ZEROS OF THE RIEMANN ZETA FUNCTION $\zeta(s)$ CAN BE FOUND ARBITRARY CLOSE TO THE LINE $\Re(s) = 1$

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Abstract

In this paper, not only did we disprove the Riemann Hypothesis (RH) but we also showed that zeros of the Riemann zeta function $\zeta(s)$ can be found arbitrary close to the line $\Re(s) = 1$. Our method to reach this conclusion is based on analyzing the fine behavior of the partial sum of the Dirichlet series with the Mobius function $M(s) = \sum_n \mu(n)/n^s$ defined over p_r rough numbers (i.e. numbers that have only prime factors greater than or equal to p_r). Two methods to analyze the partial sum fine behavior are presented and compared. The first one is based on establishing a connection between the Dirichlet series with the Mobius function $M(s)$ and a functional representation of the zeta function $\zeta(s)$ in terms of its partial Euler product. Complex analysis methods (specifically, Fourier and Laplace transforms) were then used to analyze the fine behavior of partial sum of the Dirichlet series. The second method to estimate the fine behavior of partial sum was based on integration methods to add the different co-prime partial sum terms with prime numbers greater than or equal to p_r. Comparing the results of these two methods leads to a contradiction when we assume that $\zeta(s)$ has no zeros for $\Re(s) > c$ and $c < 1$.

Keywords: Riemann zeta function, Mobius function, Riemann hypothesis, conditional convergence, Euler product.

Classification: Number Theory, 11M26

1 Introduction and Paper Outline

The Riemann zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain [2]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \tag{1}$$

where, $s = \sigma + it$ is a complex variable and $s \neq 1$.

For $\sigma > 1$ (or $\Re(s) > 1$), $\zeta(s)$ can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

or by the following product over the primes p_i's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{3}$$

where, $p_1 = 2$, $\prod_{i=1}^{\infty}(1 - 1/{p_i}^s)$ is the Euler product and $\prod_{i=1}^{r}(1 - 1/{p_i}^s)$ is the partial Euler product. The above series and product representations of $\zeta(s)$ are absolutely convergent for $\sigma > 1$.

The region of the convergence for the sum in Equation (2) can be extended to $\Re(s) > 0$ by using the alternating series $\eta(s)$ where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{4}$$

and

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\eta(s). \tag{5}$$

One may notice that the term $1 - 2^{1-s}$ is zero at $s = 1$. This zero cancels the simple pole that $\zeta(s)$ has at $s = 1$ enabling the extension (or analog continuation) of the zeta function series representation over the critical strip where $0 < \Re(s) < 1$.

It is well known that all of the non-trivial zeros of $\zeta(s)$ are located in the critical strip. Riemann stated that all non-trivial zeros were very probably located on the critical line $\Re(s) = 0.5$ [14]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function $\mu(n)$ is defined as follows
$\mu(n) = 1$, if $n = 1$.
$\mu(n) = (-1)^k$, if $n = \prod_{i=1}^{k} p_i$, p_i's are distinct primes.
$\mu(n) = 0$, if $p^2|n$ for some prime number p.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \tag{6}$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s) > 1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s) = 1$. The Riemann hypothesis is equivalent to the statement that $M(s)$ is conditionally convergent to $1/\zeta(s)$ for $\Re(s) > 0.5$. It should be pointed out that our definition of $M(s)$ is different from Mertens function (defined in the literature as $M(x) = \sum_{1 \le n \le x} \mu(n)$). If we denote $M(s; 1, N)$ as partial sum of the series $M(s)$

$$M(s; 1, N) = \sum_{n=1}^{N} \frac{\mu(n)}{n^s}, \tag{7}$$

then the Mertens function is given by $M(0; 1, N)$. On RH, we then have [18]

$$M(0; 1, N) = O(N^{1/2+\epsilon}),$$

where ϵ is an arbitrary small number. By partial summation, on RH, we also have

$$M(1; 1, N) = O(N^{-1/2+\epsilon}).$$

The irregular behavior of the Mobius function $\mu(n)$ has so far hindered the attempts to estimate the asymptotic behavior of any of the above two sums as N approaches infinity.

The Riemann hypothesis is also equivalent to another statement that involves the prime number function $\pi(x)$ (defined by the the number of primes less than x). The prime counting function can be computed using Riemann Explicit Formula

$$\pi(x) + \sum_{n=2}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n} = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

and on RH,

$$\pi(x) = \mathrm{Li}(x) - \frac{\mathrm{Li}(x^{1/2})}{2} - \sum_{\rho} \mathrm{Li}(x^{\rho}) + \text{Lesser terms}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and the sum $\sum_{\rho} \mathrm{Li}(x^{\rho})$ is performed over the non-trivial zeros $\rho_i = \alpha_i + i\gamma_i$. This sum is conditionally convergent and it should be performed over the non-trivial zeros with $|\gamma_i| \leq T$ as T approaches infinity.

The prime counting function $\pi(x)$ has a jump discontinuity at each prime number. In the literature, this function is a right continuous function given by $\pi_{rc}(x) = \sum_{p_i \leq x} 1$, where the suffix rc was added to indicate that the function is right continuous. Since the analysis of this paper employs integration methods (and specifically Lebesgue-Stieltjes integration), therefore it is more appropriate to assign the left-right average to the function value at discontinuities. In the literature, this function is referred to as $\pi_0(x) = \lim_{h\to 0}(\pi(x+h) + \pi(x-h))/2$. In this paper, we define $\pi(x)$ as $\pi_0(x)$. In fact, for the above equation, $\pi(x)$ does converge to the right-left average when x is a prime number (or at the discontinuities of the function $\pi(x)$).

The distribution of the prime numbers can be also analyzed by defining the function $\psi(x)$ as

$$\psi(x) = \psi_0(x) = \frac{1}{2}\left(\sum_{{p_i}^m \leq x} \log p_i + \sum_{{p_i}^m < x} \log p_i \right),$$

and using Von Mangoldt formula given by

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}).$$

It is well known that as x approaches infinity, the prime counting function is asymptotic to the function $\mathrm{Li}(x)$. Therefore, if we consider that $\pi(x)$ is comprised of two components, the regulator component given by $\mathrm{Li}(x)$ and the irregular component $J(x)$ given by

$$J(x) = \pi(x) - \mathrm{Li}(x),$$

then on RH, we have

$$J(x) < \frac{1}{8\pi}\sqrt{x}\log x \qquad \text{for } x > 2657.$$

On RH, the irregular component $J(x)$ is also given by [16] (refer to lemmas 5 and 6)

$$J(x) = \frac{\psi(x) - x}{\log x} + O\left(\frac{\sqrt{x}}{\log x}\right)$$

or

$$J(x) = -\frac{1}{\log x}\sum_{\rho}\frac{x^{\rho}}{\rho} + O\left(\frac{\sqrt{x}}{\log x}\right) \tag{8}$$

Our method to examine the validity of the Riemann Hypothesis (and in general, to examine the region within the critical strip where $\zeta(s)$ is void of non-trivial zeros) is based on representing variants of the Dirichlet series $M(\sigma)$ (defined by Equation (6)) in terms of variants of the integral $\int dJ(x)/x$. However, the partial sum of the series $M(\sigma)$ exhibits irregular behavior due to the irregular behavior of the Mobius function $\mu(n)$. Therefore, we need to introduce a method to smooth out the partial sum of the series $M(\sigma)$ by introducing a method to represent the series $M(s)$ in terms of the partial Euler product. This task is achieved in section 2 by first eliminating the numbers that have the prime factor 2 to generate the series $M(s,3)$ (i.e, the series $M(s,3)$ is void of any number with prime factors less than 3). For the series $M(s,3)$, we then eliminate the numbers with the prime factor 3 to generate the series $M(s,5)$, and so on, up to the prime number p_r. In other words, we have applied sieving methods to modify the series $M(s)$ to include only the numbers with prime factors greater than or equal to p_r. In the literature [10], numbers with prime factors less than y are called y-smooth while numbers with prime factors greater than y are called y-rough. In essence, our approach is to compute the Dirichlet series over p_r-rough numbers. In section 3, we have shown that the series $M(s)$ and the new series $M(s,p_r)$ have the same region of convergence (Theorem 1).

After defining the series $M(s,p_r)$ and its partial sum, we note that both the series and its partial sum has two components. The two components corresponds to the two components of the prime function $\pi(x)$. These two components are $\mathrm{Li}(x)$ an $J(x)$. For $x \geq 1$, the function $\mathrm{Li}(x)$ is differentiable and its contribution to $M(s,p_r)$ and its partial sum is well behaved and can be computed using both complex analysis and integration methods. Therefore, we call the component of the series $M(s,p_r)$ (or its partial sum) due to $\mathrm{Li}(x)$ as the regular component of series $M(s,p_r)$ (or the regular component of its partial sum). We then call the remaining component of the series $M(s,p_r)$ (and the remaining component of its partial sum) as the irregular component of the series $M(s,p_r)$ (and the irregular component of its partial sum).

We will then present two methods to represent the irregular component of the series $M(s,p_r)$ and the irregular component of its partial sum in terms of the integral $\int_{p_r}^{\infty} dJ(x)/x$. The first method is based on complex analysis (sections 4 and 6). With this method, we have provided a functional equation for $\zeta(s)$ using its partial Euler product. The second method is described in section 5 and it is based on integration methods.

It is worth noting the research done by Gonek, Hughes and Keating [5] into establishing a relationship between $\zeta(s)$ and its partial Euler product for $\Re(s) < 1$. Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation". In section 4, we will present a functional equation for $\zeta(s)$ using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\prod_{r}^{\infty}\left(1 - \frac{1}{p_i^s}\right).$$

The above equation is valid for $\Re(s) > 1$. To be able to represent $\zeta(s)$ in term of its partial Euler product for $\Re(s) \leq 1$, we need to replace the term $\prod_r^\infty (1 - 1/p_i^s)$ with an equivalent one that allows the analytic continuation for the representation of $\zeta(s)$ for $\Re(s) \leq 1$. Thus, the new term (that we need to introduce to replace $\prod_r^\infty (1 - 1/p_i^s)$) must have a zero that corresponds to the pole $\zeta(s)$ has at $s = 1$. In section 4, we will use the complex analysis to compute this new term and then represent $\zeta(s)$ in terms of its partial Euler product. This functional representation is given by Theorem 2. We will then use this theorem to represent the series $M(s, p_r)$ in terms of the integral $\int_{p_r}^\infty dJ(x)/x$ (Theorem 3).

