applies. One can therefore ask for the difference between the sum of angles of a triangle and the straight angle constructed with [1)] **angle-summation**[1)] ($\pi\lambda\Diamond\pi\lambda\oplus\pi\lambda\llcorner_1\pi\lambda\neq_2\pi\lambda$) and **angle-defection**[1)] ($\pi\lambda\Diamond\pi\lambda\Diamond\pi\lambda\llcorner_1\pi\lambda\neq_2\pi\lambda$) .

$(\lambda_1\Diamond\pi\lambda_2\oplus\pi\lambda_3\llcorner_1\pi\lambda\neq_2\pi\lambda)=$	$(\pi\lambda_1;(\pi\lambda_1;\pi\lambda_3\nabla(\pi\lambda_3;\pi\lambda_1\oplus\pi\lambda_3;\pi\lambda_2\llcorner);(\pi\lambda_1;\pi\lambda_3\oplus\pi\lambda_1;\pi\lambda_2\llcorner)\llcorner)\nabla\pi\lambda_2;$ $(\pi\lambda_1;(\pi\lambda_1;\pi\lambda_3\nabla(\pi\lambda_3;\pi\lambda_1\oplus\pi\lambda_3;\pi\lambda_2\llcorner);(\pi\lambda_1;\pi\lambda_3\oplus\pi\lambda_1;\pi\lambda_2\llcorner)\llcorner)\oplus\pi\lambda_1;\pi\lambda_3\llcorner)\llcorner)$
$(\lambda_1\Diamond\pi\lambda_2\Diamond\pi\lambda_3\llcorner_1\pi\lambda\neq_2\pi\lambda)=$	$(\pi\lambda_1;(!\pi\lambda_1;\pi\lambda_2\llcorner)\div(\pi\lambda_1\Diamond\pi\lambda_2\oplus\pi\lambda_3\llcorner)\llcorner)$

For angle-defection absolute-triangle-combination suffices, no triangle-combination is necessary as one does not exceed the straight angle. One combines the three angles and reflects the point at the perpendicular in the vertex. The angle-summation is given by the isosceles triangle $\pi\lambda_1;\pi\lambda_2;(\pi\lambda_1\Diamond\pi\lambda_2\oplus\pi\lambda_3\llcorner)$, the angle-defect is given by the isosceles triangle $\pi\lambda_1;\pi\lambda_2;(\pi\lambda_1\Diamond\pi\lambda_2\Diamond\pi\lambda_3\llcorner)$.

It took some explaining in section 3.4 to introduce the concept of area in E-geometry. It turned out that using the concept of height of a triangle one could define triangle-area-equality πε;πε;πε≈≈πε;πε;πε and triangle-area-minority πε;πε;πε⦉⦉πε;πε;πε as well as triangle-area combination (πε;πε;πε⊕♦πε;πε;πε└) so that all the usual features of intuitive geometry are fulfilled, but without extending the ontological basis.

A corresponding concept can be introduced in L-geometry. However, it turns out that the sum of angles of a triangle is the relevant feature that is used for the definition of **lo-triangle-area-equality** πλ;πλ;πλ≈~πλ;πλ;πλ and **lo-triangle-area-minority** πλ;πλ;πλ⦉~πλ;πλ;πλ. But it is closer to our intuition to use the angle-defect of a triangle, as it increases if an area is combined with another. Absolute-angle-congruity and -minority are applied:

$[\pi\lambda_1;\pi\lambda_2;\pi\lambda_3\approx\sim\pi\lambda_4;\pi\lambda_5;\pi\lambda_6]\leftrightarrow$	$[[[\pi\lambda_1\neq\pi\lambda_2]\wedge[\pi\lambda_4\neq\pi\lambda_5]]\wedge$ $[\pi\lambda_1;\pi\lambda_2;(\pi\lambda_1\Diamond\pi\lambda_2\Diamond\pi\lambda_3\llcorner)\approx\pi\lambda_4;\pi\lambda_5;(\pi\lambda_4\Diamond\pi\lambda_5\Diamond\pi\lambda_6\llcorner)]]$
$[\pi\lambda_1;\pi\lambda_2;\pi\lambda_3\prec\sim\pi\lambda_4;\pi\lambda_5;\pi\lambda_6]\leftrightarrow$	$[[[\pi\lambda_1\neq\pi\lambda_2]\wedge[\pi\lambda_4\neq\pi\lambda_5]]\wedge$ $[\pi\lambda_1;\pi\lambda_2;(\pi\lambda_1\Diamond\pi\lambda_2\Diamond\pi\lambda_3\llcorner)\prec\pi\lambda_4;\pi\lambda_5;(\pi\lambda_4\Diamond\pi\lambda_5\Diamond\pi\lambda_6\llcorner)]]$

For **lo-triangle-area-combination** (πλ;πλ;πλ◊⊕πλ;πλ;πλ└) one has to keep in mind that there is a limit to triangle area, as the combined angle must not exceed the straight angle. *I am too lazy to express the condition explicitly*. It gives the point of an isosceles triangle with base $\pi\lambda_1;\pi\lambda_2$ that consists of two triangles that each has the area of one of the two combined triangles.

$$(\pi\lambda_1;\pi\lambda_2;\pi\lambda_3\Diamond\oplus\pi\lambda_4;\pi\lambda_5;\pi\lambda_6\llcorner)=(\pi\lambda_1;(\pi\lambda_1\Diamond\pi\lambda_2\Diamond\pi\lambda_3\llcorner)\nabla(\pi\lambda_2;\pi\lambda_1\oplus\pi\lambda_4;(\pi\lambda_4\Diamond\pi\lambda_5\Diamond\pi\lambda_6\llcorner)\llcorner)$$

[1)] no corresponding functions have been included in E-geometry as they are trivially constant *pi* and *0* respectively.

The interpretation of angle-defection for area-comparing of triangles is justified by the following. Firstly one defines **levification** of a triangle (!!πλ⍮πλ⍮πλ⌊1πλ≠2πλ) that produces a light triangle with the same base and the same angle sum.

πλ4=(πλ1⌿πλ3)
πλ5=(πλ2⌿πλ3)
πλ6=(πλ4⍮πλ5⍮πλ1!⌊)
πλ7=(πλ4⍮πλ5⍮πλ2!⌊)
πλ8=(πλ4⍮πλ5⍮πλ3!⌊)
(!!πλ1;πλ2;πλ3⌊)=πλ9=(πλ1;πλ6⊕)

THEOREM	πλ1≈≈πλ2;(!!πλ1;πλ2;πλ3⌊)	light triangle
	(λ1◊πλ2⊕πλ3⌊)=(λ1◊πλ2⊕(!!πλ1;πλ2;πλ3⌊)⌊)	same angle-sum

THEOREM πλ1;πλ2;(λ1◊πλ2⊕(!!πλ1;πλ2;πλ3⌊)⌊) and πλ1;πλ2;πλ3 can be split into congruent triangles as

Proof: with triangle congruence and angle congruence

πλ3;πλ4;πλ8≅πλ1;πλ4;πλ6 πλ3;πλ5;πλ8≅πλ2;πλ5;πλ7 πλ3;πλ4;πλ8≈πλ1;πλ4;πλ6
πλ3;πλ5;πλ8≈πλ2;πλ5;πλ7 πλ3;πλ8≈πλ1;πλ6≈πλ2;πλ7

THEOREM triangles with same base and angle-sum give rise to the same light triangle

[(λ1◊πλ2◊πλ3⌊)=(λ1◊πλ2◊πλ4)]→[(!!πλ1;πλ2;πλ3⌊)=(!!πλ1;πλ2;πλ4⌊)]

THEOREM if triangles πλ1;πλ2;πλ3 and πλ4;πλ5;πλ6 have the same angle-sum their corresponding light triangles (obtained by levification) can be split into congruent triangles

Proof: thus the two light triangles, obtained by levification, share the common light angle and can be put on top of each other. One then can reason with congruent triangles.

And that is the motivation, why πλ;πλ;πλ≈~πλ;πλ;πλ and πλ;πλ;πλ⎨~πλ;πλ;πλ.are taken as lo-triangle-area-equality and lo-triangle-area-minority.

With cali-metrition (◊π;π;π′⌊) one gets the function **cali-angle-defect-metrition** (πλ′πλ◊πλ⌊) that produces a point on the proto-line between πλo and πλe . It could also be called **cali-triangle-area-metrition** . Thereby one gets a method for metering areas within calcule pilambda , no numbers are involved. Instead of numbers one uses the points of the segment πλo;πλe on the proto-line.

(πλ1′πλ2◊πλ3⌊)=(◊πλ1;πλ2;(πλ1◊πλ2◊πλ3⌊)′⌊)

Then there are the following ***THEOREM*** strings that use equality and proto-minority Defined along the proto-line, defined by proto-linity —π:

[πλ1;πλ2;πλ3≈~πλ4;πλ5;πλ6]↔ [[[[πλ1≠πλ2]∧[πλ4≠πλ5]]∧[(πλ1′πλ2◊πλ3⌊)=(πλ4′πλ5◊πλ6⌊)]]∨[[¬[∠πλ1;πλ2;πλ3]]∧[¬[∠πλ4;πλ5;πλ6]]]]

[πλ1;πλ2;πλ3⎨~πλ4;πλ5;πλ6]↔ [[[[πλ1≠πλ2]∧[πλ4≠πλ5]]∧[(πλ1′πλ2◊πλ3⌊)<(πλ4′πλ5◊πλ6⌊)]]∨[[¬[∠πλ1;πλ2;πλ3]]∧[¬[∠πλ4;πλ5;πλ6]]]]

And all is done within L-geometry, not leaving the ontological basis. No new entities 'angle' or 'area' as a 'point set of a triangle' have to be introduced, no numbers are necessary for metering, a fortiori no 'real numbers'.

4.6 Planar felix-functions of the Euclid E-geometry Klein-model

In section 2.4 ***extra-relation-constant*** cali-circlity ⊂π⊃ (for the interior of the cali-circle) has already been defined for the abstract calcule pi . Now it is shown that one can set up an abstract model of pilambda using the cali-circle of calcule piepsilon . This corresponds to what is usually known as beer-mat-geometry (in German 'Bierdeckel-Geometrie') that was invented by Felix Klein. However, now it is based on our rigorous approach. By this method one constructs an abstract model for an abstract calcule: a subcalcule of piepsilon is a model of calcule pilambda , meaning that there is an **isomorphy** in the following precise sense, where the new functa of piepsilon are yet to be defined. For the moment it will be called the Felix-model, the reason for this naming will become clear in section 4.7 . Notice that these planar functions will only be defined in some parts of the cali-circle, given by property ⊂πε⊃ that is also yet to be defined:

		extra-function-constant with asteric ∗	
(πλ;πλ⊕πλ;πλ⌊)	corresponds to	(πε;πε∗⊕πε;πε⌊)	felix-appension
(πλ;πλ⊗πλ;πλ⌊)	corresponds to	(πε;πε∗⊗πε;πε⌊)	felix-linisection
(πλ;πλ∇πλ;πλ⌊)	corresponds to	(πε;πε∗∇πε;πε⌊)	felix-circulition
(πλ;πλ•πλ⌊)	corresponds to	(πε;πε∗•πε⌊)	felix-entiration
		extra-relation-constant with a hash #	
πλ;πλ;πλ;πλ	corresponds to	#πε;πε;πε;πε	felix-appensity
πλ∤πλ∤πλ∤πλ	corresponds to	#∤πε;πε∤πε;πε∤	felix-linisectivity
πλ;πλ …∴πλ;πλ	corresponds to	#πε;πε…∴πε;πε	felix-circulity
point of pilambda	corresponds to	⊂πε⊃	cali-circlity
∀πλı[	corresponds to	∀πεı[[⊂πı⊃]→[	
∃πλı[	corresponds to	∃πεı[[⊂πı⊃]∧[	
πλo	corresponds to	πεo	
πλe	corresponds to	πεke	

One can express this exactly using metalanguage **Mencish** for the two **metacalcules** ***piepsilon*** and ***pilambda*** for the two calcules piepsilon and pilambda resp. . The technique of metalanguage is not treated in this publication.

The isomorphy is given by two metafunctions ***πεfelix(πλ)*** and ***πλxilef(πε⌊felix(ıπε))*** where the metaproperty ***felix(πε)*** that appears in the condition of the partial metafunction ***πλxilef(πε⌊)*** is defined as follows: ***felix(πε)*** is valid for those strings of piepsilon that contain only the four primary felix-functions, where all secondary functions that have been constructed by composition have to be expanded and all relations have to be replaced using proper ***formula*** strings, where only the four primary felix-functions appear.

The metafunctions ***πεfelix(πλ)*** , ***πλxilef(πε⌊))*** and the metaproperty ***felix(πε)*** can be obtained by a **limited primitive recursion** with respect to the strings of piepsilon and pilambda . The following syntactic and semantic metatheorems hold:

∀πλı[[sentence(πλı)] → [sentence(πεfelix(πλı)]]
∀πεı[[[felix(πεı)] ∧ [sentence(πεı)] → [sentence(πλxilef(πεı ⌊)]]

∀πλı[[TRUTH(πλı)] → [TRUTH(πεfelix(πλı)]]
∀πεı[[[felix(πεı)] ∧ [TRUTH(πεı)] → [TRUTH(πλxilef(πεı ⌊)]]

But it is not the place to go deeper into metalanguage use.

Together with the definition of planar **felix-functions** (marked with an asteric ∗) inside the cali-circle some **felix-relations** (marked with a hash #) have to be introduced as well. *Warning: the full expansion of some expressions can get very lengthy!*

felix-appensity	**[**#πε1;πε2;πε3;πε4**]**↔[[[[[⊂πε1⊃]∧[⊂πε2⊃]]∧[⊂πε3⊃]]∧[⊂πε4⊃]]∧ [[πε1≠πε2]∨[[πε1=πε2]∧[πε3=πε4]]]]

Cross-ratios meter 'Lobachevsky distances', where addition of distances corresponds to multiplication of cross-ratios, *one* being the neutral element. Keep this in mind as a heuristic guide-line. In section 3.6 the necessary preparations were already done. In the definition of **felix-appension** one considers two cross-ratios and requests that they are equal. It will be made use of **cali-cross-ration** (πε;πε⁄′└[#1πε]∧[#2πε]) and **cali-cross-metrition** (πε;πε′⁄└[#1πε]∧[#2πε]) of section 3.6 .

felix-appension (πε;πε∗⊕πε;πε└#1πε;2πε;3πε;4πε)
Determine the point πε15 on line πε1;πε2 so that πε2;πε15 has the same cross-ratio as πε3;πε4

πε5=(πε1;πε2®⊕πεο;πεe└)	
πε6=(πε1;πε2⊕®πεο;πεe└	
πε7=(πε3;πε4®⊕πεο;πεe└)	next to πε3
πε8=(πε3;πε4⊕®πεο;πεe└)	next toπε4
πε9=(πε7′πε3└)	for cali-pair-metrition πε3 πε4 on proto-line
πε10=πε7′πε4└)	
πε11=(πε7′πε8└)	
πε12=(πε5′πε2└)	for cali-pair-metrition πε2 πε15 on proto-line
πε13=(πε5′πε15└)	where πε15 is the desired point on line thru πε1;πε2
πε14=πε5′πε6└)	

πε13 is determined from the equality of two following cross-ratios πε16=πε17 (the equation can be solved)
πε16=((πε10×(πε11−πε9└)└)/(πε9×(πε11−πε10└)└)└) for πε3;πε4
πε17=((πε13×(πε14−πε12└)└)/(πε12×(πε14−πε13└)└)└) for πε2;πε15

solution by simple arithmetics: πε13=(((πε12×(πε14×πε16└)└)/(((πε12×πε16└)+πε14└)−πε12└)└)

Felix-appension result πε15 is given by Euclidean appension of πεο;πε13 to πε1;πε2

felix-appension	(πε1;πε2∗⊕πε3;πε4└)= (πε1;πε2⊕πεο;(((πε12×(πε14×πε16└)└)/(((πε12×πε16└)+πε14└)−πε12└)└)└)

If you wish you can successively insert for πε16 down to πε5 , however, ***it is*** *getting lengthy!*

Felix-appension induces secondary ***extra-relation*** **felix-pair-congruity** #πε1;πε2≈πε1;πε2 and secondary ***extra-function*** **felix-pair-doublition** (πε;πε∗⊕└[⊂1πε⊃]∧[⊂2πε⊃]) .