Our effort will then be centered at computing the partial sum of the series $M(1, p_r)$. Two methods will be presented to compute the irregular component of the partial sum for the series $M(1, p_r)$ (in the abstract, we referred to it as the partial sum fine behavior). In section 5, we have achieved this task using integration methods (Theorem 4). In section 6, we have used Theorem 3 and the complex analysis (Fourier and Laplace transforms) to derive a second representation for the irregular component of the series $M(1, p_r)$ partial sum. The two representations of the irregular component of the partial sum of the series $M(1, p_r)$ are then compared. We will then show that this comparative analysis leads to a contradiction when we assume that $\zeta(s)$ has no non-trivial zeros for $\Re(s) > c$ where $c < 1$. This leads to the conclusion that the Riemann Hypothesis is invalid and non-trivial zeros can be found arbitrary close to the line $\Re(s) = 1$.

To some extent, our analysis has similarities with linear time-invariant system analysis. Linear time-invariant systems can be represented either in frequency domain by its transfer function (in our case, the transfer function is represent by a functional representation of the Riemann zeta function in terms of its partial Euler product) or in time domain by its impulse response (in our case, the effect of each prime number is to flip the sign of the Mobius function for any number that is divisible by this prime number). The input for the linear time-invariant system can be represented either in frequency domain by its frequency spectrum (in our case, the absence of the non-trivial zeros for sections of the critical strip) or in time domain as a function of time (in our case, the prime numbers that generates the series $M(1, p_r)$). The system output function is then determined in frequency domain by multiplying the input signal spectrum by the system transfer function and then taking the inverse Fourier or Laplace transform. The system output function can be also determined in time domain by convolving the input signal with the system impulse response. Both methods should provide the same results.

2 Notation and Preliminaries.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s} \dots .$$

Next, we introduce the series $M(s, 3)$ by eliminating all the numbers that have a prime factor 2 (or keeping only the numbers with prime factors greater than or equal to 3). Thus, $M(s,3)$ can be written as

$$M(s,3) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s}.....$$

Our analysis to test the conditional convergence of these series ($M(s)$ and $M(s,3)$ for $\sigma \leq 1$) is based on comparing correspondent terms of these two series. Therefore, rearrangement and permutation of the terms may have a significant impact on analyzing the region of convergence of both series. Thus, it essential to have the same index for both series $M(s)$ and $M(s,3)$ refer to the same term. Hence, we will represent $M(s,3)$ as follows

$$M(s,3) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

or

$$M(s,3) = \sum_{n=1}^{\infty} \frac{\mu(n,3)}{n^s}, \tag{9}$$

where
$\mu(n,3) = \mu(n)$, if n is an odd number,
$\mu(n,3) = 0$, if n is an even number.

The above series $M(s,3)$ can be further modified by eliminating all the numbers that have a prime factor 3 (or keeping only the numbers with prime factors greater than or equal to 5) to get the series $M(s,5)$ where

$$M(s,5) = 1 - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} + \frac{0}{25^s}....,$$

or more conveniently

$$M(s,5) = 1 + \frac{0}{2^s} - \frac{0}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i \geq p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(3)\}$ is the set of square-free odd numbers), then we define the series $M(s,p_r)$ as

$$M(s,p_r) = \sum_{n=1}^{\infty} \frac{\mu(n,p_r)}{n^s}, \tag{10}$$

where
$\mu(n,p_r) = \mu(n)$, if $n \in \{1, I(p_r)\}$,
otherwise, $\mu(n,p_r) = 0$.

It can be easily shown that, for every prime number p_r, the series $M(s,p_r)$ converges absolutely for $\Re(s) > 1$. Furthermore, it can be shown that, for $\Re(s) > 1$, $M(s,p_r)$ satisfies the following equation

$$M(s) = M(s, p_r) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right). \tag{11}$$

Since

$$M(s) = \frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

therefore we conclude that, for $\Re(s) > 1$, $M(s, p_r)$ approaches 1 as p_r approaches infinity. It should be pointed out here that with this definition of $M(s, p_r)$, $M(s, 2)$ is equal to $M(s)$.

The first ingredient of our analysis is the Dirichlet series $M(s, p_r)$ partial sum defined as

$$M(s, p_r; K_1, K_2) = \sum_{n=K_1}^{K_2} \frac{\mu(n, p_r)}{n^s}, \tag{12}$$

where $K_1 \geq 1$ and $K_2 \geq p_r$. If we replace the integer K_2 with the real number x (where $x \geq p_r$), then we have the general definition for $M(s, p_r; K_1, x)$

Definition 1. *$M(s, p_r; K_1, x)$ is a function of x defined as*

$$M(s, p_r; K_1, x) = M(s, p_r; K_1, \lfloor x \rfloor).$$

where $\lfloor x \rfloor$ refers to the integer value of x and K_1 is an integer greater or equal to 1.

The partial sums of the two series $M(s, p_{r-1})$ and $M(s, p_r)$ are related by the following lemma,

Lemma 1.

$$M(s, p_{r-1}; 1, Np_{r-1}) = M(s, p_r; 1, Np_{r-1}) - \frac{1}{p_{r-1}^s} M(s, p_r; 1, N). \tag{13}$$

where N is an integer greater or equal to 1.

Proof. The proof of this lemma follows directly from the definition of the partial sum for the two series $M(s, p_{r-1})$ and $M(s, p_r)$. Alternatively, one can show that each term on the right side of Equation (13) is a term on the left side of the equation and vise versa. Furthermore, there is no duplicity for each term on both side of the equation. Each term on the left side of Equation (13) corresponds to a number in the interval $[1, Np_{r-1}]$ that has distinct prime factors greater than or equal to p_{r-1}. Each term on the first term of the right side of Equation (13) corresponds to a number in the interval $[1, Np_{r-1}]$ that has distinct prime factors greater than or equal to p_r. Each term on the second term of the right side of Equation (13) corresponds to a number in the interval $[1, Np_{r-1}]$ that has distinct prime factors greater than or equal to p_{r-1} with p_{r-1} being one of its prime factors. □

For Equation (13), if we replace the integer Np_{r-1} by the real number $x > p_{r-1}$, we then have

$$M(s, p_r; 1, x) = M(s, p_{r-1}; 1, x) + \frac{1}{p_{r-1}^s} M(s, p_r; 1, x/p_{r-1}). \tag{14}$$

Note that regardless of the value of p_r, the partial sum $M(s, p_r; p_r, p_r^2 - 1)$ comprised of terms that correspond to only prime numbers. Similarly, the partial sum $M(s, p_r; p_r^2, p_r^3 - 1)$

comprised of terms that correspond to only prime numbers or products of two prime numbers, and so on. Therefore, one might expect that regardless of the value of p_r, the partial sum $M(s, p_r; 1, p_r^a)$ exhibits certain characteristics with respect to the variable a that can be exploited. These characteristics will be discussed in details in section 5.

The second ingredient of our analysis is the partial Euler product defined as $\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right)$. Our analysis for this product will be restricted to the region $\Re(s) > 0.5$. Taking the logarithm of the partial Euler product, we then have for $\sigma > 0.5$

$$\log\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right)=\sum_{i=r1}^{r2}\log\left(1-\frac{1}{{p_i}^s}\right)+2\pi iN,$$

where N is zero, positive or negative integer to account for the ambiguity in the phase of the logarithm of complex numbers. Since $1/|p_i^s|< 1$, hence,

$$\log\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right)=\sum_{i=r1}^{r2}\left(-\frac{1}{{p_i}^s}-\frac{1}{2{p_i}^{2s}}-\frac{1}{3{p_i}^{3s}}-...\right)+2\pi iN. \tag{15}$$

We split the sum on the right side of Equation (15) into two sums. The first sum comprises of the terms of the form $1/p_i^s$ while the second sum comprises of the rest of the sum. This leads us to introduce the terms $\delta(s; p_{r1}, p_{r2})$ and $\delta(s; p_{r1})$

Definition 2. *Let $\delta(s; p_{r1}, p_{r2})$ be defined as the sum*

$$\delta(s;p_{r1},p_{r2})=\sum_{i=r1}^{r2}\left(-\frac{1}{2{p_i}^{2s}}-\frac{1}{3{p_i}^{3s}}-\frac{1}{4{p_i}^{4s}}-...\right), \tag{16}$$

and let $\delta(s; p_{r1})$ be defined as

$$\delta(s;p_{r1})=\lim_{r2\to\infty}\delta(s;p_{r1},p_{r2})=\sum_{i=r1}^{\infty}\left(-\frac{1}{2{p_i}^{2s}}-\frac{1}{3{p_i}^{3s}}-\frac{1}{4{p_i}^{4s}}-...\right). \tag{17}$$

Using Definition 2, we can write Equation (15) as

$$\log\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right)=-\sum_{i=r1}^{r2}\frac{1}{{p_i}^s}+\delta(s;p_{r1},p_{r2})+2\pi iN. \tag{18}$$

The following lemma deals with the term (or the sum) $\delta(s; p_{r1}, p_{r2})$) followed by analysis of the $\sum_{i=r1}^{r2} 1/{p_i}^s$.

Lemma 2. *For $\sigma > 0.5$,*

$$|\delta(s;p_{r1},p_{r2})|=O\left(\frac{p_{r1}^{1-2\sigma}}{2\sigma-1}\right)$$

Proof. For $\sigma > 0.5$, we have

$$\delta(s;p_{r1},p_{r2}) = \sum_{i=r1}^{r2}\left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}} + ...\right),$$

hence

$$|\delta(s;p_{r1},p_{r2})| \leq \sum_{i=r1}^{r2}\left(\left|\frac{1}{2{p_i}^{2s}}\right| + \left|\frac{1}{3{p_i}^{3s}}\right| + \left|\frac{1}{4{p_i}^{4s}}\right| + ...\right),$$

or

$$|\delta(s;p_{r1},p_{r2})| \leq \sum_{i=r1}^{r2}\left(\frac{1}{2{p_i}^{2\sigma}} + \frac{1}{3{p_i}^{3\sigma}} + \frac{1}{4{p_i}^{4\sigma}} + ...\right),$$

or

$$|\delta(s;p_{r1},p_{r2})| \leq \sum_{i=r1}^{r2}\frac{1}{{p_i}^{2\sigma}}\left(\frac{1}{2} + \frac{1}{3{p_i}^{\sigma}} + \frac{1}{4{p_i}^{2\sigma}} + ...\right).$$

However,

$$\sum_{i=r1}^{r2}\left(\frac{1}{2} + \frac{1}{3{p_i}^{\sigma}} + \frac{1}{4{p_i}^{2\sigma}} + ...\right) < \sum_{i=r1}^{r2}\left(\frac{1}{2} + \frac{1}{{p_i}^{\sigma}} + \frac{1}{{p_i}^{2\sigma}} + ...\right) < \frac{1}{2} + \frac{1}{{p_i}^{\sigma}} + \int_1^{\infty}\frac{dx}{p_i^{x\sigma}}.$$

For $p_i \geq 2$ and $\sigma > 0.5$, we then have

$$\sum_{i=r1}^{r2}\left(\frac{1}{2} + \frac{1}{3{p_i}^{\sigma}} + \frac{1}{4{p_i}^{2\sigma}} + ...\right) < 4$$

or

$$|\delta(s;p_{r1},p_{r2})| < \sum_{i=r1}^{r2}\frac{4}{{p_i}^{2\sigma}}.$$

Since

$$\sum_{i=r1}^{r2}\frac{1}{{p_i}^{2\sigma}} < \sum_{n=p_{r1}}^{\infty}\frac{1}{n^{2\sigma}} = O\left(\frac{p_{r1}^{1-2\sigma}}{2\sigma-1}\right),$$

or

$$|\delta(s;p_{r1},p_{r2})| = O\left(\frac{p_{r1}^{1-2\sigma}}{2\sigma-1}\right).$$

□

Note that for any $\sigma > 0.5$, $|\delta(s;p_{r1},p_{r2})|$ is uniformly convergent (regardless of the value of the imaginary part t, where $s = \sigma + it$). We also note that the term $\delta(s;p_{r1},p_{r2})$ has no impact on which part (of the critical strip) is void of non-trivial zeros.

Next, we turn our attention to the term $\sum_{i=r1}^{r2} 1/{p_i}^s$. This term has a direct impact on which part (of the critical strip) is void of non-trivial zeros. We will first analyze this sum on the real axis (i.e. $s = \sigma$). We will then extend this analysis to complex plain (i.e. $s = \sigma + it$). Before, we do so, we have the following definitions.

Definition 3. *We define the prime counting function $\pi(x)$ as*

$$\pi(x) = \lim_{h\to 0}(\pi_{rc}(x+h) + \pi_{rc}(x-h))/2$$

where

$$\pi_{rc}(x) = \sum_{p_i \le x} 1$$

In other words; we define $\pi(x)$ as the right-left average of the conventional prime counting function.

As mentioned in the previous section, as x approaches infinity, the prime counting function $\pi(x)$ is asymptotic to the function $\mathrm{Li}(x)$. Therefore, we can split $\pi(x)$ into two components, the regulator component given by $\mathrm{Li}(x)$ and the irregular component $J(x)$.

Definition 4. *The irregular component $J(x)$ of the prime counting function $\pi(x)$ is defined as*

$$J(x) = \pi(x) - \mathrm{Li}(x) \tag{19}$$

Definition 5. *Let $\varepsilon(s; p_{r1}, p_{r2})$ be defined as the integral*

$$\varepsilon(s; p_{r1}, p_{r2}) = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^s, \tag{20}$$

and let $\varepsilon(s; p_{r1})$ be defined as

$$\varepsilon(s; p_{r1}) = \int_{p_{r1}}^{\infty} dJ(x)/x^s. \tag{21}$$

With these definitions, we can compute the sum $\sum_{i=r1}^{r2} 1/{p_i}^{\sigma}$ using the following lemma

Lemma 3. *For $\sigma > 0.5$, the sum $\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}}$ is unconditionally given by*

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon(\sigma; p_{r1}, p_{r2}) \tag{22}$$

where, $\varepsilon(\sigma; p_{r1}, p_{r2}) = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^{\sigma}$ and $J(x) = \pi(x) - \mathrm{Li}(x)$,

Proof. Using Lebesgue-Stieltjes integral [8], we can write the sum $\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}}$ as the following integral

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^{\sigma}}$$

or

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{d\mathrm{Li}(x)}{x^{\sigma}} + \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^{\sigma}}.$$

Hence

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}\log x} dx + \varepsilon(\sigma, p_{r1}, p_{r2}).$$

For $\sigma \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du$ (where $r \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2})$$

It should be pointed out that although the functions $E_1((\sigma - 1)\log p_{r1})$ and $E_1((\sigma - 1)\log p_{r2})$ have a singularity at $\sigma = 1$, the difference has a removable singularity at $\sigma = 1$. This follows from the fact that as σ approaches 1, the difference can be written as

$$E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2}) = -\log((1 - \sigma)\log p_{r1}) - \gamma + \log((1 - \sigma)\log p_{r2}) + \gamma$$

or,

$$\lim_{\sigma \to 1} \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \lim_{\sigma \to 1} \{E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2})\} = -\log\log p_{r1} + \log\log p_{r2}$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ for $\sigma < 1$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1.$$

With the variable substantiations $w_1 = (1 - \sigma)(\log p_{r1})z_1$ and $w_2 = (1 - \sigma)(\log p_{r2})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2} - 1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1} - 1}{w_1} dw_1 +$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [1] (refer to page 230)

$$\int_0^a \frac{e^t - 1}{t} dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2}) + \varepsilon(\sigma; p_{r1}, p_{r2})$$

$\square$

The results of Lemma 3 can be extended to compute the sum $\sum_{i=r1}^{r2} \frac{1}{{p_i}^s}$ where $s = \sigma + it$ using the following lemma

Lemma 4. *For $\Re(s) > 0.5$, the sum $\sum_{i=r1}^{r2} \frac{1}{{p_i}^s}$ is unconditionally given by*

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon(s; p_{r1}, p_{r2}) \tag{23}$$

where, $\varepsilon(s; p_{r1}, p_{r2}) = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^s$ and $J(x) = \pi(x) - \mathrm{Li}(x)$,

Proof. The proof of this lemma is given in Appendix 1. Note that the Exponential Integral with the complex variable z is given by $E_1(z) = \int_1^\infty \frac{e^{-tz}}{t} dt$ (where $\Re(z) \geq 0$) □

To compute the integrals $\int_{p_{r1}}^{p_{r2}} dJ(x)/x^\sigma$ and $\int_{p_{r1}}^{p_{r2}} dJ(x)/x^s$, we need to write $J(x)$ in terms of the function $\psi(x)$

Definition 6. *We define the function $\psi(x)$ as*

$$\psi(x) = \frac{1}{2}\left(\sum_{{p_i}^m \leq x} \log p_i + \sum_{{p_i}^m < x} \log p_i\right)$$

The function $\psi(x)$ can be expressed using Von Mangoldt formula given by

Von Mangoldt formula [2].

$$\psi(x) - x = -\sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2})$$

where the sum $\sum_\rho x^\rho/\rho$ is performed over the non-trivial zeros $\rho_i = \alpha_i + i\gamma_i$. This sum is conditionally convergent and it should be performed over the non-trivial zeros with $|\gamma_i| \leq T$ as T approaches infinity.

The function $J(x)$ can be written in term of the function $\psi(x)$ using Lemma 6 of [16]

Lemma 5 ([16]). *The function $J(x)$ defined by $\pi(x) - \mathrm{Li}(x))$ is given by*

$$J(x) = -\sum_{n=2}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n} + \frac{\psi(x) - x}{\log x} + P(x),$$

where,

$$P(x) = \int_2^x \frac{\psi(u) - u}{u \log^2 u} du + \frac{2}{\log 2} - \mathrm{Li}(2),$$

Lemma 6. *On RH, $J(x)$ is given by*

$$J(x) = \frac{\psi(x) - x}{\log x} + \int_2^x \frac{\psi(u) - u}{u \log^2 u} du - \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right),$$

and

$$J(x) = \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + \int_2^x \left(\frac{1}{u \log^2 u}\sum_\rho \frac{u^\rho}{\rho}\right) du - \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right).$$

Proof. The first equation of the lemma can be driven from Lemma 5 where we have

$$\sum_{n=2}^{\lfloor \log x/\log 2\rfloor} \frac{\pi(x^{1/n})}{n} = \mathrm{Li}(x^{1/2}) + J(x^{1/2}) + \sum_{n=3}^{\lfloor \log x/\log 2\rfloor} \frac{\pi(x^{1/n})}{n}.$$

Since $\sum_{n=3}^{\lfloor \log x/\log 2\rfloor} \frac{\pi(x^{1/n})}{n} = O(\pi(x^{1/3}))$ and on RH $J(x^{1/2}) = O(x^{1/4}\log x)$, thus

$$\sum_{n=2}^{\lfloor \log x/\log 2\rfloor} \frac{\pi(x^{1/n})}{n} = \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right).$$

Referring to Lemma 5, we then have

$$J(x) = \frac{\psi(x)-x}{\log x} + \int_2^x \frac{\psi(u)-u}{u\log^2 u}du - \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right),$$

and by the virtue of Von Mangoldt formula, we then have

$$J(x) = \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + \int_2^x \left(\frac{1}{u\log^2 u}\sum_\rho \frac{u^\rho}{\rho}\right) du - \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right).$$

□

The following lemma deals with the size of $J(x)$

Lemma 7 (Size of $J(x)$). *The size of $J(x)$ is given by*

i Unconditionally,

$$J(x) = O\left(xe^{-a\sqrt{\log x}}\right), \tag{24}$$

where $a > 0$.

ii If the not-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 \le c < 1$), then

$$J(x) = O(x^c \log x).$$

iii On RH

$$J(x) = O(x^{1/2}\log x).$$

Proof. For i, refer to page 43 of [17].

For ii, refer to Theorem 5.8 of [13] which states that If the not-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$, then

$$\frac{\psi(x)-x}{\log x} = O(x^c \log x).$$

We then substitute $O(x^c\log x)$ for $\frac{\psi(x)-x}{\log x}$ in the expression for $J(x)$ in Lemma 5 to get the desired result.