[#πε1;πε2≈πε3;πε4]↔[[[[[⊂πε1⊃]∧[⊂πε2⊃]]∧[⊂πε3⊃]]∧[⊂πε4⊃]]∧[[[πε1=πε2]∧[πε3=πε4]]∨
[[πε1≠πε2]∧[(πε1;πε2∗⊕πε1;πε2└)=(πε1;πε2∗⊕πε3;πε4└)]]]]

[[⊂πε1⊃]∧[⊂πε2⊃]]→(πε1;πε2∗⊕└)=(πε1;πε2∗⊕πε1;πε2└)]

It is much simpler to define felix-linisection, as - heuristically speaking - straight lines in L-geometry correspond to straight lines in the Felix-model. It is only necessary to slightly modify the conditions. For cissectivity one has to make sure that the result lies inside the cali-circle

felix-transsectivity	[#πε1;πε2ǂπε3;πε4]↔[[[[[⊂πε1⊃]∧[⊂πε2⊃]]∧[⊂πε3⊃]]∧[⊂πε4⊃]]∧ [πε1;πε2ǂπε3;πε4]]
felix-cissectivity	[#ǂπε1;πε2;πε3;πε4ǂ]↔[[[[[⊂πε1⊃]∧[⊂πε2⊃]]∧[⊂πε3⊃]]∧[⊂πε4⊃]] [[ǂπε1;πε2;πε3;πε4ǂ]∧[⊂(1πε;2πε⊗πε3;πε4└)⊃]]
felix-linisectivity	[#ǂπε1;πε2ǂπε3;πε4ǂ]↔[[#πε1;πε2ǂπε3;πε4]∨[#ǂπε1;πε2;πε3;πε4ǂ]]
felix-linisection	[#ǂπε1;πε2ǂπε3;πε4ǂ]→[(πε1;πε2*⊗πε3;πε4└)=(πε1;πε2⊗πε3;πε4└)]

Notice that felix-linisection has been defined before felix-circulition, whereas in top-down axiomatics it was the other way round, circulition before linisection.

For **felix-circulition** (πε;πε*∇πε;πε└) one firstly has to transpose the condition using felix-appension by simply adding an asteric to the corresponding definitions of section 2.1 and 3.1 :

felix-tria-circulity [#πε1;πε2∴πε3;πε4]↔

[[[[[[[[[[⊂πε1⊃]∧[⊂πε2⊃]]∧[⊂πε3⊃]]∧[⊂πε4⊃]]∧[πε1≠πε2]]∧[πε1≠πε3]]∧[πε2≠πε4]]∧
[(πε1;((πε2;πε1*⊕πε1;πε2└);πε1*⊕πε1;πε3└)*⊕πε2;πε4└)=
(πε1;πε2*⊕πε2;(πε1;((πε2;πε1*⊕πε1;πε2└);πε1*⊕πε1;πε3└)*⊕πε2;πε4└)└)]]∧
[(πε1;πε2*⊕πε2;πε4└)=(πε1;((πε2;πε1*⊕πε1;πε2└);πε1*⊕πε1;πε3└)*⊕((πε2;πε1*⊕πε1;πε2└);
πε1*⊕πε1;πε3└);(πε1;πε2*⊕πε2;πε4└)└)]]∧
[(πε2;πε1*⊕πε1;πε3└)=(πε2;((πε1;πε2*⊕πε1;πε2└);πε2*⊕πε2;πε4└)*⊕((πε1;πε2*⊕πε1;πε2└);πε2*⊕πε2;
πε4└);(πε2;πε1*⊕πε1;πε3└)└)]]

felix-lina-circulity [#πε1;πε2…πε3;πε4]↔
[[[[[⊂πε1⊃]∧[⊂πε2⊃]]∧[⊂πε3⊃]]∧[⊂πε4⊃]]∧
[[[((πε1;πε2*⊕πε1;πε2└)=πε1;(πε1;πε2*⊕πε2;πε4└)*⊕πε1;πε3└)]∨
[(πε1;πε2*⊕πε2;πε4└)=((πε2;πε1*⊕πε1;πε2└);πε1*⊕πε1;πε3└)]]∨
[(πε2;πε1*⊕πε1;πε3└)=((πε1;πε2*⊕πε1;πε2└);πε2*⊕πε2;πε4└)]]]

felix-circulity	[#πε1;πε2…∴πε3;πε4]↔ [[[#πε1;πε2…πε3;πε4]∨[#πε1;πε2∴πε3;πε4]]∨[[[πε1=πε2]∧[πε1=πε3]]∧[πε1=πε4]]]

felix-isoscition (πε;πε*∇πε└) is defined accordingly to produce an isosceles triangle.

A circle of <u>pilambda</u> corresponds to an ellipse in the E-geometry Felix-model. Firstly such an ellipse is described by an equation for its biradical 'coordinates' of similar to the method that was used for felix-appension. Two circles lead to two ellipses that intersect in two points. The corresponding two quadratic equations are solved in biradical numbers, which immediately give rise to the protive solution.

The simplest case to which the general problem can be transformed is a Euclidean circle and a shifted flat ellipse; from this idea one shows that there can only be two solutions, one protive and one contrive.

Find the ellipse-equation for a circle of felix-pair-congruent points πε3 with center πε1 and radius πε1;πε2 i.e. #πε1;πε2≈πε1;πε3 . Without restriction πε1 and πε2 are chosen on the proto-segment πεo≤πε1<πε2<πεe . This follows immediately from the transformations given by automorphous collineations of the unit circle that map the unit circle to itself and that map straight lines inside the unit circle to straight lines, where the cross-ratio of point pairs is conserved.

For convenience conventional notation is used for the moment. The start is a circle with radius $r<1$ around πεo, which is then translated to a circle with center at $c<1$ on πεo πεe ; u,v and x,y are coordinate pairs, that are mapped onto each other.

$u^2+v^2=r^2$	circle
$x=(c+u)/(1+uc)$	transformation (a simple automorphous collineation)
$y=(v(1+c^2)\ sr(1+c^2))/(1+uc)$	$(-1,0)$ and $(1,0)$ remain fix, $(0,1)$ moved to $(c,\ sr(1+c^2)))$
$u=(x-c)/(1-xc)$	reversed
$v=y/((1-xc)\ sr(1+c^2))$	
$((x-c)/(1-xc))^2+(y/((1-xc)\ sr(1+c^2)))^2=r^2$	by insertion
$(x-c)^2+y^2/(1+c^2)=r^2(1-xc)^2$	immediately identified as an ellipse symmetric to abscissa
$(x-c)^2-r^2(1-xc)^2+y^2/(1+c^2)=0$	bringing ellipse equation into normal form:

$x^2(1-r^2c^2)+2xc(r^2-1)+c^2-r^2+y^2/(1+c^2)=0$
$x^2(1-r^2)(1+c^2)+2xc(r^2-1)+c^2-r^2+y^2/(1+c^2)=0$
$x^2-2xc/(1+c^2)+(c^2-r^2)/((1-r^2)(1+c^2))+y^2/((1+c^2)^2(1-r^2))=0$
$(x-sr(c/(1+c^2)))^2+(c^2-r^2-c(1-r^2))/((1-r^2)(1+c^2))+y^2/((1+c^2)^2(1-r^2))=0$
$(x-sr(c/(1+c^2)))^2+y^2/((1+c^2)^2(1-r^2))=(r^2+c(1-c-r^2))/((1-r^2)(1+c^2))$

center of ellipse on abscissa	$m=sr(c/(1+c^2))$
major axis	$a=sr(r^2+c(1-c-r^2))/((1-r^2)(1+c^2)))$
minor axis	$b=sr((((1+c^2)^2(1-r^2))(r^2+c(1-c-r^2)))/((1-r^2)(1+c^2)))$

For closing in on intersecting this ellipse with a circle around the origin with radius radius $s<1$.
$x^2+y^2=s^2$ $\qquad ((x-m)/a)^2+(y/b)^2=1$

It is getting a bit boring to solve the two equations for the two solutions $x_1\ y_1$ and $x_2\ y_2$ - and - by the way to show that there are at most two solutions. But it is clear that the solutions involve only biradical numbers, as they are obtained by solving a quadratic equation. With the proto-functions of sections 2.4 and 3.6 can derive the function **felix-circulition** (πε;πε∗∇πε;πε└) if one also makes use of the automorphous collineations. The essential calculation is done in section 4.7 when determining **felix-jacition** (πε∗√πε/πε└).

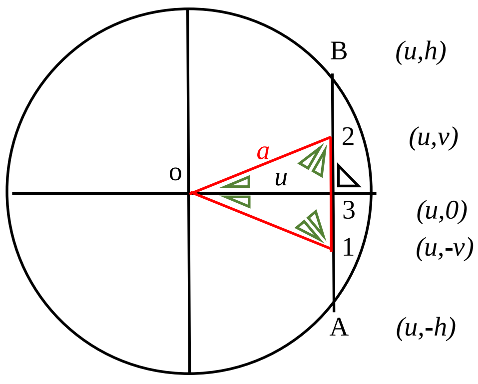
For the determination of the **klein-proto-end** πεke the **octimidial equilateral triangle** πεo , πε1 , πε2 is constructed where the cross-ratio $(h+v)/(h-v)$ of πε1,πε2 equals the cross-ratio $(a+1)/(a-1)$ of πεo ,πε1 and πεo ,πε2 , starting with parameter a that is to be determined by the octimidial condition.

$u=a.sr(2+sr(2))/2$ with quarterright angle for o13
$v=a.sr(2-sr(2))/2$
$h=sr(1-u^2)\ sr(1-a^2(2+sr(2))/4)$

inserting h and v
$(1+a)/(1-a)=(sr(1-a^2(2+sr(2))/4)+a.(2-sr(2))/2)/$
$(sr(1-a^2(2+sr(2))/4)-a.sr(2-sr(2))/2)$

quadratic equation for a^2 :
$a^4-a^2(2sr(2)-1)+(2sr(2)-2)=0$
solution $a^2=2sr(2)-2$

giving distance $a=sr(2sr(2)-2)$ $\qquad =0{,}910\ldots$



translated to proper form with proto-functions:

πεke=(√((√(πεe+└)└)−(πεe+└)└)

4.7 Linic felix-functions as proto-functions in the cali-circle

In sections 2.4 and 3.6 the so-called proto-functions have been introduced that were only defined on the proto-line given by property proto-linity ⊆π and ⊆πε resp. . The proto-linity ⊆πλ has to be mapped by the isomorphy of section 4.6 accordingly. It is then given by property proto-diametrity #πε# . Just as appension, linisection and isoscelition induced proto-functions both in N-geometry and E-geometry, one can find felix-functions starting from appension, linisection and circulation in L-geometry (they are given the new names in order not to be confused with normal arithmetic functions):

<u>primary</u>			
felix-annition	(πε*+πε⌊	[#1πε#]∧[#2πε#])	
felix-menition	(*−πε⌊	#1πε#)	
felix-evition	(πε*×πε/πε⌊	[[[≤1πε#]∧[≤2πε#]]∧[≤3πε#]]∧ [(((1πε×2πε⌊)/(2πε+πε3⌊)⌊)×⌊)<πεe]	
felix-riation	(πε*√+πε⌊	[#1πε#]∧[#2πε#])	
felix-clarition	(πε*√−πε⌊	2πε≤1πε)	condition implies proto-diametrity #πε #
felix-jacition	(πε*√πε/πε⌊	[[[[[[≤1πε#]∧[≤2πε#]]∧[≤3πε#]]∧[1πε≤(2πε*+3πε⌊)]]∧ [2πε≤(3πε*+1πε⌊)]]∧[3πε≤(1πε*+2πε⌊)]])	
<u>secondary</u>			
felix-mariation	(πε*+⌊	#1πε#)	
felix-matition	(πε*−πε⌊	[#1πε#]∧[#2πε#])	
felix-luisition	(πε*/⌊	#1πε#)	
felix-carlition	(πε*°+πε⌊	[≤1πε<]∧[≤2πε<])	conditions implies proto-diametrity #πε #
felix-rudition	(πε*°+⌊	≤1πε<)	
felix-hansition	(*°−πε⌊	≤1πε<)	
felix-petrition	(πε*°−πε⌊	[≤1πε<]∧[≤2πε<])	
felix-franition	(πε*°/⌊	≤1πε<)	

They are defined with the use of proto-functions of section 3.6 - that's the beauty!

felix-annition [[#πε1#]∧[#πε2#]]→[(πε1*+πε2⌊)=((πε1+πε2⌊)/(πεe+(πε1×πε2⌊)⌊)⌊)
felix-proto-addition conventional *(x+y)/(1+xy)*

felix-menition [#πε1#]→[(*−πε1⌊)=(−πε1)
felix-proto-negativation conventional *-x*

felix-evition [[[[[≤πε1#]∧[≤πε2#]]∧[≤πε3#]]]∧[πε2≠(−πε3)]]∧[((πε1×⌊)+(πε2×⌊))⌊)<πεe]]→
felix-proto-proportion [(πε1*×πε2/πε3⌊)=((πε2+πε3⌊);πεo↓πεo↓(πε1×πε2⌊)/(πε2+πε3⌊)⌊)
conventional *xy/(y+z)* with *y+z* not *0* and $sr(x^2+y^2)<1$

felix-riation [[#πε1#]∧[#πε2#]]→[(πε1*√+πε2⌊)=
felix-proto-pro-hypotion [√((((πε1×⌊)+((πε2×⌊)⌊)−((πε1×⌊)×(πε2×⌊)⌊)⌊)⌊)⌊)]
felix-pythagoras-a conventional $sr(x^2+y^2-x^2y^2)$

felix-clarition [[≤πε1<]∧[≤πε2<]]→[(πε1*°+πε2⌊)=
felix-proto-pro-cathetion √((((πε1×⌊)−(πε2×⌊)⌊)/(πεe −(πε2×⌊)⌊)⌊)⌊)]
felix-pythagoras-b conventional $sr((x^2-y^2)/(1-y^2))$

felix-jacition it could also be called f**elix-proto-tripedition**, determining the foot (pes), where the perpendicular from circulation point πε4 of the triangle πεo πε1 πε4 hits the proto-radius.