For iii, we refer to [15], where on RH, $J(x)$ is given by

$$J(x) < \frac{1}{8\pi}\sqrt{x}\log x \qquad \text{for } x > 2657$$

□

Lemma 8. *If the not-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 \le c < 1$), then $J(x)$ is given by*

$$J(x) = \frac{\psi(x) - x}{\log x} + \int_2^x \frac{\psi(u) - u}{u \log^2 u} du - \mathrm{Li}(x^{1/2}) + O\left(x^{\max(1/3, c/2)}\right),$$

and

$$J(x) = \frac{1}{\log x} \sum_\rho \frac{x^\rho}{\rho} + \int_2^x \left(\frac{1}{u \log^2 u} \sum_\rho \frac{u^\rho}{\rho} \right) du - \mathrm{Li}(x^{1/2}) + O\left(x^{\max(1/3, c/2)}\right).$$

Proof. The first equation of the lemma can be driven from Lemma 5 where we have

$$\sum_{n=2}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n} = \mathrm{Li}(x^{1/2}) + J(x^{1/2}) + \sum_{n=3}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n}.$$

Since $\sum_{n=3}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n} = O(\pi(x^{1/3}))$ and by the virtue of Lemma 7 $J(x^{1/2}) = O(x^{c/2})$, thus

$$\sum_{n=2}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n} = \mathrm{Li}(x^{1/2}) + O\left(x^{\max(1/3, c/2)}\right).$$

Referring to Lemma 5, we then have

$$J(x) = \frac{\psi(x) - x}{\log x} + \int_2^x \frac{\psi(u) - u}{u \log^2 u} du - \mathrm{Li}(x^{1/2}) + O\left(x^{\max(1/3, c/2)}\right),$$

and by the virtue of Von Mangoldt formula, we then have

$$J(x) = \frac{1}{\log x} \sum_\rho \frac{x^\rho}{\rho} + \int_2^x \left(\frac{1}{u \log^2 u} \sum_\rho \frac{u^\rho}{\rho} \right) du - \mathrm{Li}(x^{1/2}) + O\left(x^{\max(1/3, c/2)}\right).$$

□

We now use Lemma 7 to estimate the size of $\varepsilon(\sigma; p_{r1}, p_{r2})$ (or size of integral $\int_{p_{r1}}^{p_{r2}} dJ(x)/x^\sigma$)

Lemma 9 (Size of $\int dJ(x)/x^\sigma$).
$\varepsilon(\sigma; p_{r1}, p_{r2})$ is given by

i If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 \le c < 1$), then for $\sigma > c$ we have

$$\varepsilon(\sigma; p_{r1}, p_{r2}) = O\left(\frac{{p_{r1}}^{c-\sigma} \log p_{r1}}{(\sigma - c)^2} \right)$$

ii On RH, we have for $\sigma > 0.5$

$$\varepsilon(\sigma; p_{r1}, p_{r2}) = O\left(\frac{{p_{r1}}^{0.5-\sigma} \log p_{r1}}{(\sigma - 0.5)^2} \right)$$

Proof.

$$\varepsilon(\sigma; p_{r1}, p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x)$$

By the method of integration by parts, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = \frac{J(p_{r2})}{(p_{r2})^\sigma} - \frac{J(p_{r1})}{(p_{r1})^\sigma} - \int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^\sigma}\right)$$

The function $x^{-\sigma}$ is a monotone decreasing strictly positive function. Thus, for i and referring to Lemma 5 to substitute $O\left(x^c \log x\right)$ for $J(x)$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = \frac{O\left({p_{r2}}^c \log p_{r2}\right)}{{p_{r2}}^\sigma} - \frac{O\left({p_{r1}}^c \log p_{r1}\right)}{{p_{r1}}^\sigma} - \int_{p_{r1}}^{p_{r2}} O\left(x^c \log x\right) d\left(\frac{1}{x^\sigma}\right)$$

Since $x > 0$, thus

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = \frac{O\left({p_{r2}}^c \log p_{r2}\right)}{{p_{r2}}^\sigma} - \frac{O\left({p_{r1}}^c \log p_{r1}\right)}{{p_{r1}}^\sigma} - O\left(\int_{p_{r1}}^{p_{r2}} x^c \log x \, d\left(\frac{1}{x^\sigma}\right)\right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} x^c \log x \, d\left(\frac{1}{x^\sigma}\right) = -\int_{\log p_{r1}}^{\log p_{r2}} \sigma y e^{(c-\sigma)y} dy.$$

Since

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax},$$

therefore

$$\int_{p_{r1}}^{p_{r2}} x^c \log x \, d\left(\frac{1}{x^\sigma}\right) = -\sigma\left(\frac{\log p_{r2}}{c-\sigma} - \frac{1}{(c-\sigma)^2}\right) {p_{r2}}^{c-\sigma} + \sigma\left(\frac{\log p_{r1}}{c-\sigma} - \frac{1}{(c-\sigma)^2}\right) {p_{r1}}^{c-\sigma}.$$

Hence, for $\sigma > c$, we then have

$$\varepsilon(\sigma; p_{r1}, p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = O\left(\frac{{p_{r1}}^{c-\sigma} \log p_{r1}}{(\sigma - c)^2}\right) \tag{25}$$

For ii, we set $c = 0.5$ in the above equation to obtain

$$\varepsilon(\sigma; p_{r1}, p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = O\left(\frac{{p_{r1}}^{0.5-\sigma} \log p_{r1}}{(\sigma - 0.5)^2}\right) \tag{26}$$

□

The following lemma deal with the size of $\varepsilon(s; p_{r1}, p_{r2})$ (or size of integral $\int_{p_{r1}}^{p_{r2}} dJ(x)/x^s$) when s is a complex variable.

Lemma 10 (Size of $\int dJ(x)/x^s$).
$\varepsilon(s; p_{r1}, p_{r2})$ where $s = \sigma + it$ is given by

i If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $\sigma > c$ we have

$$|\varepsilon(s;p_{r1},p_{r2})|= O\left(|s|\ \frac{{p_{r1}}^{c-\sigma}\log p_{r1}}{(\sigma-c)^2}\right).$$

ii On RH, we have for $\sigma > 0.5$

$$|\varepsilon(s;p_{r1},p_{r2})|= O\left(|s|\ \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

Proof.

$$\varepsilon(s;p_{r1},p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x)$$

By the method of integration by parts, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x) = \frac{J(p_{r2})}{(p_{r2})^s} - \frac{J(p_{r1})}{(p_{r1})^s} - \int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right) \tag{27}$$

or,

$$\left|\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x)\right| \leq \left|\frac{J(p_{r2})}{(p_{r2})^s}\right| + \left|\frac{J(p_{r1})}{(p_{r1})^s}\right| + \left|\int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right)\right|$$

Thus, for i and referring to Lemma 5 to substitute $O\left(x^c \log x\right)$ for $J(x)$, we then have

$$\left|\frac{J(p_{r2})}{(p_{r2})^s}\right| = \frac{O\left({p_{r2}}^c \log p_{r2}\right)}{{p_{r2}}^\sigma}$$

$$\left|\frac{J(p_{r1})}{(p_{r1})^s}\right| = \frac{O\left({p_{r1}}^c \log p_{r1}\right)}{{p_{r1}}^\sigma}$$

and

$$\left|\int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right)\right| = O\left(\left|\int_{p_{r1}}^{p_{r2}} x^c \log x\, d\left(\frac{1}{x^s}\right)\right|\right)$$

or

$$\left|\int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right)\right| = O\left(\left|\int_{p_{r1}}^{p_{r2}} x^c \log x\ s x^{-s-1} dx\right|\right)$$

Hence

$$\left|\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x)\right| = \frac{O\left({p_{r2}}^c \log p_{r2}\right)}{{p_{r2}}^\sigma} + \frac{O\left({p_{r1}}^c \log p_{r1}\right)}{{p_{r1}}^\sigma} + O\left(|s| \int_{p_{r1}}^{p_{r2}} x^c \log x\, |x^{-s-1}| dx\right)$$

Hence, for $\sigma > c$, we have

$$\varepsilon(\sigma;p_{r1},p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = O\left(|s| \frac{{p_{r1}}^{c-\sigma}\log p_{r1}}{(\sigma-c)^2}\right) \tag{28}$$

For ii, we follow the same steps with $O\left(x^{0.5}\log x\right)$ is substituted for $J(x)$ to obtain on RH and for $\sigma > 0.5$

$$\varepsilon(\sigma;p_{r1},p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dJ(x) = O\left(|s| \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right) \tag{29}$$

□

Lemmas 9 and 10 provide strict upper boards for the integrals $\int dJ(x)/x^\sigma$ and $\int dJ(x)/x^s$. The following lemma provides a more relaxed upper bound for the integral $\int |dJ(x)/x|$. This lemma will be useful in our analysis in later sections.

Lemma 11. *Unconditionally and for any prime number p_r, the integral $\int_1^a (|dJ(p_r^x)| / p_r^x)$ is given by*

$$\int_1^a \frac{|dJ(p_r^x)|}{p_r^x} \leq 2\log(a) + O(1/\log p_r)$$

or

$$\int_1^a \frac{|dJ(p_r^x)|}{p_r^x} = O(\log(a))$$

Proof. We first note that

$$dJ(p_r^x) = d\pi({p_r}^x) - d\mathrm{Li}({p_r}^x),$$

or

$$\frac{dJ({p_r}^x)}{dx} = \frac{d\pi({p_r}^x)}{dx} - \frac{1}{\log {p_r}^x}\frac{d{p_r}^x}{dx}.$$

Hence

$$\frac{dJ({p_r}^x)}{dx} = \sum_{i=1}^{\infty} \delta({p_r}^x - p_i) - \frac{p_r^x}{x},$$

or

$$\frac{|dJ({p_r}^x)|}{dx} \leq \frac{p_r^x}{x} + \sum_{i=1}^{\infty} \delta({p_r}^x - p_i). \tag{30}$$

Hence

$$\int_1^a \frac{|dJ(p_r^x)|}{p_r^x} \leq \int_1^a \frac{dx}{x} + \int_1^a \frac{\sum_{i=1}^{\infty} \delta({p_r}^x - p_i)}{p_r^x} dx,$$

or

$$\int_1^a \frac{|dJ(p_r^x)|}{p_r^x} \leq \log a + \sum_{p_r \leq p_i \leq p_r^a} \frac{1}{p_i}$$

By the virtue of Mertens' theorem [12] (which states $\sum_{p_i<x} 1/p_i = \log\log x + b + O(1/\log x)$ where b is a constant), we then have

$$\int_1^a \frac{|dJ(p_r^x)|}{p_r^x} \leq 2\log(a) + O(1/\log p_r)$$

or

$$\int_1^a \frac{|dJ(p_r^x)|}{p_r^x} = O(\log(a))$$

□

For the remaining of this section, we will present some of the well-known results in number theory and complex analysis that we will use in our analysis.