Using felix-jacition and felix-riation one can also determine the distance from the foot to πε4 (that can be metered by proto-ortho-metrition) and therefore do the complete construction within the cali-circle.

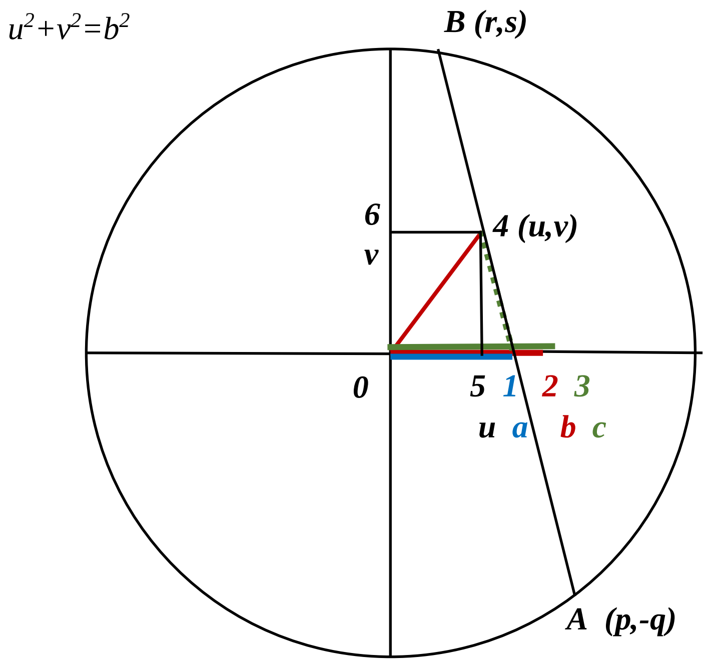
For shortness and better understanding conventional notation with Times Roman italics is used for the moment, replacing the Euclidean distances πεο,πε1 , πεο,πε2 and πεο,πε3 by a , b and c with condition $0<=a<1$, $0<=b<1$ and $0<=c<1$. The circulity condition is constructed as follows. Take the cross-ratios $\underline{a}=(a+1)/(a-1)$, $\underline{b}=(b+1)/(b-1)$ and $\underline{c}=(c+1)/(c-1)$ and express triangle-condition $\underline{c}<=\underline{a}\ \underline{b}$, $\underline{b}<=\underline{c}\ \underline{a}$ and $\underline{a}<=\underline{b}\ \underline{c}$.

$1:\ (a,0)$
$4:\ (u,v)$
$A:\ (p,-q)\quad 0<p\ \ 0<q$
$B:\ (r,s)$

$a=01$
$b=02$
$c=03$

for cross-ratio: use projection on ortho-axis
$A4=q+v$
$B4=s-v$
$A1=q$
$B1=s$

cross ratio equality:
$\underline{c}=(1+c)/(1-c)=$
$(A4.B1)/(A1.B4)=$
$s(q+v)/(q(s-v))$



Determine s and q with $0<q<=1$ and $0<s<=1$ from a and a coordinate pair u,v with $0<v$ and $u^2+v^2<1$ by intersection of unit circle $x^2+y^2=1$ and straight line thru 14 $x=y(u-a)/v+a$

intersection of circle $u^2+v^2=b^2$ and straight line to determine u
by equating cross ratios $\underline{c}=(1+c)/(1-c)$ and cross ratio $\underline{14}$ given by $(A4.B1)/(A1.B4)=s(q+v)/(q(s-v))$

$(y(u-a)/v+a)^2+y^2=1$ $\qquad (y(u-a)+av)^2+v^2y^2=v^2$ $\qquad y^2(v^2+(a-u)^2)-2y(a-u)av+v^2(a^2-1)=0$

$s,-q=((a-u)av\ \ +,-\ \ sr(((a-u)av)^2+(v^2+(a-u)^2)(v^2(1-a^2))))/(v^2+(a-u)^2)=$
$v((a-u)a\ \ +,-\ \ sr((1-a^2)v^2+(a-u)^2))/(v^2+(a-u)^2)$
$s=v((a-u)a\ +sr((1-a^2)v^2+(a-u)^2))/(v^2+(a-u)^2)$
$q=v(-(a-u)a\ +sr((1-a^2)v^2+(a-u)^2))/(v^2+(a-u)^2)$

$s+q=2v.sr((1-a^2)v^2+(a-u)^2))/(v^2+(a-u)^2)$
$s-q=2v((a-u)a/(v^2+(a-u)^2)$
$qs=v^2((1-a^2)v^2+(a-u)^2-(a-u)^2a^2)/(v^2+(a-u)^2)^2=v^2(1-a^2)/(v^2+(a-u)^2)=$

using the cross-ratio equality
$(1+c)q(s-v)=(1-c)s(q+v)$
$(1+c)(qs-qv)=(1-c)(qs+sv)$
$2cqs-v(q+s)+cv(s-q)=0$
$2c(v^2(1-a^2)/(v^2+(a-u)^2))-v(2v.sr((1-a^2)v^2+(a-u)^2))/(v^2+(a-u)^2))+cv(2v((a-u)a/(v^2+(a-u)^2))=0$
$c(1-a^2)-sr((1-a^2)v^2+(a-u)^2))+ac(a-u)=0$

eliminating square root

$c((1-a^2)+a(a-u))^2=(1-a^2)(b^2-u^2)+(a-u)^2$
$c^2((1-a^2)^2+a^2(a-u)^2+2a(1-a^2)(a-u))=(1-a^2)(b^2-u^2)+(a-u)^2$
$c^2(1+a^4-2a^2+a^4+u^2a^2-2a^3u+2a^2-2au-2a^4+2a^3u)=(1-a^2)(b^2-u^2)+(a-u)^2$

Therefore one has the quadratic equation $a^2(1-c^2)u^2-2a(1-c^2)u-c^2+b^2-a^2b^2=0$

with solution: $u_{1,2}=(a(1-c^2) +- sr((a^2(1-c^2)^2-a^2(1-c^2)(b^2-a^2b^2)))/(a^2(1-c^2))$

With a little rearrangement one gets the desired expression in conventional form $u(a,b,c)$ where the negative sign has to be taken, so that u may become negative:

$u=(1-sr((1-b^2(1-a^2))/(1-c^2)))/a$ if not $a=0$ and $u=0$ for $a=0$

Translate this conventional form back into the proper language, also taking care of case πεο :

felix-jacition [[[[[[≤πε1#]]∧[≤πε2#]]∧[≤πε3#]]∧[πε1≤(πε2*+πε3⎿)]]∧[πε2≤(πε3*+πε1⎿)]]∧ [πε3≤(πε1*+πε2⎿)]]→[(πε1*√πε2/πε3⎿)=

(πε1;πεο↓πεο↓((πεe−√(((πεe−((πε2×⎿)×(πεe−(πε1×⎿)⎿)⎿)⎿)/(πεe−(πε1×⎿)⎿)⎿)⎿)⎿)/πε1⎿))

Now one can write down the **secondary felix-functions**:

felix-mariation [#πε1#]→[(πε1*+⎿)=((πε1*+πε1⎿) =((πε1+πε1⎿)/((πεe+(πε1×πε1⎿)⎿)⎿)
felix-proto-duplication conventional $2x/(1+x^2)$

felix- matition [#πε1#]→[(πε1*+⎿)=((πε1*−πε1⎿) =((πε1*+(−πε2⎿)⎿)
felix-proto-subtraction conventional $(x-y)/(1-xy)$

By putting πε1=πε2=πε3 one gets immediately from felix-jacition:

felix-luisition [#πε1#]→[(πε1*/⎿)=((πεe−(√(πεe−πε1⎿)⎿)⎿)/(πε1+⎿)⎿)
felix-proto-bisection conventional $(1-sr(1-x^2))/x$ if not $x=0$ and 0 for $x=0$

Via proportition (πε;πε∫πε;πε⎿) a distance of Euclid E-geometry can be cut in every proportion of two numbers. This is different in Lobachevsky L-geometry. The only possible intersections are dyadic, i.e. multiples of successive dichotomitions. If trisection of a distance would be possible in L-geometry the following equation would have biradical solutions:

trisection $a=(3t+t^3)/(1+3t^2)$

However, a solution of **cubic** equation $t^3-3at^2+3t-a=0$ is not possible with biradical numbers! Trisection of distances is not possible in L-geometry! Otherwise one could transfer the result of felix-functions back to Euclid E-geometry as a solution for a cubic equation. This leads to:

THEOREM only dichotomition-produced intersections of segments (using entiration)

∃πλ1[∃πλ2[¬[∃πλ3[[(π1;(π1;π3⊕)⎿)⦃ [π1;π2]∧
[(πλ1;((πλ2⊕πλ1;πλ1;πλ3⎿)⊕πλ1;πλ1;λπ3⎿)•πλ3⎿)=((πλ2⊕πλ1;πλ1;πλ3⎿)⊕πλ1;πλ1;πλ3⎿)]]∧
[(πλ1;λπ2•πλ3⎿)=πλ2]]]]]

E.g. there is no trichotomition.

Just as for E-geometry the proto-angle-functions of section 2.7 could be directly transferred to L-geometry as they have been defined on the proto-cycle-segment in the first place. However, when mapping Lobachevsky to the cali-circle one has to notice, that the functions involved have to be replaced by their felix-counterparts and that the proto-end πλe is mapped to a value on the proto-radius πεke .
For proto-angle-bisection (πε1°/⎿)=(◊πεo;πεe;(πεo∕('◊πε1⎿))'⎿) one needed cali-pro-angulation ('◊πε⎿) , dichotomition (πε∕πε⎿) and cali-angle-metrition (◊πε;πεe;πε'⎿) and the two proto-points πεo;πεe . As the proto-origin πεo stays the same all one has to do is to replace πεe by πεke .

felix-franition [≤πε1<]→[(πε1*°/⎿)=◊πεo;πεke;(πεo∕('◊πε1⎿))'⎿)
felix-proto-angle-bisection

felix-rudition [≤πε1<]→[(πε1*°+⎿)=(◊πεo;πεke;(πεo;πεke;(◊πε1'⎿⊕◊⎿)'⎿)
felix-proto-angle-duplication

felix-carlition [[≤πε1<]∧[≤πε2<]]→
felix-proto-angle-addition (◊επo;πεke;((◊πεo;πεke;(πεo;πεke;('◊πε1⎿)⊕◊πεo;πεke;('◊πε2⎿)⎿)'⎿)⊕⎿)'⎿)

felix-hansition [[≤πε1<]→[(πε1*°−⎿)=(◊πεo;πεke;(πεo÷πεke;('◊πε1⎿)⎿)'⎿)
felix -proto-angle-negativation

felix-petrition [[≤πε1<]∧[≤πε2<]]→[(πε1*°−πε2⎿)=(πε1*°+(*°−πε2⎿)⎿)
felix-proto-angle-subtraction

Triangle-dichotomition (π∕π;π⎿[1π≠2π]∧[1π≠3π]) of N-geometry allows for dividing an angle into two equal parts, one could not do other divisions in Euclid E-geometry, nor can one do better in Lobachevsky L-geometry. Applied to proto-functions this means that there is **only dyadic intersection of angles**. This means that there is only felix-franition for intersection of angle-meters.

4.8 Felix-is-not-Klein-conjecture

All necessary planar felix-functions (section 4.7) have been defined in the cali-circle ⊂πε⊃ and all necessary linic felix-functions (section 4.8) have been defined inside the proto-diameter #πε# of E-geometry. One could stop now and claim that this is the desired isomorphic representation of planar Lobachevsky L-geometry in planar Euclid E-geometry, as all the functa have been defined explicitly and there are one-to-one corresponding ***Axiom*** strings of L-geometry and ***THEOREM*** strings of E-geometry.

However, there is a **catch**: the ***Axiom*** **mater of inductivity** of section 4.2 which requests that all points that can be constructed from the proto-endπλe by means of the three functions appension, linisection and circulition belong to the plane and only those points. What does that mean for the Felix-model ?

The cali-circle looked upon in the semiconcrete calcule PIdelta of decimal planar Descartes E-geometry (section 3.9) consists of pairs of biradical numbers of semiconcrete calcule DELTAalpha . It is easier to look at the proto-diameter of E-geometry whose points from πεm to πεe (exclusive limits) correspond bijectively to biradical numbers conventionally denoted by *-1<x<1* or in the language of section 3.9 with the somewhat strange looking ***formula*** [8419<Δα1]∧[Δα1<1] .

But this would mean that **all** biradical numbers between *-1* and *1* are included in the isomorphic model. This would include the following examples:

- the simplest case rational number *1/2*
- all rational numbers *x* with *-1<x<1*

If one looks at the primary six linic felix-functions annition (πε*⊕πε⌊) , menition (*−πε⌊) , evition (πε*×πε/πε⌊), jacition (πε*√πε/πε⌊), clarition (πε*√−πε⌊) and riation (πε*√+πε⌊) and the starting point πεke one gets very strong doubts that they all biradical points can be constructed from them. The constructible points on calcule piepsilon are called **Klein-circle-points** with **klein-circlity** ⊂⊂πε⊃⊃ , **Klein-diameter-points** with **klein-diametrity** ##πε## and the corresponding numbers in calcule deltaalpha **Klein-numbers** [1)] with the property **klein-diametrity** ##δα## . The necessary recursive [1)] definitions:

klein-circlity from planar felix-functions

[⊂⊂πε1⊃⊃]↔
[[πε1=πεke]∨[∃πε2[[⊂⊂πε2⊃⊃]∧[∃πε3[[⊂⊂πε3⊃⊃]∧[∃πε4[[⊂⊂πε4⊃⊃]∧[∃πε5[[⊂⊂πε5⊃⊃]∧
[[[[[πε2≠πε3]∨[[πε2=πε3]∧[πε4=πε5]]]∧[πε1=(πε2;πε3*⊕πε4;πε5⌊)]]∨
[[#πε2;πε3… ∴ πε4;πε5]∧[πε1=(πε2;πε3*∇πε4;πε5⌊)]]]∨
[[#⫞πε2;πε3⫞πε4;πε5⫞]∧[πε1=(πε2;πε3*⊗πε4;πε5⌊)]]]]]]]]]]]

and very simple by conjunction **klein-diametrity** [##πε1##]↔[[⊂⊂πε1⊃⊃]∧[#πε1#]] or alternatively from linic felix-functions which has the advantage that it can be directly transferred to calcule deltaalpha:

[##πε1##]↔[[πε1=πεke]∨[∃πε2[[#πε2#]∧[[πε1=(*−πε2⌊)]∨
[∃πε3[[#πε3#]∧[[[[πε1=(πε2*+πε3⌊)]∨[πε1=(πε2*√+πε3⌊)]]∨[[πε3≤πε2]∧[πε1=(πε2*√−πε3⌊)]]]∨
[∃πε4[[#πε4#]∧[[[(πε1*+πε2⌊)≠πεo]∧[πε1=(πε2*×πε3/πε4⌊)]]∨
[[[[[[[≤πε1#]∧[≤πε2#]]∧[≤πε3#]]∧[πε1≤(πε2*+πε3⌊)]]∧[πε2≤(πε3*+πε1⌊)]]∧[πε3≤(πε1*+πε2⌊)]]∧
[πε1=(πε2*√πε3/πε4⌊)]]]]]]]]]]]]

[1)] 'Klein-biradical number' may lead to the **pun** in German 'kleine biradikale Zahl' in German 'klein' means 'small' ; it also nice to talk about 'Klein-Pythagoras' or 'kleiner Pythagoras' i.e. 'small Pythagoras'

[2)] recursive definitions of relations do not contradict our self-imposed rule on defining functions by composition only

Corresponding to **Klein-number** as special biradical numbers with (−δαu)<δα1 and δα1<δαu

[##δα1##]↔[[δα1=δαke]∨[∃δα2[[#δα2#]∧[[δα1=(∗−δα2⎿)]∨
[∃δα3[[#δα3#]∧[[[[δα1=(δα2∗+δα3⎿)]∨[δα1=(δα2∗√+δα3⎿)]]∨[[δα3≤δα2]∧[δα1=(δα2∗√−δα3⎿)]]]∨
[∃δα4[[#δα4#]∧[[[(δα1∗+δα2⎿)≠δαo]∧[δα1=(δα2∗×δα3/δα4⎿)]]∨
[[[[[[[≤δα1#]∧[≤δα2#]]∧[≤δα3#]]∧[δα1≤(δα2∗+δα3⎿)]]∧[δα2≤(δα3∗+δα1⎿)]]∧[δα3≤(δα1∗+δα2⎿)]]∧
[δα1=(δα2∗√δα3/δα4⎿)]]]]]]]]]]]

The three primary planar and the six primary linic **klein-functions** of piepsilon are simply obtained by restricting the corresponding felix-functions to klein-circlity rather than cali-circlity. Secondary klein-functions can be introduced accordingly. With [[[[##πε1##]∧[##πε2##]]∧[##πε3##]]∧[##πε4##]] one has together with the conditions of the felix-functions

klein-appension	→(πε1;πε2⊕⊕πε3;πε4⎿)=(πε1;πε2∗⊕πε3;πε4⎿)
klein-linisection	→(πε1;πε2⊗⊗πε3;πε4⎿)=(πε1;πε2∗⊗πε3;πε4⎿)
klein-circulation	→(πε1;πε2∇∇πε3;πε4⎿)=(πε1;πε2∗∇πε3;πε4⎿)
klein-annition	→(πε1++πε2⎿)=(πε1∗+πε2⎿)
klein-menation	→(− −πε1⎿)=(∗−πε1⎿)
klein-evition	→(πε1××πε2//πε⎿)=(πε1∗×πε2/πε3⎿)
klein-jacition	→(πε1√√πε2//πε⎿)=(πε1∗√πε2/πε3⎿)
klein-clarition	→(πε1√√− −πε2⎿)=(πε1∗√−πε2⎿)
klein-riation	→(πε1√√++πε2⎿)=(πε1∗√+πε2⎿)

The six primary linic **klein-functions** of deltaalpha correspond to the linic felix-functions of section 4.7 with equal construction from addition, negativation, multiplication, division and biradication. Secondary functions can be introduced accordingly.

klein-annition	(δα1++δα2⎿[##1δα##]∧[##2δα##])
klein-menation	(− −δα1⎿##1δα##)
klein-evition	(δα1××δα2//δα1⎿([[##1δα##]∧[##2δα##]]∧[##3δα##]]∧[(1δα++2δα⎿)≠δαn]))
klein-riation	(δα1√√++δα2⎿[##1δα##]∧[##2δα##])
klein-clarition	(δα1√√− −δα2⎿2δα≤≤1δα) condition implies ##δα##
klein-jacition	(δα1√√δα2//δα⎿[[[##1δα##]∧[##2δα##]]∧[##3δα##]]∧[[[[[[≤1δα#]∧[≤2δα#]]∧ [≤3δα#]]∧[1δα≤(2δα++3δα⎿)]]∧[2δα≤(3δα++1δα⎿)]]∧[3δα≤(1δα++2δα⎿)]]])

It is straightforward to supplement the proto-relations of sections 2.4 by felix-properties and klein-properties by restricting them to the proto-diameter or the klein-diameter:

felix	klein		proto
#πε#	##πε##	-diametrity	#πε#
πε#<πε	πε<<πε	-minority	πε<πε
πε#≤πε	πε≤≤πε	-equal-minority	πε≤πε
#<πε	<<πε	-positivity	<πε
#≤πε	≤≤πε	-nonnegativity	≤πε
πε#<	πε<<	-negativity	πε<
πε#≤	πε≤≤	-nonpositivity	πε≤
#≤πε<	≤≤πε<<	-cyclicity	≤πε<
πε#∈	πε∈∈	-dyadicity	πε∈
#≤πε∈<	≤≤πε∈∈<<	-dyadic-cyclicity	≤πε∈<

In section 3.8 the semiconcrete calcule DELTAalpha of ***Decimal-biradical-number*** strings were defined as the ***Individual*** strings. On this basis the semiconcrete calcule PIdelta of decimal planar Descartes E-geometry was introduced in section 3.9, with ***Descartes-point*** as ***Individual*** strings. Descartes E-geometry constituted a model of Euclid E-geometry; a semiconcrete calcule as a model for an abstract calcule.

In the same way one could introduce two calcules with regard to Lobachevsky L-geometry via the ideas of the Euclidean Klein-model:

- semiconcrete calcule DELTAkappa of ***Klein-number*** strings as the ***Individual*** strings; one could use the same coding procedure as in DELTAalpha .
- semiconcrete calcule PIkappa of ***Klein-point*** strings as the ***Individual*** strings

The semiconcrete calcules DELTAkappa and PIkappa are subcalcules of the semiconcrete calcules DELTAalpha and PIalpha . ***Klein-number*** strings are ***Decimal-biradical-number*** strings, ***Klein-point*** strings are ***Descartes-point*** strings, but the very promising conjecture says that it is not so the other way round. This can be formulated in various ways for calcule piepsilon and calcule deltaalpha as well, from the general case to the simplest case:

conjectures in conventional fashion

Construct Klein-numbers starting from *sr(2sr(2)-2)* using
annition *(x+y)/(1+xy)* , menition *-x* , evition *xy/(y+z)*with $sr(x^2+y^2)<1$ and not *y+z=0* ,
riation $sr(x^2+y^2-x^2y^2)$, clarition $sr((x^2-y^2)/(1-y^2))$ with not $x^2< y^2$ and
jacition $(1-sr((1-y^2(1-x^2))/(1-z^2)))/x$ if not *x=0* else *0*

Not all biradical numbers between -1 and 1 are Klein-numbers.

1/2 is not a Klein-number.

conjectures with respect to calcule piepsilon

Not all points of the cali-circle are Klein-points.
¬[∀πε1[[⊂πε1⊃]→[##πε1##]]]

Not all points of the proto-diameter are Klein-points.
¬[∀πε1[[#πε1#]→[##πε1##]]

Proto-dimi πεc=(πεo/πεe) is not a Klein-point.
¬[##πεc##]

conjectures with respect to calcule deltaalpha

Not all biradical numbers with diametrity #δα# are Klein-numbers.
¬[∀δα1[[[[δαm< δα1]∧[δα1<δαu]]∧[δα1≠δαn]]→[##δα1##]]]

The nonvanishing rational numbers with diametrity #δα# are not Klein-numbers.
∀δα1[[[[[δαm< δα1]∧[δα1<δαu]]∧[δα1≠δαn]]∧[∈δα1]]→[¬[##δα1##]]]

Dimi δαc=(δαu/) is not a Klein-number.
¬[##δαc##]

4.9 Non-standard L-geometry

In section 4.2 the axioms of the abstract calcule pilambda of planar Lobachevsky L-geometry were introduced, including the so-called ***Proto-octomidial-axiom A26*** ('proto-octomidial equilateral triangle') without a correspondence in E-geometry. Now it is the time to justify the inclusion of this ***Axiom*** and its special choice.

One can define various L-geometries with the same axioms ***A1*** to ***A25*** but each with a different additional axiom with respect to equilateral triangles. The simplest choice is the abstract calcule pinualpha of planar **Nikolai-7** septimidial L-geometry. For its expression some definition work has to be done in pilambda:

Firstly two cali-functions are defined as special variants of the Liebmann-functions
angle-angle-angle-base-triangulation (πλ;πλ;πλ∇πλ;πλ;πλ∇πλ;πλ;πλ⎿)
angle-angle-angle-tip-triangulation (πλ;πλ;πλ∇∇πλ;πλ;πλ∇πλ;πλ;πλ⎿)

cali-liebmann-base-triangulation (πλ′∇πλ∇πλ⎿)
cali-liebmann-tip-triangulation (πλ∇′πλ∇πλ⎿)

both with the condition [[[[⎛πλo;πλe;1πλ]∧[⎛πλo;πλe;2πλ]]∧[⎛πλo;πλe;3πλ]]∧ [[⎛πλo;1πλ;2πλ]∧[⎛πλo;2πλ;3πλ]]]

which means that πλo;(πλ1′∇πλ2∇πλ3⎿);(πλ1∇′πλ2∇πλ3⎿) form a protive triangle with a base point (πλ1′∇πλ2∇πλ3⎿) on the proto-ray. The three angles are given by πλo;πλe;πλ1 and πλo;πλ1;πλ2 and πλo;πλ2;πλ3 where care is taken that their sum is less than the straight angle. The cali-liebmann-functions are defined:

(πλ1′∇πλ2∇πλ3⎿)=(πλo;πλe;πλ1∇πλo;πλ1;πλ2∇πλo;πλ2;πλ3⎿)
(πλ1∇′πλ2∇πλ3⎿)=(πλo;πλe;πλ1∇∇πλo;πλ1;πλ2∇πλo;πλ2;πλ3⎿)

The Liebmann construction has a deep meaning. In E-geometry one chooses a proto-pair of different points πεo;πεe without any requirements. Generally speaking: there are no properties of pairs of points that can be expressed in E-geometry. In L-geometry, however, pairs of points[1)] have a property that can be expressed in Funcish. As a funny consequence with respect to the geometry of the physical world Gauss has suggested to do triangular measurements to find out what kind of geometry is realized in the physical world (but he knew that there was an accuracy problem).

Assume a person can communicate with some intelligent being somewhere in the universe and he wants to tell his partner how tall[2)] he is. Assume further that so far they could not communicate about physics but perfectly about mathematics. Now there are two possibilities: if the physical geometry is

- Euclidean, there is no chance to communicate the size
- Lobachevskyan, one sends the angle-defect of the equilateral triangle to that size (more exactly: a series of dyadic angle defects, for a given accuracy)

Return to serious business. Calcule pinualpha has the same axioms ***A1*** to ***A25*** as pilambda but in addition the ***Proto-septimidial-axiom*** :

A26a	proto-septimidial equilateral triangle (πναo∇(πναo∇(πναo∇(πναo∇(πναo∇(πναo∇(πναo;πναe∇πναe⎿)⎿)⎿)⎿)⎿)⎿)⎿)=πναe

The ***Axiom*** mater of inductivity is taken directly from pilambda .That is all. But what does it mean ?

1) By the way, single points have no properties in both planar geometries without reference to the proto-pair.

2) By no means he can tell his partner that he is right-handed, they just can agree that there are two sense of direction

Every ***THEOREM*** of pilambda that does not involve the proto-octomidial triangle is also a ***THEOREM*** of pinualpha . Otherwise πναο and πναe can be used like πεο and πεe . But now one can make use of the cali-liebmann-functions in pinualpha as well and construct the **proto-octimidial-unit-point** πναee on the proto-ray for the base and one point for the tip of the proto-octimidial triangle in pinualpha . This means that every***THEOREM*** of pilambda that does involve the proto-octimidial triangle is also a ***THEOREM*** of pinualpha , one just has to replace πλe by πναee . On the other hand: There is no ***THEOREM*** of pilambda that corresponds to the ***Proto-septimidial-axiom*** . No **septisection** of angles is possible in pilambda ; this is inferred from the Klein-model: a septisection in pilambda would require septisection in E-geometry. The only possible parts of the straight angle that can be obtained in E-geometry are given as products of Fermat-primes and powers of two (seven is not among them). Therefore one has:

Every ***THEOREM*** of pilambda is a ***THEOREM*** of pinualpha

Not every ***THEOREM*** of pinualpha is a ***THEOREM*** of pilambda

These are two metatheorems, using the string-replacement metafunctions ***(πλ;*** πλ∫πνα***)*** and ***(πνα;*** πνα∫πλ***)*** one gets in proper Mencish :

$\forall \pi\lambda_1[\ \forall \pi\nu\alpha_1[\,[\,[\mathrm{THEOREM}(\pi\lambda_1)\,] \wedge [\,\pi\nu\alpha_1 = (\pi\lambda_1;\pi\lambda\int\pi\nu\alpha)\,]\,] \rightarrow [\,\mathrm{THEOREM}(\pi\nu\alpha_1)\,]\,]\,]$

$\neg[\ \forall \pi\nu\alpha_1[\ \forall \pi\lambda_1[\,[\,[\mathrm{THEOREM}(\pi\nu\alpha_1)\,] \wedge [\,\pi\lambda_1 = (\pi\nu\alpha_1;\pi\nu\alpha\int\pi\lambda)\,]\,] \rightarrow [\,\mathrm{THEOREM}(\pi\lambda_1)\,]\,]\,]\,]$

$\forall \pi\lambda_1[\,[\,\mathrm{THEOREM}(\pi\lambda_1)\,] \rightarrow [\,\mathrm{THEOREM}((\pi\lambda_1;\pi\lambda\int\pi\nu\alpha))\,]\,]$

$\neg[\ \forall \pi\nu\alpha_1[\,[\,\mathrm{THEOREM}(\pi\nu\alpha_1)\,] \rightarrow [\,\mathrm{THEOREM}((\pi\nu\alpha_1;\pi\nu\alpha\int\pi\lambda))\,]\,]\,]$

The ***Proto-septimidial-axiom*** is not the only choice for an ***Axiom*** that can replace ***Proto-octomidial-axiom*** as ***A26*** . One certainly can take all dividings of the straight angle that cannot be obtained constructively in E-geometry. This means that one can use all partings that do not correspond to products of Fermat-primes and powers of two (seven is not among them) i.e. 7, 11, 13, 19, 21 … one can write down a corresponding ***Proto-midial-axiom A26a*** , ***A26b*** , ***A26c*** , ***A26d*** , ***A26e*** … . This means that there are infinitely man nonstandard L-geometries, each with a mutually exclusive ***Proto-midial-axiom*** .

Of course this means that the calcule pibeta of **Bolyai**[1)] L-geometry that only comprises ***A1*** to ***A25*** is not **complete** , meaning that it contains ***limbHOOD sentence*** strings that are neither a ***TRUTH*** nor a ***FALSEHOOD*** because one can add the standard ***Proto-octomidial-axiom*** or any nonstandard ***Proto-midial-axiom*** as described above. It may well be that there are more nonstandard choices, perhaps the ***Proto-nonomidial-axiom*** , but this not relevant at this moment.