Weiestrass theorem [4]. *If the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then f is also analytic on Ω and $f_n^{'}$ converges uniformly to $f^{'}$ on Ω*

Cramer 's theorem on the gap between primes [3]. *On RH, the gap between the prime numbers p_{r-1} and p_r is less than $k\sqrt{p_r}\log p_r$ for some constant k*

Average difference between consecutive prime numbers. *There are infinitely many primes p_r such that $p_r - p_{r-1}$ is less than or equal to $\log p_r$ (this result follows directly from the Prime Number Theorem).*

It should be pointed out that in Lemma 11, we used the Dirac delta function to represent the function $dJ(x)/dx$. The discontinuities in the function $J(x)$ are step functions at each prime number. In this paper, we will consider the Dirac delta function (that corresponds to these discontinuities at the prime numbers) as the limit of a Gaussian distribution (i.e. $\delta(x) = \lim_{\epsilon\to+0}\ e^{-x^2/4\epsilon}/(2\sqrt{\pi\epsilon})$). All of the discontinuities encountered in this paper can be traced back to step functions at integers and the derivative at these discontinuities can be represented by a Dirac delta function as the limit of a Gaussian distribution.

3 The region of convergence for the series $M(s)$ and $M(s, p_r)$.

In this section, we will deal with the question of the relationship between the conditional convergence of the two series $M(s, p_r)$ and $M(s)$ over the strip $0.5 < \Re(s) \leq 1$. Theorem 1 establishes this relationship.

Theorem 1. *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(s)$ converges conditionally if and only if the series $M(s, p_r)$ converges conditionally. Furthermore, within the region of convergence, $M(s)$ and $M(s, p_r)$ are related as follows*

$$M(s) = M(s, p_r)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right). \tag{31}$$

Proof. The proof of this theorem can be achieved either by applying the Cauchy convergence criteria or more conveniently by applying the complex analysis where we take advantage of the fact that both functions $\zeta(s)$ and $\zeta(s)\prod_{i=1}^{r-1}(1 - 1/p_i^s)$ have the same zeros (and a simple pole at $s = 1$) to the right of the line $\Re(s) = 0$.

In the following, we will use the complex analysis to prove Theorem 1 by using a method similar to the one outlined by Littlewood theorem that shows that the Riemann Hypothesis is valid if and only if the sum $\sum_{n=1}^{\infty}\mu(n)/n^s$ is convergent to $1/\zeta(s)$ for every s with $\sigma > 0.5$. The proof of this theorem can be found in [18] (refer to Theorem 14.12) and it depends mainly on Lemma 3.12 of the same reference [18]. This Lemma states: Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$, where $\sigma > 1$, $a_n = O(\psi(n))$ being non-decreasing and $\sum_{n=1}^{\infty}|a_n|/n^\sigma = O(1/(\sigma-1)^\alpha)$ as $\sigma \to 1$. Then, if $c > 0$, $\sigma + c > 1$, x is not an integer and N is the integer nearest to x, we have

$$\sum_{n<x}\frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}dw + O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) + O\left(\frac{\psi(2x)x^{1-\sigma}\log x}{T}\right) + O\left(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right)$$

To prove the first part of Theorem 1 (i.e. for $s = \sigma + it$ and $0.5 < \sigma \leq 1$, the series $M(s, p_r)$ converges conditionally if $M(s)$ converges conditionally), we note that for $\sigma > 1$,

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

and

$$M(s,p_r) = \sum_{n=1}^{\infty} \frac{\mu(n,p_r)}{n^s} = \frac{1}{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)}.$$

If we assume that $M(s)$ is convergent for $\sigma > h > 0.5$, then $\zeta(s)$ has no zeros in the complex plane to the right of the line $\Re(s) = h$ [18] (refer to Theorem 14.12). Consequently, the function $\zeta(s)\prod_{i=1}^{r-1}(1-1/p_i^s)$ has no zeros in the complex plane to the right of the line $\Re(s) = h$. Thus, we may apply Lemma 3.12 [18] with $a_n = \mu(n,p_r)$, $f(s) = 1/\Big(\zeta(s)\prod_{i=1}^{r-1}(1-1/p_i^s)\Big)$, $c = 2$ and x half an odd integer to obtain [18] (refer to Theorem 14.12)

$$\sum_{n<x} \frac{\mu(n,p_r)}{n^s} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{s+w}}\right)}\frac{x^w}{w}dw + O\left(\frac{x^2}{T}\right)$$

However, by the calculus of residues we have

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{s+w}}\right)}\frac{x^w}{w}dw = \frac{1}{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)} +$$

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{h-\sigma+\gamma-iT} + \int_{h-\sigma+\gamma-iT}^{h-\sigma+\gamma+iT} + \int_{h-\sigma+\gamma+iT}^{2+iT}\right)\frac{1}{\zeta(s+w)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{s+w}}\right)}\frac{x^w}{w}dw$$

where, $0 < \gamma < \sigma - h$. Since, along the line of integration and for an arbitrary small ϵ, we have $1/\zeta(\sigma+iT) = O(T^{\epsilon})$ [18], therefore the first and third integrals on right side of the above equation are given by $O(T^{-1+\epsilon}x^2)$ while the second integral is given by $O(x^{h-\sigma+\gamma}T^{\epsilon})$. Hence

$$\sum_{n<x} \frac{\mu(n,p_r)}{n^s} = \frac{1}{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)} + O(T^{-1+\epsilon}x^2) + O(T^{\epsilon}x^{h-\sigma+\gamma})$$

Taking $T = x^3$, the $O-$terms tend to zero as x approaches infinity. Consequently, the partial sum $\sum_{n<x}\mu(n,p_r)/n^s$ is convergent as x approaches infinity and it is given by

$$M(s,p_r) = \sum_{n=1}^{\infty} \frac{\mu(n,p_r)}{n^s} = \frac{1}{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)}.$$

or

$$M(s) = M(s,p_r)\prod_{i=1}^{r-1}\left(1-\frac{1}{{p_i}^s}\right).$$

Similarly, we can prove the second part of Theorem 1 (i.e. for $s = \sigma + it$ and $0.5 < \sigma \leq 1$, the series $M(s)$ converges conditionally if $M(s,p_r)$ converges conditionally). Alternatively, the second part of the theorem can be also proved by refereeing to Lemma 1 where

$$M(s,p_{r-1};1,Np_{r-1}) = M(s,p_r;1,Np_{r-1}) - \frac{1}{p_{r-1}^s}M(s,p_r;1,N).$$

Since the series $M(s,p_r)$ is conditionally convergent, then the partial sums $M(s,p_r;1,Np_r)$ and $M(s,p_r;1,N)$ are both convergent to $M(s,p_r)$ as N approaches infinity. Furthermore, the partial sum $M(s,p_r;Np_{r-1},Np_{r-1}+k)$ (for any integer k in the range $1 \leq k \leq p_{r-1}$) approaches zero as N approaches infinity (note that $|M(s,p_r;Np_{r-1},Np_{r-1}+k)| < \frac{1}{Np_{r-1}} + \frac{1}{Np_{r-1}+1} + ... + \frac{1}{Np_{r-1}+p_{r-1}} < 1/N$). Hence, as N approaches infinity, we obtain

$$M(s,p_{r-1}) = \lim_{x\to\infty} M(s,p_{r-1};1,x) = M(s,p_r)\left(1 - \frac{1}{p_{r-1}^s}\right).$$

By repeating this process $r-1$ times, we then obtain

$$M(s) = M(s,p_r)\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right).$$

□

4 Functional representation of $\zeta(s)$ using its partial Euler product.

In this section, we will use the prime counting function to derive a functional representation for $\zeta(s)$ using its partial Euler product. We will then use it to find a functional representation for the Dirichlet series $M(s,p_r)$.

We will start this task by first writing $\zeta(s)$ in terms of its Euler product for $\sigma > 1$ as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\prod_{r}^{\infty}\left(1 - \frac{1}{p_i^s}\right). \tag{32}$$

Taking the logarithm of both side,

$$-\log\zeta(z) = \sum_{i=1}^{r-1}\log\left(1 - \frac{1}{{p_i}^s}\right) + \sum_{i=r}^{\infty}\log\left(1 - \frac{1}{{p_i}^s}\right) + 2\pi i N,$$

where N is zero, positive or negative integer to account for the ambiguity in the phase of the logarithm of complex numbers. Referring to Lemmas 2 and 4, we then have

$$-\log\zeta(z) = \sum_{i=1}^{r-1}\log\left(1 - \frac{1}{{p_i}^s}\right) - E_1((s-1)\log p_r) - \varepsilon(s;p_r) + \delta(s;p_r) + 2\pi i N.$$

Rearranging the terms of the above equation and then taking the exponential of both sides, we then have for $\sigma > 1$

$$\zeta(s)\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right) = e^{\varepsilon(s;p_r)-\delta(s;p_r)}. \tag{33}$$

Our task in this section is to show that Equation 33 is also valid not only for $\sigma > 1$ but also it is valid for the region of convergence of the Dirichlet series $M(s,p_r)$. This task will be achieved by first proving that, for the region of convergence of the Dirichlet series $M(s, p_r)$, we have

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\}=1. \tag{34}$$

Toward this task, we first define the functions $G(s, p_r)$ and $G(s)$

Definition 7. *For any prime number p_r, let the function $G(s, p_r)$ be defined as*

$$G(s,p_r)=\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right) \tag{35}$$

also for any integer n, let $G(s, n)$, be defined as

$$G(s,n)=G(s,p_r)$$

where, p_r is the largest prime number that is less than or equal to n. Furthermore, let the function $G(s)$ be defined as

$$G(s)=\lim_{r\to\infty}G(s,p_r) \tag{36}$$

Note that, for every p_r, the function $\exp(-E_1((s-1)\log p_r))$ is an entire function with a zero at $s = 1$, the function $\zeta(s)$ is analytic everywhere except at $s = 1$ and the function $\prod_{i=1}^{r-1}(1 - 1/p_i^s)$ is analytic for $\Re(s) > 0$. Thus, for any $\sigma > 1$, the function $G(s, p_r)$ can be considered as a sequence of analytic functions. We will show that this sequence of analytic functions is convergent to the analytic function one. Furthermore, we will show that $G(s, p_r)$ has a removable singularity at $s = 1$ and for the region of convergence of the Dirichlet series $M(s, p_r)$, the sequence of analytic functions $G(s, p_r)$ is convergent to the analytic function one (i.e $G(s) = 1$).