From all possible Lobachevsky L-geometry calcules pilambda was picked by ***Proto-length-axiom A26*** as a **standard**, as it is the simplest and smallest. Smallest in the sense that it is a genuine subcalcule of the other calcules (like the one with the septimidial angle), that one could call **nonstandard-L-geometries**.

So this chapter led to two very important results:

- L-geometry, as it is usually defined is **not complete**, there are nonstandard L-geometries
- All **nonstandard** L-geometries contain Lobachevsky L-geometry, which can be taken as **standard**

1) honoring poor, partially forgotten Bolyai; it could also be called absolute or neutral L-geometry

5. Extending F-geometry

5.1 Archimedes axiom, polygons and Fermat-primes

So far little use was made of Archimedes-ordering as it has been mentioned first in section 2.2 . The function **entiration** $(\pi;\pi\bullet\pi\lfloor)$ was chosen so that one could talk about Archimedes without actually introducing natural numbers. It seems that this was sufficient for the only purpose, i.e. the plausibility argument for metering the area of a triangle in section 3.4 .

Looking at the development of planar Euclidean geometry one observes hat it talks about polygons and especially about regular polygons. This leads to the question, how to incorporate that into the strict axiomatic approach. To make it more precise: Carl Friedrich Gauss proved the constructibility of the regular 17-gon in 1796. Five years later, he developed the theory of Gaussian periods. This theory allowed him to formulate a sufficient condition for the constructibility of regular polygons: A regular *n*-gon can be constructed with compass and straightedge if *n* is the product of a power of 2 and **Fermat primes** (none, one or many). Gauss stated without proof that this condition was also necessary, but never published his proof. A full proof of necessity was given by Pierre Wantzel in 1837. The result is known as the **Gauss–Wantzel theorem**. This is all well understood, but it is yet to be formulated in an abstract calcule piomega that meets our rigor. In the calcules that have been introduced so far there is no way to talk about numbers. This is changed by including the calcule ALPHABETA of bi-Robinson arithmetics (with dual natural power) . One has to enrich the ontological basis by natural numbers with corresponding functa and some functa that refer to both points and numbers. The calcule piepsilon is replaced by abstract calcule piomega of planar Wantzel E-geometry that includes ALPHABETA (one could do a similar extension of Lobachevsky L-geometry) in addition to Euclid E-geometry:

sort natural-number	AB		
basis-individual-constant	***nullum*** ABn	***unus*** ABu	
extra-individual-constant	***duo*** ABb	ABb=(ABu+ABu)	
basis-function-constant			
natural addition	(AB+AB)		
natural multiplication	(AB×AB)		
bi-ponentiation 2^x	(↑AB)	(↑ABn)=ABu	$(\uparrow(AB_1+ABu))=((\uparrow AB_1)\times ABb)$
natural production	$(AB\times\pi\omega;\pi\omega)$	e.g. $(3\times\pi\omega_1;\pi\omega_2)=(\pi\omega_1;(\pi\omega_1;\pi\omega_2\oplus\lfloor)\oplus\pi\omega_1;\pi\omega_2\lfloor)$	
basis-relation-constant			
minority	AB<AB		

With this enrichment of the ontological basis and corresponding trivial ***Axiom*** strings of bi-Robinson arithmetics and ***Axiom*** strings for Archimedes replacing ***A22*** , ***A23*** and ***A24*** one has the proper base. One then can do the proper definitions of ***extra-functum-constant extra-*** strings that are used in the relevant ***THEOREM*** strings for regular polygons etc. .

extra-relation-constant		
oddity	Odd(AB)	$[Odd(AB_1)]\leftrightarrow [\exists AB_2[[AB_2<AB_1]\wedge[AB_1=((AB_2+AB_2)+ABu)]]]$
primality	Prime(AB)	$[Prime(AB_1]\leftrightarrow[[ABu<AB_1]\wedge[\forall AB_2[[AB_2<AB_1]\rightarrow [\forall AB_3[[AB_3<AB_1]\rightarrow[AB_1\neq(AB_2\times AB_3)]]]]]]$
fermat-primality	Fermprime(AB)	$[Fermprime(AB_1)]\leftrightarrow [[Prime(AB_1)]\wedge[\exists AB_2[AB_1=((\uparrow(\uparrow AB_2))+ABu)]]]$

One can express the construction-condition for AB_1 as count of corners of a regular polygon as:
$[\exists AB_2[\exists AB_3[[Fermprime(AB_2)]\wedge[AB_1=(AB_2\times(\uparrow AB_3))]]]]$

5.2 Isoscition paradox

What happens if one simply replaces ***Axiom A21e*** of unique parallels in planar E-geometry calcule piepsilon by ***Axiom A21l*** of multiple parallels of planar L-geometry calcule pilambda ? Call this abstract calcule pichi .There is only isoscition (πχ;πχ∇πχ⌊) in pichi and you cannot define circulition (πχ;πχ∇πχ;πχ⌊) as the ***THEOREM*** in section 3.3 depended on the ***Axiom A21e*** of unique parallels. It seems that you can prove very little in abstract calcule pichi . And strangest of all: given three pairs of points: there is not a unique protive triangle with corresponding sides, because that is what circulition would do! There may be none or sometimes there may be many - you cannot prove the unique existence of such a triangle. It is questionable if pichi is a **complete** calcule.

5.3 Higher dimension and elliptic geometries of O

Going to higher dimensions one gets at first the calcule pipi of spatial or 3-dimensional neutral geometry, with sort ππ . The ontological basis has to be amended by a sexary function **pilation** (ππ;ππ;ππ∇ππ;ππ;ππ⌊) with a proper condition that the three spheres intersect and give a tetrahedron with positive orientation. The condition for **circulation** (ππ;ππ∇ππ;ππ⌊) has to be adapted too.

Going one step further one gets the calcule pipipi of hyperspatial or 4-dimensional neutral geometry with sort πππ . The ontological basis has to be amended by a sexary function **hyperpilation** (ππ;ππ;ππ;ππ∇ππ;ππ;ππ;ππ⌊) with a proper condition that the four hyperspheres intersect and give a tetrahedron with positive orientation. The conditions for **circulation** (πππ;πππ∇πππ;πππ⌊) and **pilation** (πππ;πππ;πππ∇πππ;πππ;πππ⌊) have to be adapted too.

The question of parallelity has to be investigated anew in every higher-dimensional geometry. The Euclidean case is straightforward. The good news is that biradical functions are sufficient for Euclidean geometries of higher dimension. No use has to be made of higher functions as shortly described in the section 5.4.

Axiom A7 'directivity appension' excludes geometries with looping lines, i.e. so-called elliptic S-geometries (S refers to sphere). One can set up these geometries as geometries of O as well. So-called elliptic geometry is a non-Euclidean-geometry where there are no parallels. It was not considered in this publication. Although it is two-dimensional it is not a planar geometry as lines are not 'unlimited'

5.4 Higher radical and algebraic numbers

Biradical numbers can be generalized to higher radical numbers in a straightforward fashion. E.g. **quinradical numbers** of calcule deltadelta with δδ contain square roots (√δδ⌊) , cubic roots (√√δδ⌊) , quadric roots (√√√√δδ⌊) and quintic roots (√√√√δδ⌊) . For **radical numbers** in general one need the inclusion of power-Robinson arithmetic into calcule deltarho with δρ and AE . Besides addition (AE+AE) and multiplication (AE×AE) power-Robinson arithmetic contains exponentiation (AE↑AE) . Unary function square root (√δα⌊) of calcule deltaalpha is replaced by binary root (√δρ;AE⌊) with base and power as arguments, that is the inverse of potentiation. It is important to notice that radical numbers are recursively defined starting from a unit δρu and successive application of the basic functions.

Radical numbers are not to be confused with **algebraic numbers**, that are given as solutions of polynomial expressions. The solutions of polynomial expressions with integer coefficients comprise more than roots, starting with degree 5 where elliptic functions become necessary as Charles Hermite has shown. One also has to enter the world of complex numbers with a calcule etaalpha of algebraic numbers. It seems that they are not relevant in the geometries of O.

5.5 Final remarks

A remark on terminology: the Greek γεωμετρια for geometry contains the words γη and μετρειν which mean earth and metering. However, the abstract calcules pi , piepsilon and pilambda are purely mathematical systems. Only by a theory of the physiacl world and its application to observations and experiments a road can be opened to physical geo-metry. Metering implies in our normal understanding a correspondence of certain physical objects with some kind of numbers. The funny thing about abstract planar F-geometry is, that it can do without any numbers: it talks about points, lines, circles, segments, triangles, angles et cetera; and no metering, neither of distances or angles or areas is necessary. Only in so-called analytical geometry (and in extended axiomatic geometry) one has numbers for all kinds of meterings, usually they are taken to be real numbers, which by the way is pretty awkward, as real numbers are far too rich. Metering is a somewhat strange procedure in geometry, it is not decidable if a distance or an angle or an area is less or equal to another one resp. as the necessary numbers are not 'decidable'.

Why investing so much time in good old geometry? The answer: it is not just a 'Glasperlenspiel' for the entertainment of a private scholar. After all, geometry is so important for our world, it is at the very basis of our understanding of the world. Whereas Kant hat thought that Euclid's geometry was an a priori notion for human understanding, since Lobachevsky one knows that in principle other geometries of the world are possible as well. Eventually Einstein taught us that the natural science of geometry cannot be based on an a priori notion of space-time either. We rather have to observe eclipses of the sun and so forth and put forward new theories without prejudging geometry etc. .

However, it seems that Einstein had an understanding that some kind of mathematics is taken **a priori.** Not the so-called Euclidean space, as he discovered different metrics both in special and general relativity. But he stuck to **real numbers** and all the transcendental **calculus jazz** that is on the base of his Riemannian geometries like on many other domains of mathematics. Perhaps it is there that future generations of physicists (and mathematicians) have to start the lever. After all quantum physics also tells us that something strange happens at fermi-distances. Perhaps the incompatibility of quantum physics and gravitational theory stems from inadequate mathematical ontology of geometrical points and physical particles, physical fields and metrics.

The so-called Euclidean space of *n* dimensions, denoted by $\boldsymbol{R}^n$ is not Euclidean in the axiomatic sense. It is a tuple-system based on real numbers using limits and more of calculus. As there is not even a semiconcrete model of **real** numbers there is a principal problem for applying $\boldsymbol{R}^3$ to the physical world.

The beauty of Lobachevsky's geometry lies in the fact that it gives rise to think about ontology with respect to mathematics in general and in reconsidering concepts of equivalence and ordering with and without the use of numbers. It helps us to remove all of our supposed a priori notions about the physical world: 'it ain't necessarily so' (*Sporting Life* in 'Porgy and Bess').

Appendix A History table and literature

	publication (*with reference number underlined: actually used*)
BC Thales of Milet 624/23-548/544	…
BC Pythagoras of Samos 570-510	…
BC Aristotle 384-322	*syllogism, actual versus potential infinity …*
BC Euclid of Alexandria ca. 360-300	*Ευκλειδης Στομχεια , Euclidis – Elementorum libri,*
	(1) Euclid Elements, book 1-6
BC Archimedes of Syracuse ca.287-212	…
Wilhelm Ockham 1288-1347	*nominalism, Ockham's razor (novacula Occami)*
Ludolph van Ceulen1540-1610	*Ludolph number pi*
René Descartes 1596-1650	*cartesian coordinates ?*
Pierre de Fermat 1607–1665	*cartesian coordinates (better fermat-coordinates)*
1794 Adrien-Marie Legendre 1752-1833	*Éléments de géométrie*
1654 Johannes Clauberg 1622-1665	*'Entia non sunt multiplicanda sine necessitate'*
1733 Girolamo Saccheri 1667–1733	*Euclides ab Omni Naevo Vindicatus*
1708 Bishop George Berkeley 1685-1753	*Philosophical Commentaries*
1710	*A Treatise Concerning the Principles of Human Knowledge*
1766 Johann Heinrich Lambert 1728-1777	…
1781 Immanuel Kant 1724-1804	*(2) Kritik der reinen Vernunft*
1826 Nikolai Ivanovich Lobachevsky 1792-1856	*Complete Collected Works*
1829–1830	*Vol. I:. Geometrical Investigations on the theory of parallel lines; on the foundation of geometry*
1835-1838	*Vol. II: New foundations of geometry with a complete theory of parallels*
1832 János Bolyai 1775-1856	*treatise on a complete system of non-Euclidean geometry, published as an appendix to a math-textbook by his father*
1854 Johann Carl Friedrich Gauss 1777-1855	*no publication concerning non-Euclidean geometry*
Pierre Wantzel 1814-1848	…
Arthur Cayley 1821-1895	…
Eugenio Beltrami1835-1900	…
Felix Klein 1880-1975	…
Henri Poincaré 1854-1912	…
Hermann Minkowski 1864-1910	…
1879 Gottlob Frege 1848 – 1925	*'Begriffsschrift' ('Concept script')*
1882 Moritz Pasch 1843-1930	*Vorlesung über Neuere Geometrie*
1891 Giuseppe Veronese 1854-1917	*Fondamenti di geometria*
1899 David Hilbert 1862-1943	*(3) Grundlagen der Geometrie 13. Auflage 1987*
1922 Ludwig Wittgenstein 1889-1951	*(4) Tractatus logico-philosophicus*
1922 Heinrich Liebmann1874-1939	*(5) Nichteuklidische Geometrie*
1926 Alfred Tarski 1901-1983	*Givant, Steven, 'Tarski's system of geometry' Bulletin of Symbolic Logic 5*: 175-214*(1999)*
1961 Max Frisch 1911-1991	*(6) Don Juan oder Die Liebe zur Geometrie*
1962 Oskar Perron 1880-1975	*(7) Nichteuklidische Elementargeometrie der Ebene*
1964 Richard Baldus 1885-1945 Frank Löbell 1893-1964	*(8) Nichteuklidische Geometrie , Hyperbolische Geometrie der Ebene*
1983 W.Schwabhäuser, W.Szmielew and A. Tarski	*(9) Metamathematische Methoden in der Geometrie*
2006 the author, *arxiv.org cs/0610038*	*(10) Church's thesis is questioned by new calculation paradigm*
forthcoming With the Calculation Criterion of Truth	*(11) Opus logico-mathematicus*
2007 Marvin Greenberg 1935-2017	*(12) Euclidean and Non-Euclidean Geometries …*

Appendix B Index of individual-, function- and relation-constants

In alphabetic order within calcules <u>pi</u> , <u>piepsilon</u> , <u>pilambda</u> , <u>piomega</u> and <u>deltaalpha</u> for ***basis-*** and ***extra-*** (for quadrangles, felix and klein-cases some have no references to sections).

There are many newly coined names of constants, some are deviant from conventional usage.

all calcules	***basis function-constant***	*comment*	*ref.*
decision	(ϕ;ϕ↓ϕ↓ϕ)	*logical projection function with* ↓	*2.1*
calcule <u>pi</u>	***basis-individual-constant***		*ref.*
proto-end	πe	*necessary reference point*	*2.1*
proto-origin	πo	*necessary reference point*	*2.1*
calcule <u>pi</u>	***basis function-constant***	⊕ ∇ ⊗ ●	*ref.*
appension	(π;π⊕π;π└)	*segment-segment-attachment*	*2.1*
circulition	(π;π∇π;π└)	*circle-circle-intersection*	*2.2*
entiration	(π;π●π└)	*maximum of multiples of a pair*	*2.1*
isoscition	(π;π∇π└)	*isosceles triangle construction*	*3.1*
linisection	(π;π⊗π;π└)	*line-line-intersection*	*2.1*
calcule <u>pi</u>	***extra-individual-constant***		*ref.*
cali-above	πa	*above unit*	*2.3*
cali-halfright	πx	*intersection of* πe;πi *and* πo;πa	*2.3*
cali-imago	πi	*unit on ortho-line, not imaginary*	*2.3*
cali-low	πl	*below unit*	*2.3*
proto-dimi	πc	*center (half) of proto-segment*	*2.3*
proto-full	πf	*double-unit*	*2.3*
proto-halfright	πw	*for halfright angle*	*2.3*
proto-minus-end	πm	*minus-unit*	*2.3*
proto-quarter	πv	*quarter of proto-segment*	*2.3*
proto-right	πr	*for right angle*	*2.3*
proto-three-half	πg	*three-half-unit*	*2.3*
calcule <u>pi</u>	***extra-function-constant***	⊕ ∇ ⊗ ® ◊ ⍀ ⇓ ⇑ ÷ ! ∪ ∩ ′ *cali* +−×/√°	*ref.*
absolute-triangle-combination	(π;π;π⊕π;π;π└)	*vertices at* 1π 4π *, no sensitivity*	*2.6*
absolute-triangle-dichot.	(⍀π;π;π└)	*absolute dichotomition of angle*	*2.6*
adipension	(⊕π;π;π;π└)	*absolute difference of two pairs*	*2.3*
angle-metrition	(π;π;π◊π;π└)	*see cali-angle-metrition*	*2.6*
angle-side-angle triangul.	(π;π;π∇π;π∇π;π;π└)	*triangle construction*	*2.5*
angle-side-side-triangul.	(π;π;π∇π;π∇π;π└)	*triangle construction*	*2.5*
anti-ortho-parallelation	(π;π⇓⇑π└)	*opposite direction*	*2.3*
anti-perculation	(!π;π└)	*right at* 1π	*2.3*
anti-resection	(π;π⊕®π;π└)	*second circle12-line34-intersection*	*2.3*
anti-riscolation	(!!π;π└)	*right-isoscelation at* 1π	*2.3*
cali-abscissation	(′π)	*abscissa on* πo πe	*2.4*
cali-adipension	(⊕π;π′└)	*absolute difference from proto-pair*	*2.4*
cali-angle-metrition	(◊π;π;π′└)	*angle-meter on proto-line*	*2.7*
cali-contra-angulation	(◊′π└)	*value on cali-circle*	*2.7*
cali-contra-cathetetion	(π!′π└)	*contrive, E-geometry for sr(abs(1-x*^{2}*))*	*2.4*
cali-contra-hypothetion	(!π;π′└)	*contrive triangle, E-geom.for sr(1+x*^{2}*)*	*2.4*

cali-cosinition	(′◊π;π⎿)	*distance cosine from* πo	2.7
cali-mabation	(″π)	*metered absolute abscissa*	2.4
cali-mabotion	(π″)	*metered absolute ordinate*	2.4
cali-meter-angulation	(◊π′⎿)	*value on cali-circle*	2.7
cali-opposition	(′π÷π)	*sort of translation of* π1 *by* πo;π2	2.7
cali-ordination	(π′)	*ordinate on* πo πe	2.4
cali-pair-metrition	(π′π)	*pair mapped on proto-line*	2.5
cali-pro-angulation	(′◊π⎿)	*value on cali-circle*	2.7
cali-pro-cathetetion	(π′!π⎿)	*protive, E-geometry for sr(abs(1-x^2))*	2.4
cali-pro-hypothetion	(′π;π!⎿)	*protive triangle, E-geom.for sr(1+x^2)*	2.4
cali-reflection	(′÷π)	*mirroring with respect to proto-line*	2.7
cali-rotation	(′π⊕◊π)	*rotation of* π1 *around* πo	2.7
cali-sinition	(◊′π;π⎿)	*distance sine from* πo	2.7
center-rotation	(π⊕◊π;π;π⎿)	*induces a rigid motion*	2.6
contra-cathetetion	(π;π!π⎿)	*hypotenuse cathetus, contrive*	2.3
contra-hypothetion	(π;π!!π⎿)	*cathetus cathetus contrive*	2.3
dispension	(π⊕π;π;π⎿)	*appending from first point*	2.3
emaxation	(π∪π;π)	*greater or equal distance*	2.2
emination	(π∩π;π)	*smaller or equal distance*	2.2
equi-triangulation	(π∇π)	*equilateral triangle*	2.3
opposition	(π÷π;π)	*rotation by straight angle around center,*	2.3
pair-dichotomition	(π⁄π)	*bisection with respect to appension*	2.3
pair-doublition	(π;π⊕)	*appending itself*	2.3
perpendiculation	(π;π;π!⎿)	*from* 3π *to* 1π 2π	2.3
pro-cathetetion	(π!π;π⎿)	*hypotenuse cathetus, protive*	2.3
pro-hypothetion	(π!!π;π⎿)	*cathetus cathetus protive*	2.3
proto- triplication	(π+⎿) , (π++⎿)	*3x*	2.6
proto-absolute-subtraction	(+π−π⎿)	*abs(x-y)*	2.6
proto-absolution	(+π⎿)	*abs(x)*	2.6
proto-addition	(π+π⎿)	*x+y*	2.6
proto-angle-addition	(π°+π⎿)	*added angles*	2.7
proto-angle-bisection	(π°/⎿)	*half angle*	2.6
proto-angle-duplication	(π°+⎿)	*double angle*	2.7
proto-angle-negativation	(°−π⎿)	*other direction*	2.7
proto-angle-subtraction	(π°−π⎿)	*subtracted angles*	2.7
proto-bisection	(π/⎿)	*x/2*	2.6
proto-cathecation	(√−π⎿)	*in Euclid sr(1-x^2)*	2.6
proto-cathetion	(π√−π⎿)	*in Euclid sr(x^2- y^2) Pythagoras-subtraction-square-root*	2.6
proto-cisition	(π×/π;π⎿)	*special proportition with cissection*	2.6
proto-duplication	(π+⎿)	*2x*	2.6
proto-hypocation	(√+π⎿)	*in Euclid sr(1+x^2),*	2.6
proto-hypotion	(π√+π⎿)	*in Euclid sr(x^2+y^2) Pythagoras-addition-square-root*	2.6
proto-negativation	(−π⎿)	*-x*	2.6
proto-subtraction	(π−π⎿)	*x-y*	2.6
proto-transition	(π/×π;π⎿)	*special proportition with transsection*	2.6
rectification	(!π;π;π⎿)	*right angle* 1π *Same height*	2.3
rectification	(!π;π;π⎿)	*right angle* 1π *Same height*	2.3
reflection	(π;π÷π⎿)	*of* 3π *at* 1π 2π	2.3

segment-rotation	(π;π⊕◊π;π;π⌊)	*induces a rigid motion, angle separate*	*2.6*
side-angle-angle-triangul.	(π;π∇π;π;π∇π;π;π⌊)	*triangle construction*	*2.5*
side-angle-side-triangul.	(π;π∇π;π;π∇π;π⌊)	*triangle construction*	*2.5*
side-side-side-triangul.	(π;π∇π;π∇π;π⌊)	*triangle construction*	*2.5*
suspension	(π;π;π⊕π⌊)	*appending other direction*	*2.3*
syn-ortho-parallelation	(π;π⇑⇓π⌊)	*same direction, translation*	*2.3*
syn-perculation	(π;π!⌊)	*right at 2π*	*2.3*
syn-resection	(π;π®⊕π;π⌊)	*circle12-line34-intersection*	*2.3*
syn-riscolation	(π;π!!⌊)	*right-isoscelation at 2π*	*2.3*
triangle-combination	(π;π;π⊕◊π;π;π⌊)	*sensitivity taken into account*	*2.6*
triangle-dichotomition	(π∤π;π⌊)	*dichotomition of angle*	*2.6*
triangle-doublition	(π;π;π⊕◊⌊)	*sensitivity taken into account*	*2.6*
calcule <u>pi</u>	***extra-relation-constant***	≈ ├ ┤ ≅ ⎠ ⎛ — ∠ ⊂ ⊃ ╡ | … ∴ *cali* ≤< ⊆#∈	*ref.*
absolute-angle-congruity	π;π;π≈π;π;π	*vertices at 1π and 4π, up to straight angle*	*2.5*
absolute-angle-congru-minority	π;π;π├ ≈π;π;π	*vertices at 1π and 4π*	*2.5*
absolute-angle-minority	π;π;π├ π;π;π	*vertices at 1π and 4π*	*2.5*
acute-angularity	Aca(π;π;π)	*vertex 1π , protive or contrive*	*2.3*
angle-congruity	π;π;π≈∠π;π;π	*vertices at 1π and 4π*	*2.5*
angle-minority	π;π;π├ ∠π;π;π	*vertices at 1π and 4π*	*2.5*
angularity	∠π;π;π—	*tri- or line-angularity*	*2.3*
anti-contra-parallelity	π;π┤⎠π;π	*one of 4 combinations*	*2.3*
anti-line-equality	π;π┤—┤π;π	*with opposite direction*	*2.3*
anti-parallelity	π;π┤┤π;π	*opposite direction, regular*	*2.3*
anti-pro-parallelity	π;π┤⎛ππ	*one of 4 combinations*	*2.3*
appensity	π;π;π;π	*line arrangement* — ├	*2.3*
betweenity (as in <u>pitau</u>)	—π;π;π—	*internity or second point equal*	*1.4*
cali-circlity	⊂π⊃	*points inside cali-circle*	*2.4*
circle-area-particity	CIA(π;π;π)	*relative 3π*	*2.5*
circle-line-particity	CIL(π;π;π)	*relative 3π*	*2.5*
circulity	π;π… ∴π;π	*two circles intersect or touch*	*2.1*
cissectivity	π;π╫π;π	*cis-position of pairs; with* ***entitor***	*2.1*
collinicity	π;π;π—	*three different points on a line*	*2.3*
concave-quadrangularity	π;π;π∠π	*usual meaning*	
contra-parallelity	π;π⎠⎠π;π	*contrivity 1π 2π 3π*	*2.3*
contrivity	⎠π;π;π	*counter-clockwise*	*2.3*
convex-quadrangularity	π∠π;π;π	*usual meaning*	
ellipse-area-particity	ELA(π;π;π;π)	*relative 4π*	*2.5*
ellipse-line-particity	ELL(π;π;π;π)	*relative 4π*	*2.5*
equilaterality	π≈π≈π	*equilateral triangle*	*2.3*
fringe-non-linisectivity	π;π|π;π	*neither parallelity nor linisectivity*	*2.3*
fringe-transsectivity	π;π╫π;π	*at most one pair, collinic or equal*	*2.3*
halfright-angularity	Hra(π;π;π)	*vertex 1π , protive or contrive*	*2.3*
hyperbola-area-particity	HYA(π;π;π;π)	*relative 4π*	*2.5*
hyperbola-line-particity	HYL(π;π;π;π)	*relative 4π*	*2.5*
internity	π—π—π	*three different points with inner point*	*2.3*
isoscelity	π;π≈π	*vertex 1π*	*2.3*
isoscity	π;π… ∴π	*same radii intersect or touch*	*2.3*
lina-circulity	π;π…π;π	*two circles touch (3 ways)*	*2.1*

line-angularity	——π;π;π	*zero- or straight-angular*	2.3
line-equality	π;π—π;π	*two pairs of points constitute same line*	2.3
linicity	—π;π;π	*two points may be the same on the line*	2.3
lini-contrivity	⎠—π;π;π	*contrive or linic*	2.3
lini-protivity	⎛—π;π;π	*protive or linic*	2.3
linisectivity	π;π⧻π;π	*transsectivity or cissectivity*	2.1
non-linisectivity	π;π⦀π;π	*parallel or fringe-non-linisectivity*	2.3
oblique-angularity	Ola(π;π;π)	*vertex* 1π *, protive or contrive*	2.3
obtuse-angularity	Ota(π;π;π)	*vertex* 1π *, protive or contrive*	2.3
pair-congruity	π;π≈π;π	*usual meaning*	2.1
pair-conminority	π;π⎨≈π;π	*pair-congruity or -minority*	2.1
pair-minority	π;π⎨π;π	*usual meaning*	2.1
parallelity	π;π⊢⊣π;π	*regular-parallelity or line-equality,* *with* ***entitor***	2.3
perpendicular-equidistancy	π;π;π≈≈π	*defines equidistant-line relative* 4π	2.5
pervex-quadrangularity	π;π∠π;π	*crossing sides*	
prima-isoscity	π;π…π	*same radii*	2.3
pro-parallelity	π;π⎛⎛π;π	*protivity* 1π 2π 3π	2.3
protivity	⎛π;π;π	*sense of orientation clockwise (convention)*	2.3
proto- nonnegativity	≤π	*proto-ray including* πo	2.4
proto-cisity	π⊆π⊆π	*cissectivity in cali-circle , points on non-negative proto-ray*	2.4
proto-cyclicity	≤π<	***cycle-segment*** πo*(included) to* πf *(excluded)*	2.4
proto-diametrity	#π#	***diameter-segment*** πm*(excluded) to* πe*(excluded)*	2.4
proto-dyadic-cyclicity	≤π∈<	*dyadic in cycle-segment*	2.4
proto-dyadicity	π∈	*recursively defined by proto-bisection, proto-addition and proto-negativation, start from* πe	2.4
proto-equal-minority	π≤π	*proto-minority or equality*	2.4
proto-linity	⊆π	*points on proto-line:* —πo;πe;π1	2.4
proto-minority	π<π	*points on proto-line* ⊆π	2.4
proto-negativity	π<	*negative-proto-ray excluding* πo	2.4
proto-nonpositivity	π≤	*negative-proto-ray including* πo	2.4
proto-positivity	<π	*proto-ray excluding* πo	2.4
proto-radity	≤π#	***radius-segment*** πo *(included) to* πe *(excluded)*	2.4
quadrangle-congruity	π;ππ;π;π≅π;π;π;π	*without sensitivity*	
quadrangle-contrivity	⎠π;π;π;π	*counter-clockwise*	
quadrangle-protivity	⎛π;π;π;π	*sense of orientation clockwise*	
quadrangle-sense-congr.	π;ππ;π;π≅≅π;π;π;π	*with sensitivity*	
quadrangularity	∠π;π;π;π	*genuine quadrangle*	
regular-parallelity	π;π‖π;π	*no common point, no line-equality, with* ***entitor***	2.3
regular-transsectivity	π;π⧺π;π	*two genuine pairs*	2.3
right-angularity	π;π⊥π	*vertex* 1π *, protive or contrive*	2.3
riso-angularity	π⊥π≈π	*vertex* 1π *, right isoscelic*	2.3
straight-angularity	π;π——π	2π—1π—3π	2.3
straight-line-particity	STL(π;π;π)	*relative* 3π	2.5
syn-contra-parallelity	π;π⎨⎠π;π	*one of 4 combinations*	2.3
syn-line-equality	π;π⎨—⎨π;π	*with same direction*	2.3
syn-parallelity	π;π⎨⎨π;π	*same direction, regular*	2.3
syn-pro-parallelity	π;π⎨⎛π;π	*one of 4 combinations*	2.3