Lemma 12. *For $\sigma > 1$, $G(s) = 1$*

Proof. For $\sigma > 1$, we have (refer to Equation 33)

$$\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)=e^{\varepsilon(s;p_r)-\delta(s;p_r)}.$$

For $\sigma > 1$, by the virtue of Lemma 2, we have

$$\lim_{r\to\infty}|\delta(s;p_r)|=\lim_{r\to\infty}O(p_r^{1-2\sigma})=0$$

Furthermore, referring to Equation (27) of Lemma 10, we also have for $\Re(s) > 1$

$$\varepsilon(s;p_r)=\int_{p_r}^{\infty}\frac{1}{x^s}dJ(x)=\frac{J(x)}{x^s}\bigg|_{p_r}^{\infty}-\int_{p_r}^{\infty}J(x)d\left(\frac{1}{x^s}\right)$$

where $J(x)$ is unconditionally given by (refer to Equation (24))

$$J(x)=O\left(xe^{-a\sqrt{\log x}}\right).$$

Hence

$$|\varepsilon(s;p_r)|=\left|\int_{p_r}^{\infty}\frac{1}{x^s}dJ(x)\right|=\frac{O\left(p_r e^{-a\sqrt{\log p_r}}\right)}{{p_r}^{\sigma}}+O\left(|s|\int_{p_r}^{\infty}xe^{-a\sqrt{\log x}}\,|x^{-s-1}|dx\right)$$

Thus, for $\sigma>1$, we then have

$$\lim_{r\to\infty}|\varepsilon(s;p_r)|=0.$$

Therefore for $\Re(s)>1$, we then have

$$G(s)=\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\}=1$$

Alternatively, we may prove this lemma by noting that for $\sigma>1$ we have

$$\zeta(s)=\lim_{r\to\infty}\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)$$

Furthermore, for $\sigma>1$, the asymptotic expansion for $E_1(s)$ is given by [1]

$$E_1(s)=\frac{e^{-s}}{s}\left(1+O\left(\frac{1}{s}\right)\right)$$

Hence, for $\Re(s)>1$

$$\lim_{r\to\infty}|\exp\left(-E_1((s-1)\log p_r)\right)|=1$$

Consequently,

$$G(s)=\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\}=1$$

□

Our next task is to extend the results of Lemma 12 to the line $s=1+it$.

Lemma 13. *For line $s=1+it$, $G(s)=1$*

Proof. We will first show that although both $\zeta(s)$ and $E_1((s-1)\log p_r)$ have a singularity at $s=1$, the product $G(s,p_r)$ has a removable singularity at $s=1$ for every p_r. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity at $s=1$

$$\zeta(s)=\frac{1}{s-1}+\gamma-\gamma_1(s-1)+\gamma_2\frac{(s-1)^2}{2!}-\gamma_3\frac{(s-1)^3}{3!}+...,\tag{37}$$

where γ is the Euler-Mascheroni constant and γ_i's are the Stieltjes constants. For $s=1+\epsilon$ (where $\epsilon=\epsilon_1+i\epsilon_2$, ϵ_1 and ϵ_2 are arbitrary small numbers), the above equation can be written as

$$\zeta(s)=\frac{1}{\epsilon}+\gamma-\gamma_1\epsilon+\gamma_2\frac{\epsilon^2}{2!}-\gamma_3\frac{\epsilon^3}{3!}+...\tag{38}$$

Furthermore, using the definition of the Exponential Integral, we may write $E_1(s)$ as

$$E_1(s) = -\gamma - \log s + s - \frac{s^2}{2\,2!} + \frac{s^3}{3\,3!} - \frac{s^4}{4\,4!} + ... \tag{39}$$

Thus, for $s = 1 + \epsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\epsilon\, \log p_r\, \exp\left(-\epsilon \log p_r + \frac{(\epsilon \log p_r)^2}{2\,2!} - \frac{(\epsilon \log p_r)^3}{3\,3!} + ...\right). \tag{40}$$

By taking the product $\zeta(s)\exp\left(-E_1((s-1)\log p_r)\right)$ and allowing $|\epsilon|$ to approach zero, we then have

$$\lim_{s\to 1}\left\{\zeta(s)\, \exp\left(-E_1((s-1)\log p_r)\right)\right\} = e^{\gamma}\, \log p_r. \tag{41}$$

However, it is well known that the partial Euler product at $s = 1$ can be written as [11]

$$\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_{r-1}} + O\left(\frac{1}{(\log p_{r-1})^2}\right). \tag{42}$$

Multiplying Equations (41) and (42), we then conclude that at $s = 1$, $G(s, p_r)$ approaches 1 as p_r approaches infinity.

Furthermore, for $s = 1 + it$ and $t \neq 0$, the value of $\exp(-E_1(it \log p_r))$ approaches 1 as p_r approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1 - \frac{1}{{p_i}^s}\right)\right\} = 1,$$

therefore, for $s = 1 + it$, we have the following

$$G(s) = \lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\} = 1.$$

□

So far, we have shown that the function $G(s, p_r)$ is uniformly convergent to 1 when $\Re(s) > 1 + \delta > 1$ (where δ is an arbitrary small number). We have also shown that $G(s, p_r)$ is convergent to 1 for $\Re(s) = 1$. In the following theorem, we will show that, assuming the validity of the Riemann Hypothesis, the function $G(s, p_r)$ is uniformly convergent to 1 for every value of s with $\Re(s) > 0.5 + \epsilon$, where ϵ is an arbitrary small number.

Theorem 2. *On RH and for $\sigma > 0.5$, we have*

$$G(s) = \lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\} = 1, \tag{43}$$

and

$$\lim_{r\to\infty}\left\{M(s, p_r)\exp\left(E_1((s-1)\log p_r)\right)\right\} = 1. \tag{44}$$

Proof. We first write the expression for $G(s,p_{r1})$ and $G(s,p_{r2})$ where $r2$ is an arbitrary large number greater than $r1$

$$G(s,p_{r1}) = \zeta(s)\exp\left(-E_1((s-1)\log p_{r1})\right)\prod_{i=1}^{r1-1}\left(1-\frac{1}{p_i^s}\right), \tag{45}$$

$$G(s,p_{r2}) = \zeta(s)\exp\left(-E_1((s-1)\log p_{r2})\right)\prod_{i=1}^{r2-1}\left(1-\frac{1}{p_i^s}\right). \tag{46}$$

Since the function $G(s,p_r)$ is analytic and it is not equal to 0 for $\sigma > 0.5$, hence we can divide Equation (46) by Equation (45) and then take the logarithm to obtain

$$\log\left(\frac{G(s,p_{r2})}{G(s,p_{r1})}\right) = E_1\left((s-1)\log p_{r1}\right) - E_1\left((s-1)\log p_{r2}\right) + \log\left(\prod_{i=r1}^{r2-1}\left(1-\frac{1}{{p_i}^s}\right)\right) + 2i\pi N_1. \tag{47}$$

where N_1 is zero, positive or negative integer.

Referring to Equation (15), we then have

$$\log\frac{G(s,p_{r2})}{G(s,p_{r1})} = -\varepsilon(s;p_{r1},p_{r2-1})+\delta(s;p_{r1},p_{r2-1})+E_1((s-1)\log p_{r2-1})-E_1((s-1)\log p_{r2})+2i\pi N_1.$$

Taking the exponential of both sides, we then have

$$\frac{G(s,p_{r2})}{G(s,p_{r1})} = \exp\left(-\varepsilon(s;p_{r1},p_{r2-1})+\delta(s;p_{r1},p_{r2-1})+E_1((s-1)\log p_{r2-1})-E_1((s-1)\log p_{r2})\right).$$

or

$$G(s,p_{r2}) = G(s,p_{r1})\ e^{-\varepsilon(s;p_{r1},p_{r2-1})+\delta(s;p_{r1},p_{r2-1})}\ e^{E_1((s-1)\log p_{r2-1})-E_1((s-1)\log p_{r2})}. \tag{48}$$

However on RH, the absolute value of difference $E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})$ is bounded and it approaches zero as p_{r2} approaches infinity. This can be proved by recalling Cramer's theorem on the gap between consecutive primes [3]. By the virtue of Cramer's theorem, we have on RH

$$p_r = p_{r-1} + p_{r-1}\,O\left(\frac{1}{\sqrt{p_{r-1}}}\right),$$

or

$$\log(p_r) = \log\left(p_{r-1} + p_{r-1}\,O\left(\frac{1}{\sqrt{p_{r-1}}}\right)\right).$$

Hence

$$\log(p_r) = \log(p_{r-1}) + \log\left(1 + O\left(\frac{1}{\sqrt{p_{r-1}}}\right)\right).$$

Since $\log(1+x) = O(x)$ for $x << 1$, thus

$$\log(p_r) = \log(p_{r-1}) + O\left(\frac{1}{\sqrt{p_{r-1}}}\right).$$

Furthermore, since the function $E_1(z)$ is analytic, therefore

$$E_1(z+\Delta z) - E_1(z) = \frac{dE_1(z)}{dz}\Delta z = E_0(z)\Delta z \quad as \ \ |\Delta z| \to 0$$

Hence

$$E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2}) = -E_0((s-1)\log p_{r2-1})\,(s-1)\,O\left(\frac{1}{\sqrt{p_{r2-1}}}\right),$$

where $E_0(z) = e^{-z}/z$. Thus,

$$|E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})| \le \left|\frac{e^{-(s-1)\log p_{r2-1}}}{(s-1)\log p_{r2-1}}\right| \ |s-1|\, O\left(\frac{1}{\sqrt{p_{r2-1}}}\right).$$

Hence

$$|E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2-1})| = O\left(\frac{1}{\log p_{r2-1}}\right).$$

Consequently on RH and for $\Re(s) > 0.5$, $|E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})|$ is bounded. Moreover, as p_{r2} approaches infinity, $|E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})|$ approaches zero and the term $|e^{E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})}|$ approaches 1.