transsectivity	π;π∤π;π	*regular or fringe, not line-equality no* ***entitor***	*2.1*
tria-circulity	π;π∴π;π	*two circles intersect*	*2.1*
tria-isoscity	π;π∴π	*same radii intersect*	*2.3*
triangle-anchor-congruity	π;π;π≅≈π;π;π	*correspondingly congruent without sensitivity*	*2.3*
triangle-anchor-sense-congruity	π;π;π∠≅π;π;π	*with sensitivity*	*2.3*
triangle-congruity	π;π;π≅π;π;π	*without sensitivity (mirror included)*	*2.3*
triangle-sense-congruity	π;π;π≅∠π;π;π	*with sensitivity*	*2.3*
tri-angularity	∠π;π;π	*protive or contrive, not linic, 3 diff. points*	*2.3*
zero-angularity	π—π;π	1π—2π—3π or 1π—3π—2π or 1π≠2π=3π	*2.3*
calcule piepsilon	***extra-individual-constant***		*ref.*
klein-proto-end	πεke	*for Klein in Euclid*	*4.6*
klein-proto-full	πεkf	*for Klein in Euclid*	*4.6*
klein-proto-halfright	πεkw	*for Klein in Euclid*	*4.6*
klein-proto-minus-end	πεkm	*for Klein in Euclid*	*4.6*
klein-proto-right	πεkr	*for Klein in Euclid*	*4.6*
calcule piepsilon	***extra-function-constant***	*additional* ♦ ∫ *	*ref.*
area-endometrition	(♦πε;πε;πε⎿)	*intrinsic, Fermat-cathetus*	*3.4*
area-exometrition	(πε;πε;πε♦πε;πε⎿)	*with respect to pair*	*3.4*
area-right-combination	(πε;πε;πε♦⊕πε;πε;πε⎿)	*ditto but same base, right angle*	*3.4*
area-triangle-combination	(πε;πε;πε⊕♦πε;πε;πε⎿)	*for triangle with combined area, same base and adjacent angle*	*3.4*
cali-anti-resection	(′πε;πε⊕®⎿)	*opposite direction*	*3.6*
cali-area-endometrition	(♦πε;πε;πε′⎿)	*mapping to proto-line*	*3.5*
cali-area-exometrition	(′πε;πε;πε♦⎿)	*special cali-proportion*	*3.5*
cali-area-quad-metrition	(πε♦πε′πε⎿)	*square thereof*	*3.6*
cali-biradication	(′πε;πε′)	*sr(abs(x))*	*3.5*
cali-cross-metrition	(πε;πε′⁄⎿)	*result on proto-radius* ≤πε⊃	*3.6*
cali-cross-ration	(πε;πε⁄′⎿)	*result on proto-ray* πεe<πε	*3.6*
cali-divix-portion	(∫πε;πε′⎿)	*special cali-proportion*	*3.5*
cali-endo-biradication	(′πε;πε⎿)	*sr(x) with 1<=x*	*3.5*
cali-eu-motion	(′πε⊕◊πε;πε÷πε)	*Euclid symmetry transformation*	*2.9*
cali-exo-biradication	(πε;πε′⎿)	*sr(x) with 0<= x<=1*	*3.5*
cali-multix-portion	(′πε;πε∫)	*special cali-proportion*	*3.5*
cali-quadrix-portion	(∫πε′)		*3.5*
cali-recix-portion	(∫πε′⎿)	*special cali-proportion*	*3.5*
cali-syn-resection	(′πε;πε®⊕⎿)	*intersects cali-circle with pair*	*3.6*
cali-tension	(′πε×πε⎿)	*stretching with center* πεo	*3.6*
felix-annition	(πε*+πε⎿)	*proto-addition*	*4.7*
felix-appension	(πε;πε*⊕πε;πε⎿)	*represents Lobachevsky appension*	*4.6*
felix-carlition	(πε*°+πε⎿)	*proto-angle-addition*	*4.7*
felix-circulition	(πε;πε*∇πε;πε⎿)	*represents Lobachevsky circulition*	*4.6*
felix-clarition	(πε*√−πε⎿)	*proto-pro-cathetion*	*4.7*
felix-dyadic-angle-metrition	(*°πε°⎿)	*recursive definition, only for* ***dyadic*** *points in cycle-segment,values between* πεo *and* πεf	*4.7*
felix-entiration	(πε;πε*●πε⎿)	*represents Lobachevsky entiration*	*4.6*
felix-evition	(πε*×πε/πε⎿)	*proto-lisition*	*4.7*
felix-franition	(πε*°/⎿)	*proto-angle-bisection*	*4.7*
felix-hansition	(*°−πε⎿)	*proto-angle-negativation*	*4.7*
felix-isoscition	(πε;πε*∇πε⎿)	*represents Lobachevsky isoscition*	*4.6*

felix-jacition	(πε*√πε/πε⌊)	*proto-tripedition*	4.7
felix-linisection	(πε;πε*⊗)πε;πε⌊)	*represents Lobachevsky linisection*	4.6
felix-luisition	(πε*/⌊)	*bisection with respect to annition*	4.7
felix-mariation	(πε*+⌊)	*duplication with respect to annition*	4.7
felix-matition	(πε*–πε⌊)	*subtraction with respect to annition*	4.7
felix-menition	(*–πε⌊)	*proto-negativation*	4.7
felix-pair-doublition	(πε;πε*⊕⌊)	*represents Lobachevsky doublition*	4.6
felix-petrition	(πε*°–πε⌊)	*proto-angle-subtraction*	4.7
felix-riation	(πε*√+πε⌊)	*proto-pro-hypotion*	4.7
felix-rudition	(πε*°+⌊)	*proto-angle-duplication*	4.7
klein-annition	(πε++πε⌊)	*proto-addition*	4.7
klein-appension	(πε;πε⊕⊕πε;πε⌊)	*represents Lobachevsky appension*	4.8
klein-carlition	(πε°++πε⌊)	*klein-angle-addition*	
klein-circulition	(πε;πε∇∇πε;πε⌊)	*represents Lobachevsky circulition*	4.8
klein-clarition	(πε√√–πε⌊)	*proto-pro-cathetion*	4.7
klein-dyadic-angle-metrition	(°°πε°°⌊)	*recursive definition, only for* ***dyadic*** *points in cycle-segment,values between* πεo *and* πεkf	
klein-entiration	(πε;πε●●πε⌊)	*represents Lobachevsky entiration*	4.8
klein-evition	(πε××πε/πε⌊)	*proto-lisition*	4.7
klein-franition	(πε°//⌊)	*klein-angle-bisection*	
klein-hansition	°– –πε⌊)	*klein-angle-negativation*	
klein-isoscition	(πε;πε∇∇πε⌊)	*represents Lobachevsky isoscition*	4.8
klein-jacition	(πε√√πε/√√ε⌊)	*proto-tripedition* $(1\text{-}sr((1\text{-}y^2(1\text{-}x^2)/(1\text{-}z^2)))/x$	4.7
klein-linisection	(πε;πε⊗⊗)πε;πε⌊)	*represents Lobachevsky linisection*	4.8
klein-luisition	(πε//⌊)	*klein-bisection*	
klein-mariation	(πε++⌊)	*klein-duplication*	
klein-matition	(πε– –πε⌊)	*klein-subtraction*	
klein-menition	(– –πε⌊)	*proto-negativation*	4.7
klein-pair-doublition	(πε;πε⊕⊕⌊)	*represents Lobachevsky doublition*	4.8
klein-petrition	(πε°– –πε⌊)	*klein-angle-subtraction*	
klein-riation	(πε√√++πε⌊)	*proto-pro-hypotion*	4.7
klein-rudition	(πε°++⌊)	*klein-angle-duplication*	
locustition	(πε;πε∇πε;πε⌊)	***form-different from N-geometry***	3.3
proportition	(πε;πε∫πε;πε⌊)	*cross-multiplication*	3.3
proto-biradication	(√πε⌊)	*sr(x) square root*	3.6
proto-cross-ration	(πε;πε×/×πε;πε⌊)	*((u-y)(v-x))/((v-y)(u-x))*	3.6
proto-division	(πε/πε⌊)	*x/y*	3.6
proto-dyadic-angle-metrition	(°πε°⌊)	*recursive definition, only for* ***dyadic*** *points in cycle-segment, values between* πεo *and* πεf	3.6
proto-multiplication	(πε×πε⌊)	*x.y*	3.6
proto-ortho-immetrition	(πε1√–×πε2⌊)	$y.sr(1\text{-}x^2)$ *in open -1 to 1*	3.6
proto-ortho-metrition	(πε1√–/πε2⌊)	$y/sr(1\text{-}x^2)$ *in open -1 to 1*	3.6
proto-para-immetrition	(πε1–/+πε2⌊)	$y(1\text{-}x^2)/(1+xy)$ *in open -1 to 1*	3.6
proto-para-metrition	(πε1–+/πε2⌊)	*y/(1-x(x+y)) in open -1 to 1*	3.6
proto-quadration, -cubation	(πε×⌊) , (πε××⌊)	x^2 , x^3	3.6
proto-reciprocation	(/πε⌊)	*1/x*	3.6
proto-reduction	(πε–/+⌊)	*(x-1)/(x+1)with x not -1*	3.6
proto-upduction	(πε+/–⌊)	*(x+1)/(x-1)with x not 1*	3.6
square-division	(πε;πε∫πε⌊)	*special case of cross-multiplication*	3.3

calcule piepsilon	***extra-relation-constant***		*ref.*
anti-contra-mono-parallelity	⎬πε;πε;πε;πε⎠	*one of 4 combinations*	*3.3*
anti-mono-parallelity	⎬πε;πε;πε;πε⎬	*opposite direction*	*3.3*
anti-pro-mono-parallelity	⎬πε;πε;πεπε⎛	*one of 4 combinations*	*3.3*
contra-mono-parallelity	⎠πε;πε;πε;πε⎠	*contrivity* 1πε 2πε 3πε	*3.3*
eu-cissectivity	┤πε;πε;πε;πε┤	*eu-linisectivity, not transsectivity, no* **entitor**	*3.3*
eu-linisectivity	┤πε;πε┤πε;πε┤	*no eu-parallelity or fringe-non-linisectivity, no* **entitor**	*3.3*
eu-parallelity	∣πε;πε—πε;πε∣	*mono-parallelity or line-equality, no* **entitor**	*3.3*
felix-appensity	#πε;πε;πε;πε	*represents Lobachevsky-appensity*	*4.6*
felix-circulity	#πε;πε… ∴πε;πε	*represents Lobachevsky-circulity*	*4.6*
felix-cissectivity	#┤πε;πε;πε;πε┤	*represents Lobachevsky-cissectivity*	*4.6*
felix-cyclicity	#≤πε<	***cycle-segment*** πεo *(included) to* πεkf *(excluded);*	
felix-dyadic-cyclicity	#≤πε∈<	*felix-dyadic in cycle-segment,*	
felix-dyadicity	πε#∈	*recursively by felix-annition, -menition, -luisition from* πεke	
felix-equal-minority	πε#≤πε	*represents Lobachevsky-equ.-minority*	
felix-isoscelity	#πε;πε… ∴πε	*represents Lobachevsky-isoscelity*	*4.6*
felix-lina-circulity	#πε;πε…πε;πε	*represents Lobachevsky-lina-circulity*	*4.6*
felix-linisectivity	#┤πε;πε┤πε;πε┤	*represents Lobachevsky-linisectivity*	*4.6*
felix-minority	πε#<πε	*represents Lobachevsky-minority*	
felix-negativity	πε#<	*represents Lobachevsky-negativity*	
felix-nonnegativity	#≤πε	*represents Lobachevsky-nonnegativity*	
felix-nonpositivity	πε#≤	*represents Lobachevsky-nonpositivity*	
felix-pair-congruity	#πε;πε≈πε;πε	*represents Lobachevsky-pair-congr.*	*4.6*
felix-positivity	#<πε	*represents Lobachevsky-positivity*	
felix-radity	#≤πε#	***radius-segment*** πεo *(included) to* πεke*(excluded);*	
felix-transsectivity	#πε;πε┤πε;πε	*represents Lobachevsky-transsectivity*	*4.6*
felix-tria-circulity	#πε;πε∴πε;πε	*represents Lobachevsky-tria-circulity*	*4.6*
klein-appensity	#πε;πε;πε;πε	*represents Lobachevsky-appensity*	
klein-circlity	⊂⊂πε⊃⊃	*Klein points inside Euclid cali-circle, recursively defined*	
klein-circulity	#πε;πε… ∴πε;πε	*represents Lobachevsky-circulity*	
klein-cissectivity	#┤πε;πε;πε;πε┤	*represents Lobachevsky-cissectivity*	
klein-congruity	#πε;πε≈πε;πε	*represents Lobachevsky-pair-congr.*	
klein-diametrity	##πε##	*Klein points inside Euclid diameter-segment recursively defined*	
klein-isoscelity	#πε;πε… ∴πε	*represents Lobachevsky-isoscelity*	
klein-lina-circulity	#πε;πε…πε;πε	*represents Lobachevsky-lina-circulity*	
klein-linisectivity	#┤πε;πε┤πε;πε┤	*represents Lobachevsky-linisectivity*	
klein-transsectivity	#πε;πε┤πε;πε	*represents Lobachevsky-transsectivity*	
klein-tria-circulity	#πε;πε∴πε;πε	*represents Lobachevsky-tria-circulity*	
mono-parallelity (eu-regular-)	∣πε;πε;πε;πε∣	*no common point, no line-equality, no* **entitor**	*3.3*
pro-mono-parallelity	⎛πε;πε;πε;πε⎛	*protivity* 1πε 2πε 3πε	*3.3*
proto-klein-cyclicity	#≤πε<	***Klein-cycle-segment*** πεo *(included) to* πεkf *(excluded);*	*4.8*
proto-klein-dyadic-cyclicity	#≤πε∈<	*klein-dyadic in cycle-segment*	*4.8*