For a fixed p_{r1} and arbitrary $p_{r2}(> p_{r1})$, the term $G(s, p_{r1})$ is fixed and bounded. Furthermore, on RH and by the virtue of Lemma 10, the term $e^{-\varepsilon(s;p_{r1},p_{r2-1})+\delta(s;p_{r1},p_{r2-1})}$ is also bounded for $\Re(s) > 0.5$ (note that as p_{r2} approaches infinity, the term $\delta(s; p_{r1}, p_{r2-1})$ is unconditionally convergent for $\Re(s) > 0.5$ by the virtue of Lemma 2). Hence, by the virtue of Equation (48), $G(s, p_{r2})$ is also bound for $\Re(s) > 0.5$.

In the following, using Cauchy convergence criteria, we will show that $G(s, p_r)$ convergences as p_r approaches infinity. First we recall that on RH and for $\sigma > 0.5 + \epsilon$, the term $|\varepsilon(s; p_{r1})| + |\delta(s; p_{r1})|$ can be made arbitrary small by choosing p_{r1} sufficiently large (refer to Lemmas 2 and 10). Let p_{r1a} and p_{r1b} be any two prime numbers greater than p_{r1}. Choose p_{r2} such that $p_{r1a} < p_{r2}$ and $p_{r1b} < p_{r2}$. Thus

$$G(s, p_{r1a}) = G(s, p_{r2})\, e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}) + \Delta(s;p_{r2}))}$$

$$G(s, p_{r1b}) = G(s, p_{r2})\, e^{(\varepsilon(s;p_{r1b},p_{r2-1}) - \delta(s;p_{r1b},p_{r2-1}) + \Delta(s;p_{r2}))}$$

where,

$$\Delta(s; p_{r2}) = -E_1((s-1)\log p_{r2-1}) + E_1((s-1)\log p_{r2}).$$

Thus,

$$|G(s,p_{r1a}) - G(s,p_{r1b})| = |G(s,p_{r2})||e^{\Delta(s;p_{r2})}| \left| e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}))} - e^{(\varepsilon(s;p_{r1b},p_{r2-1}) - \delta(s;p_{r1b},p_{r2-1}))} \right|$$

Since for $|x| < 1$, $e^x = 1 + O(x)$ and for sufficiently large p_{r1}, $|\varepsilon(s; p_{r1a}, p_{r2-1})|$, $|\delta(s; p_{r1a}, p_{r2-1})|$, $|\varepsilon(s; p_{r1b}, p_{r2-1})|$ and $|\delta(s; p_{r1b}, p_{r2-1})|$ are less than 1, therefore

$$\left| e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}))} - e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}))} \right| =$$

$$O\left(|\varepsilon(s; p_{r1a}, p_{r2-1})|\right) + O\left(|\delta(s; p_{r1a}, p_{r2-1})|\right) + O\left(|\varepsilon(s; p_{r1b}, p_{r2-1})|\right) + O\left(|\delta(s; p_{r1b}, p_{r2-1})|\right).$$

Since $p_{r1a}, p_{r1b} > p_{r1}$, hence on RH and for $\Re(s) > 0.5$ (refer to lemma 10)

$$\left| e^{(\varepsilon(s;p_{r1a},p_{r2-1})-\delta(s;p_{r1a},p_{r2-1}))} - e^{(\varepsilon(s;p_{r1a},p_{r2-1})-\delta(s;p_{r1a},p_{r2-1}))} \right| = O\left(|s| \frac{{p_{r1}}^{0.5-\sigma} \log p_{r1}}{(\sigma - 0.5)^2} \right)$$

Moreover, since $|G(s, p_{r2})|$ is bounded and $|e^{\Delta(s;p_{r2})}|$ approaches 1 as p_{r2} approaches infinity, hence $|G(s, p_{r1a}) - G(s, p_{r1b})|$ can be made arbitrarily small by selecting p_{r1} sufficiently large. Consequently by the virtue of Cauchy convergence criteria, $G(s, p_r)$ (or $G(s, n)$) is convergent,

$$G(s) = \lim_{r\to\infty} G(s, p_r) \tag{49}$$

It should be noted that, while the function sequence $G(s, p_r)$ (or $G(s, n)$) is not uniformly convergent when the region of convergence is extended all the way to the line $\sigma = 0.5$, it is however uniformly convergent for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > 0.5+\epsilon$ (for any arbitrary small ϵ). This follows from Lemma 10 where on RH, $|\varepsilon(s; p_r)|$ is convergent and bounded (uniformly convergent) for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > 0.5 + \epsilon$ (for any arbitrary small ϵ). Since $G(s, p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > 0.5 + \epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) > 0.5 + \epsilon$ (Weiestrass theorem [4]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, therefore on RH and for $\Re(s) > 0.5$, we then have the desired result, i.e

$$G(s) = 1.$$

Note that for a fixed p_{r1} and as p_{r2} approaches infinity, Equation (48) can be then written as

$$G(s, p_{r1}) = e^{\varepsilon(s;p_{r1})+\delta(s;p_{r1})}.$$

In Theorem 3, we will extend the above equation to the region where the series $M(s, p_r)$ converges without the assumption of RH.

□

Corollary 1. *For the region of convergence of the series $M(s, p_r)$, we have*

$$G(s) = \lim_{r\to\infty} \left\{ \zeta(s) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_r)\right) \right\} = 1, \tag{50}$$

and

$$\lim_{r\to\infty} \left\{ M(s, p_r) \exp\left(E_1((s-1)\log p_r)\right) \right\} = 1. \tag{51}$$

Proof. If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then our task is to show that Equation (50) holds for $\Re(s) > c$. This task can be achieved by following the same steps we used to prove Theorem 2 and writing the ratio $G(s, p_{r2})/G(s, p_{r1})$ as

$$\frac{G(s, p_{r2})}{G(s, p_{r1})} = \exp\left(-\varepsilon(s; p_{r1}, p_{r2-1}) + \delta(s; p_{r1}, p_{r2-1}) + E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})\right).$$

where $p_{r1} < p_{r2}$. In the proof of Theorem 2, we let p_{r2} approaches infinity to show that $G(s, p_{r2})$ is bounded for every p_{r2}. This was achieved using Cramer's Theorem to compute $\Delta(s; p_{r2})$ for every p_{r2}. We then showed that $|\Delta(s; p_{r2})|$ approached zero as p_{r2} approached infinity. In fact, since the selection of p_{r2} is independent of the selection of p_{r1}, therefore, we only need to compute $\Delta(s;p_{r2})$ for infinitely many p_{r2}'s (and not necessarily for every p_{r2}) and then show that $|\Delta(s;p_{r2})|$ (for the selected infinitely many p_{r2}'s) approaches zero as p_{r2} approaches infinity. For the proof of this corollary, we only select p_{r2}'s that satisfy the following

$$p_{r2} - p_{r2-1} \leq \log p_{r2}$$

The prime number theorem asserts the presence of infinity many primes that satisfy the above inequality. With this selection of p_{r2}, we then have

$$\log(p_{r2}) = \log(p_{r2-1}) + O\left(\frac{\log p_{r2}}{p_{r2}}\right).$$

or

$$E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2}) = E_0((s-1)\log p_{r2-1})\,(s-1)\,O\left(\frac{\log p_{r2}}{p_{r2}}\right),$$

Hence

$$|E_1((s-1)\log p_{r2-1}) - E_1((s-1)\log p_{r2})| = O\left(\frac{1}{p_{r2}^c}\right).$$

Thus, $|\Delta(s;p_{r2})|$ approaches zero as p_{r2} approaches infinity. Consequently, $|G(s,p_{r2})|$ is bounded for infinitely many p_{r2}'s.

For the next step (as it was the case with the proof of Theorem2), we select p_{r1a} and p_{r1b} any two prime numbers greater than p_{r1} and choose p_{r2} such that $p_{r1a} < p_{r2}$ and $p_{r1b} < p_{r2}$ (where $p_{r2} - p_{r2-1} \leq \log p_{r2}$). Thus

$$G(s,p_{r1a}) = G(s,p_{r2})\, e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}) + \Delta(s;p_{r2}))}$$

$$G(s,p_{r1b}) = G(s,p_{r2})\, e^{(\varepsilon(s;p_{r1b},p_{r2-1}) - \delta(s;p_{r1b},p_{r2-1}) + \Delta(s;p_{r2}))}$$

Hence,

$$\left|e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}))} - e^{(\varepsilon(s;p_{r1b},p_{r2-1}) - \delta(s;p_{r1b},p_{r2-1}))}\right| =$$

$$O\left(|\varepsilon(s;p_{r1a},p_{r2-1})|\right) + O\left(|\delta(s;p_{r1a},p_{r2-1})|\right) + O\left(|\varepsilon(s;p_{r1b},p_{r2-1})|\right) + O\left(|\delta(s;p_{r1b},p_{r2-1})|\right).$$

or,

$$\left|e^{(\varepsilon(s;p_{r1a},p_{r2-1}) - \delta(s;p_{r1a},p_{r2-1}))} - e^{(\varepsilon(s;p_{r1b},p_{r2-1}) - \delta(s;p_{r1b},p_{r2-1}))}\right| = O\left(|s|\,\frac{{p_{r1}}^{c-\sigma}\log p_{r1}}{(\sigma-c)^2}\right)$$

Since $|G(s,p_{r2})|$ is bounded and $|e^{\Delta(s;p_{r2})}|$ approaches 1 as p_{r2} approaches infinity, hence $|G(s,p_{r1a}) - G(s,p_{r1b})|$ can be made arbitrarily small by selecting p_{r1} sufficiently large. Consequently by the virtue of Cauchy convergence criteria, $G(s,p_r)$ (or $G(s,n)$) is convergent,

$$G(s) = \lim_{r\to\infty} G(s,p_r) \tag{52}$$

As it was the case with Theorem 2, the function sequence $G(s,p_r)$ (or $G(s,n)$) is uniformly convergent for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > c+\epsilon$ (for any arbitrary small ϵ). This follows from Lemma 10 where $|\varepsilon(s;p_r)|$ is uniformly convergent for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > c+\epsilon$ (for any arbitrary small ϵ). Since $G(s,p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > c+\epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) >$

$c + \epsilon$ (Weiestrass theorem [4]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, therefore we have the desired outcome for $\Re(s) > c$, i.e

$$G(s) = 1.$$

□

In the following, we use Theorem 2 and Corollary 1 to compute $M(s, p_r)$ for any prime number p_r

Theorem 3. *For the region of convergence of the series $M(s, p_r) = \sum_1^\infty \mu(n, p_r)/n^s$, we have*

$$M(s, p_r) = e^{-E_1((s-1)\log p_r) - \varepsilon(s;p_r) + \delta(s;p_r)}, \tag{53}$$

where $\varepsilon(s; p_r) = \int_{p_r}^\infty dJ(x)/x^s$, $J(x) = \pi(x) - \mathrm{Li}(x)$ *and* $\delta(s; p_r) = \sum_{i=r}^\infty \left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}} \cdots\right)$.