proto-klein-dyadicity	#πε∈	*recursively defined by klein-bisection, klein-addition and klein-negativation from* πεke	*4.8*
proto-klein-equal-minority	πε#≤πε	*analogously*	*4.8*
proto-klein-minority	πε#<πε	[[#πε1#]∧[#πε2#]]∧[πε1<πε2]	*4.8*
proto-klein-negativity	πε#<	*analogously*	*4.8*
proto-klein-nonnegativity	#≤πε	*analogously*	*4.8*
proto-klein-nonpositivity	πε#≤	*analogously*	*4.8*
proto-klein-positivity	#<πε	*analogously*	*4.8*
proto-klein-radity	#≤πε⊃	***Klein-radius-segment*** πεo *(included) to* πεke*(excluded);*	*4.8*
quadrangle-area-equality	πε;πε;πε;πε≈≈πε;πε;πε;πε	*usual meaning*	
quadrangle-area-minority	πε;πε;πε;πε⎨⎨πε;πε;πε;πε	*usual meaning*	
rectangularity	πε⊥πε;πε;πε	*rectangle*	
squarity	πε⊥πε;πε≈πε	*square*	
syn-contra-mono-parallelity	⎨πε;πε;πε;πε⎠	*one of 4 combinations*	*3.3*
syn-mono-parallelity	⎨πε;πε;πε;πε⎨	*same direction*	*3.3*
syn-pro-mono-parallelity	⎨πε;πε;πε;πε⎛	*one of 4 combinations*	*3.3*
triangle-anchor-sense-similarity	πε;πεπε∠~πε;πε;πε	*with sensitivity*	
triangle-anchor-similarity	πε;πεπε~≈πε;πε;πε	*correspondingly similar without sensitivity*	
triangle-area-equality	πε;πε;πε≈≈πε;πε;πε	*usual meaning*	
triangle-area-minority	πε;πε;πε⎨⎨πε;πε;πε	*usual meaning*	
triangle-sense-similarity	πε;πεπε~∠πε;πε;πε	*with sensitivity*	
triangle-similarity	πε;πεπε~πε;πε;πε	*without sensitivity (mirror included)*	
tri-quadrangle-area-equality	πε;πε;πε≈≈πε;πε;πε;πε	*usual meaning*	
tri-quadrangle-area-minority	πε;πε;πε⎨⎨πε;πε;πε;πε	*usual meaning*	
calcule <u>pilambda</u>	***extra-function-constant***		*ref.*
angle-angle-angle-base-triangulation	(πλ;πλ;πλ∇πλ;πλ;πλ∇πλ;πλ;πλ⎿)	*not in E ! Liebmann-function*	*4.4*
angle-angle-angle-tip-triangulation	(πλ;πλ;πλ∇∇πλ;πλ;πλ∇πλ;πλ;πλ⎿)	*not in E ! Liebmann-function*	*4.4*
angle-angle-right-triang.	(πλ;πλ;πλ∇πλ;πλ;πλ⎿)	*not in E !*	*4.4*
angle-defection	(πλ◊πλ◊πλ⎿)	*difference of straight and sum*	*4.5*
angle-right-triangulation	(∇πλ;πλ;πλ⎿)	*not in E ! riso-triangulation*	*4.4*
angle-summation	(πλ◊πλ⊕πλ⎿)	*usual meaning*	*4.5*
anti-angle-horo-parallelation	(πλ;πλ;πλ⟱⎿)	*distance for which the angle is horo*	*4.3*
anti-horo-parallelation	(πλ;πλ⇓πλ⎿)	*horoparallel to pair thru point*	*4.3*
archimedes-triangulation	(∇∇πλ;πλ⎿)	*not in E ! octomidial-equilateral triangle (three halfright angles)*	*4.4*
cali-angle-defect-metrition	(πλ′πλ◊πλ)⎿	*calibrated angle-defection*	*4.5*
cali-liebmann-base-triangulation	(πλ′∇πλ∇πλ⎿)	*calibrated Liebmann base*	*4.9*
cali-liebmann-tip-triangul.	(πλ∇′πλ∇πλ⎿)	*calibrated Liebmann tip*	*4.9*
cali-lo-motion	(″πλ⊕◊πλ;πλ÷πλ)	*Lobachevsky symmetry transformation*	*2.9*
contra-horo-parallel-angulation	(⇓πλ;πλ⎿)	*parallel angle for distance*	*4.3*
levification	(!!πλ;πλ;πλ⎿)		*4.4*

lo-area-triangle-combination	(πλ;πλ;πλ◊⊕πλ;πλ;πλ∟)	*the condition of angle sum less straight is not written explicitly*	*4.5*
pro-horo-parallel-angulation	(⥣πλ;πλ∟)	*parallel angle for distance*	*4.3*
side-angle-angle-triangulation	(πλ;πλ∇πλ;πλ;πλ∇πλ;πλ;πλ∟)	*usual meaning*	*4.4*
syn-angle-horo-parallelation	(πλ;πλ;πλ⥣∟)	*distance for which the angle is horo*	*4.3*
syn-horo-parallelation	(πλ;πλ⥣πλ∟)	*horoparallel to pair thru point*	*4.3*
calcule pilambda	***extra-relation-constant***		*ref.*
anti-contra-horo-parallelity	πλ⎬πλ;πλ⎠πλ	*one of 4 combinations*	*4.3*
anti-contra-hyper-parallelity	πλ⎬⎬πλ;πλ⎠⎠πλ	*one of 4 combinations*	*4.3*
anti-horo-parallelity	πλ⎬πλ;πλ⎬πλ	*opposite direction*	*4.3*
anti-hyper-parallelity	πλ⎬⎬πλ;πλ⎬⎬πλ	*opposite direction*	*4.3*
anti-pro horo-parallelity	πλ⎬πλ;πλ⎛πλ	*one of 4 combinations*	*4.3*
anti-pro-horo-parallelity	πλ⎬⎬πλ;πλ⎛⎛πλ	*one of 4 combinations*	*4.3*
Bolyai-quadrangularity	πλ⊥πλ;πλ⊥πλ	*opposite right-angles*	*4.4*
Bolyai-quadrity	πλ⊥πλ≈πλ⊥πλ	*legs equal*	*4.4*
contra-horo-parallelity	πλ⎠πλ;πλ⎠πλ	*contrivity* 1πλ 2πλ 3πλ	*4.3*
contra-hyper-parallelity	πλ⎠⎠πλ;πλ⎠⎠πλ	*contrivity* 1πλ 2πλ 3πλ	*4.3*
Gauss-quadrity	πλ≈πλ≈πλ≈πλ	*equal angles, equal sides*	*4.4*
horo-parallelity	πλ∣πλ,πλ∣πλ	*no* ***entitor***	*4.3*
hyper-parallelity	πλ∥πλ,πλ∥πλ	*between syn-horo and anti-horo, no* ***entitor***	*4.3*
Lambert-quadrangularity	πλ⊥πλ⊥πλ⊥πλ	*three right-angles*	*4.4*
Lambert-quadrity	πλ≈πλ⊥πλ≈πλ	*legs equal*	*4.4*
light-angularity	πλ≈≈πλ;πλ	*vertex* 1π	*4.4*
liso-angularity	πλ≈≈πλ≈πλ	*vertex* 1π *, light isoscelic*	*4.4*
lo-cissectivity	πλ⟊πλ,πλ⟊πλ	*no* ***entitor***	*4.3*
lo-linisectivity	πλ⟊πλ⟊πλ⟊πλ	*no* ***entitor***	*4.3*
lo-parallelity	πλ⫴πλ—πλ⫴πλ	*horo- or hyper-parallelity or line-equality*	*4.3*
lo-quadrangle-area-equality	πλ;πλ;πλ;πλ≈~πλ;πλ;πλ;πλ	*usual meaning*	*4.4*
lo-quadrangle-area-minority	πλ;πλ;πλ;πλ⎨~πλ;πλ;πλ;πλ	*usual meaning*	*4.4*
lo-triangle-area-equality	πλ;πλ;πλ≈~πλ;πλ;πλ	*usual meaning*	*4.4*
lo-triangle-area-minority	πλ;πλ;πλ⎨~πλ;πλ;πλ	*usual meaning*	*4.4*
lo-tri-quadrangle-area-equality	πλ;πλ;πλ≈~πλ;πλ;πλ;πλ	*usual meaning*	*4.4*
lo-tri-quadrangle-area-minority	πλ;πλ;πλ⎨~πλ;πλ;πλ;πλ	*usual meaning*	*4.4*
poly-parallelity (lo-regular-)	πλ⫴πλ;πλ⫴πλ	*horo- or hyper-parallelity*	*4.3*
pro-horo-parallelity	πλ⎛πλ;πλ⎛πλ	*protivity* 1πλ 2πλ 3πλ	*4.3*
pro-hyper-parallelity	πλ⎛⎛πλ;πλ⎛⎛πλ	*protivity* 1πλ 2πλ 3πλ	*4.3*
Saccheri-quadrangularity	πλ⊥πλ⊥πλ;πλ	*adjacent right-angles*	*4.4*
Saccheri-quadrity	πλ⊥πλ⊥πλ≈πλ	*legs and base equal*	*4.4*
syn-contra hyper-parallelity	πλ⎨⎨πλ;πλ⎠⎠πλ	*one of 4 combinations*	*4.3*
syn-contra-horo-parallelity	πλ⎨πλ;πλ⎠πλ	*one of 4 combinations*	*4.3*
syn-horo-parallelity	πλ⎨πλ;πλ⎨πλ	*same direction*	*4.3*

syn-hyper-parallelity	πλ⎨⎨πλ;πλ⎨⎨πλ	*same direction*	*4.3*
syn-pro-horo-parallelity	πλ⎨πλ;πλ⎛πλ	*one of 4 combinations*	*4.3*
syn-pro-hyper-parallelity	πλ⎨⎨πλ;πλ⎛⎛πλ	*one of 4 combinations*	*4.3*
calcule piomega	***basis-individual-constant***	*in addition to* piepsilon	*ref.*
nullum	ABn	*zero*	*5.1*
unus	ABu	*one*	*5.1*
calcule piomega	***basis function-constant***	*in addition to* piepsilon	*ref.*
natural-addition	(AB+AB)	*natural number Robinson arithmetic*	*5.1*
natural-multiplication	(AB×AB)	*ditto*	*5.1*
natural-production	(AB×πω;πω)	*note* $(0\times\pi_1;\pi_2)=\pi_1$	*5.1*
calcule piomega	***basis-relation-constant***	*in addition to* piepsilon	*ref.*
minority	AB<AB	*usual meaning*	*5.1*
calcule piomega	***extra-individual-constant***	*in addition to* piepsilon	*ref.*
duo	ABb	*two*	*5.1*
calcule piomega	***extra-function-constant***	*in addition to* piepsilon	*ref.*
bi-ponentiation	(↑AB)	2^n *recursively defined*	*5.1*
calcule piomega	***extra-relation-constant***	*in addition to* piepsilon	*ref.*
fermat-primality	Fermprime(AB)	*usual meaning*	*5.1*
oddity	Odd(AB)	*usual meaning*	*5.1*
primality	Prime(AB)	*usual meaning*	*5.1*
calcule deltaalpha	***basis-individual-constant***	*biradical number*	*ref.*
nullum	δαn	*0*	*3.8*
unus	δαu	*1*	*3.8*
calcule deltaalpha	***basis-function-constant***	+ – × / √ ↑	*ref.*
addition	(δα+δα)	*x+y*	*3.8*
biradication	(√δα⌊)	*sr(x)*	*3.8*
multiplication	(δα×δα)	*xy*	*3.8*
negativation	(–δα)	*-x*	*3.8*
reciprocation	(/δα⌊)	*1/x*	*3.8*
calcule deltaalpha	***basis-relation-constant***	<	*ref.*
positivity	<δα	*usual meaning*	*3.8*
calcule deltaalpha	***extra-individual-constant***		*ref.*
dimi	δαc	*1/2*	*3.8*
duo	δαb	*2*	*3.8*
klein-biradical-end	δαke	*sr(2sr(2)-2)*	*4.8*
klein-biradical-full	δαkf	*(2sr(2sr(2)-2))/(2sr(2)-1)*	*4.8*
klein-biradical-minus-end	δαkm	*- sr(2sr(2)-2)*	*4.8*
minus-one	δαm	*-1*	*3.8*
calcule deltaalpha	***extra-function-constant***	*additional* °	*ref.*
absolution	(+δα)	*abs(x)*	*3.8*
angle-addition, cyclation	(δα°+δα⌊)	*with respect to cyclic angle-metering*	*3.8*
angle-bisection	(δα°/⌊)	*with respect to angle-addition*	*3.8*
angle-duplication	(δα°+⌊)	*with respect to angle-addition*	*3.8*
angle-negativation	(°–δα⌊)	*with respect to angle-addition*	*3.8*
angle-subtraction	(δα°–δα⌊)	*with respect to angle-addition*	*3.8*
bisection	(δα/)	*1/x*	*3.8*
cathecation	(√δα–⌊)	*sr(1-*x^2*)*	*3.8*
cathetion	(δα√–δα⌊)	*sr(*x^2*-*y^2*)*	*3.8*
division	(δα/δα⌊)	*x/y*	*3.8*
duplication	(δα+)	*1/x*	*3.8*

hypocation	(√δα+)	*$sr(1+x^2)$*	*3.8*
hypotion	(δα√+δα)	*$sr(x^2+y^2)$*	*3.8*
klein-annition	(δα+δα⌊)	*(x+y)/(1+xy)*	*4.8*
klein-clarition	(δα√–δα⌊)	*$sr((x^2-y^2)/(1-y^2))$*	*4.8*
klein-evition	(δα××δα/δα⌊)	*xy/(y+z)* *with $sr(x^2+y^2)<1$ and not y+z=0*	*4.8*
klein-jacition	(δα√√δα/√√ε⌊)	*$(1-sr((1-y^2(1-x^2)/(1-z^2)))/x$* if not *x=0* else *0*	*4.8*
klein-menition	(– –δα⌊)	*-x*	*4.8*
klein-riation	(δα√√++δα⌊)	*$sr(x^2+y^2-x^2y^2$*	*4.8*
quadration	(δα×)	*x^2*	*3.8*
subtraction	(δα–δα)	*x-y*	*3.8*
calcule deltaalpha	***extra-relation-constant***	*additional ≤ # ∈*	*ref.*
cyclity	≤δα<	*number in cycle-segment between 0 and 2*	*3.8*
diametrity	#δα#	*number in diameter-segment between -1 and 1*	*3.8*
dyadic-cyclity	≤δα∈<	*dyadic number constr.by bisection, addition and negativ. from 1 in cycle-s. between 0 and 2*	*3.8*
dyadicity	δα∈	*number constructed by bisection, addition and negativation from 1*	*3.8*
equal-minority	δα≤δα	*usual meaning*	*3.8*
klein-cyclity	≤≤δα<<	*combined with cyclity*	*4.8*
klein-diametrity	##δα##	***recursively** constructed by klein-functions from* δαke, *diameter-segment between -1 and 1*	*4.8*
klein-dyadic-cyclity	≤≤δα∈∈<<	*klein-dyadicity combined with cyclity*	*4.8*
klein-dyadicity	δα∈∈	*recursively constructed by klein-bisection, klein-addition and klein-negativation from* δαke, *in k**lein-cycle-segment** between 0 and* δαkf	*4.8*
klein-radity	≤≤δαπ##	*combined with radity*	*4.8*
minority	δα<δα	*usual meaning*	*3.8*
negativity	δα<	*usual meaning*	*3.8*
nonnegativity	≤δα	*usual meaning*	*3.8*
nonpositivity	δα≤	*usual meaning*	*3.8*
radity	≤δα#	*number in radius-segment between 0 and 1*	*3.8*
rationality	∈δα	*rational number*	*3.8*