Proof. Equation (50) of Corollary 1 can be written as follows

$$\log \zeta(s) + \log \prod_{i=1}^{r2-1} \left(1 - \frac{1}{p_i^s}\right) - E_1\left((s-1)\log p_{r2}\right) + 2\pi i N_2 = 0 \quad \text{as } r_2 \to \infty$$

where N_2 is zero, positive or negative integer. Notice that the equality of both sides of the above equation is attained as r_2 (or p_{r2}) approaches infinity (or more appropriately, the right side can be made arbitrary close to zero by choosing p_{r2} sufficiently large). For $r < r2$, the above equation can be then written as

$$\log \zeta(s) = E_1\left((s-1)\log p_{r2}\right) - \sum_{i=1}^{r-1} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r}^{r2-1} \log\left(1 - \frac{1}{p_i^s}\right) + 2\pi i N_3 \quad \text{as } r_2 \to \infty$$

where N_3 is zero, positive or negative integer and

$$-\sum_{i=r}^{r2-1} \log\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r}^{r2-1} \frac{1}{{p_i}^s} - \delta(s; p_r, p_{r2-1}) + 2\pi i N_4$$

where N_4 is zero, positive or negative integer. For the region of convergence of the series $M(s, p_r)$, we also have (refer to lemma 4)

$$\sum_{i=r}^{r2-1} \frac{1}{{p_i}^s} = E_1\left((s-1)\log p_r\right) - E_1\left((s-1)\log p_{r2-1}\right) + \varepsilon(s; p_r, p_{r2-1})$$

Therefore, $\zeta(s)$ can be written as

$$\zeta(s) = \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right)^{-1} \lim_{p_{r2}\to\infty} e^{E_1((s-1)\log p_r) + E_1((s-1)\log p_{r2}) - E_1((s-1)\log p_{r2-1}) + \varepsilon(s;p_r,p_{r2}) - \delta(s;p_r,p_{r2})}$$

As it is the case with Corollary 1, the above equation is valid for every p_r and p_2 (where $p_r < p_{r2-1}$). If we strict our selection for p_{r2} to the prime numbers such that $p_{r2} - p_{r2-1}$ is less than or equal to than $\log p_{r2}$, then $|E_1\left((s-1)\log p_{r2}\right) - E_1\left((s-1)\log p_{r2-1}\right)|$ approaches zero as p_{r2} approaches infinity. Thus, for every p_r, we have

$$\zeta(s) = \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right)^{-1} \lim_{p_{r2}\to\infty} e^{E_1((s-1)\log p_r) + \varepsilon(s;p_r,p_{r2}) - \delta(s;p_r,p_{r2})}$$

or

$$M(s,p_r) = e^{-E_1((s-1)\log p_r) - \varepsilon(s;p_r) + \delta(s;p_r)}$$

□

So far, we have used the complex analysis to compute $M(s,p_r)$. For the remaining of the paper, our efforts will be dedicated toward the computation of the partial sum $M(1,p_r;1,p_r^a)$ (i.e. the partial sum of the series $M(s,p_r)$ at $s=1$). In the following two section, we will use integration methods and complex analysis methods to compute the partial sum $M(1,p_r;1,p_r^a)$. In section 7, we will compare the results of these methods and then show that this comparative analysis will lead to a contradiction every time we assume that $\zeta(s)$ has no trivial zeros for $\Re(s) > c$ where $c < 1$.

5 The series $M(s,p_r)$ at $s=1$.

In this section, we will compute the partial sum $M(1,p_r;1,{p_r}^a)$ using integration methods. Before we present the details of our method, it is important to mention that the partial sum $M(1,p_r;1,{p_r}^a)$ can be also generated using y-smooth numbers. The y-smooth numbers are the numbers that have only prime factors less than or equal to y. These numbers have been extensively analyzed in the literature [6] [9]. In [6], a method was presented to generate the partial sum $M(1,p_r;1,{p_r}^a)$. With this method and using the inclusion-exclusion principle [6] (refer to page 284), one can then provide an estimate for the partial sum $M(1,p_r;1,{p_r}^a)$. In this section, we will provide a more general approach to compute $M(1,p_r;1,{p_r}^a)$. The main advantage of our approach is the ability to extend it to compute the partial sum for values of s other than 1. We will present our method in the following four steps.

- In the first step of our approach, we will show that, for every a and as p_r approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ approaches a function that is dependent on only a (independent of p_r). We will then show that this function is the Dickman function $\rho(a)$. It should be noted that the results of this step are well known in the literature. In this step, we are only rephrasing these results in terms of the integral $\int dJ(p_r^y)/p_r^y$.

Toward this end, we define the function $f(a,p_r)$ as

$$f(a,p_r) = M(1,p_r;1,{p_r}^a) = \sum_{n=1}^{{p_r}^a} \frac{\mu(n,p_r)}{n}.$$

We will then show that, for every a and as p_r approaches infinity, the function $f(a,p_r)$ approaches a deterministic function $\rho(a)$. In other words; if we plot $M(1,p_r;1,N)$ (where $N={p_r}^a$) as a function of $a=\log N/\log p_r$, then for each value of a and as p_r approaches infinity, $f(a,p_r)$ approaches a unique value $\rho(a)$. This is equivalent to the statement

$$\rho(a) = \lim_{p_r\to\infty} f(a,p_r) = \lim_{p_r\to\infty} M(1,p_r;1,{p_r}^a).$$

Lemma 14. *For* $1 \le a < 2$

$$M(1,p_r;1,{p_r}^a) = 1 - M_1(1,p_r;1,{p_r}^a), \tag{54}$$

where

$$M_1(1,p_r;1,{p_r}^a)=\sum_{p_r\le p_i\le {p_r}^a}\frac{1}{p_i}=\log(a)+g_1(p_r,a),$$

and the terms of sum $M_1(1,p_r;1,{p_r}^a)$ are placed in descending order. Furthermore

$$g_1(p_r,a)=\varepsilon(1;p_r,p_r^a)=\int_1^a\frac{dJ({p_r}^y)}{{p_r}^y},\tag{55}$$

and

$$\lim_{p_r\to\infty}M(1,p_r;1,{p_r}^a)=1-\log a.$$

Proof. This result can be achieved by first noting that the partial sum $M(1,p_r;1,{p_r}^a)$ for $1\le a<2$ is given by

$$M(1,p_r;1,{p_r}^a)=1-\sum_{p_r\le p_i\le {p_r}^a}\frac{1}{p_i}.$$

Since

$$M_1(1,p_r;1,{p_r}^a)=\sum_{p_r\le p_i\le {p_r}^a}\frac{1}{p_i},$$

therefore, using Stieltjes integral, we obtain

$$M(1,p_r;1,{p_r}^a)=1-M_1(1,p_r;1,{p_r}^a)=1-\int_{x=p_r}^{{p_r}^a}\frac{d\pi(x)}{x}=1-\int_{y=1}^{a}\frac{d\pi({p_r}^y)}{{p_r}^y}.$$

Since

$$d\pi({p_r}^y)=d\mathrm{Li}({p_r}^y)+dJ(p_r^y),$$

therefore

$$d\pi({p_r}^y)=\frac{1}{\log({p_r}^y)}d{p_r}^y+dJ({p_r}^y)=\frac{p_r^y}{y}dy+dJ({p_r}^y),$$

Hence, for $1\le a<2$, we have

$$M(1,p_r;1,{p_r}^a)=1-\int_1^a\frac{dy}{y}-\int_1^a\frac{dJ({p_r}^y)}{{p_r}^y}=1-\log(a)-g_1(p_r,a),$$

where

$$M_1(1,p_r;1,{p_r}^a)=\log(a)+g_1(p_r,a),$$

and

$$g_1(p_r,a)=\varepsilon(1;p_r,p_r^a)=\int_1^a\frac{dJ({p_r}^y)}{{p_r}^y}.$$

Referring to Lemma 9, on RH or if $\zeta(s)$ has no zeros for $\Re(s)<c<1$, then as p_r approaches infinity, $g_1(p_r,a)$ approaches zero. In fact, we can show the same results unconditionally using PNT where $J(x)=O(xe^{-b\sqrt{\log x}})$ and $b>0$. For this case

$$g_1(p_r,a)=\int_{p_r}^{p_r^a}\frac{dO\left(xe^{-b\sqrt{\log x}}\right)}{x}$$

Using the method of integration by parts, we then have

$$g_1(p_r, a) = \frac{O\left(xe^{-b\sqrt{\log x}}\right)}{x}\Bigg|_{p_r}^{p_r^a} + \int_{p_r}^{p_r^a} O\left(xe^{-b\sqrt{\log x}}\right) d\left(\frac{1}{x}\right).$$

Since, for $x \geq p_r$, the function $1/x$ is a positive monotone decreasing function, thus

$$g_1(p_r, a) = O\left(e^{-b\sqrt{\log p_r}}\right) + O\left(\int_{p_r}^{p_r^a} xe^{-b\sqrt{\log x}} d\left(\frac{1}{x}\right)\right)$$

or

$$g_1(p_r, a) = O\left(e^{-b\sqrt{\log p_r}}\right) + O\left(\int_{p_r}^{p_r^a} \frac{e^{-b\sqrt{\log x}}}{x} dx\right)$$

Substituting y for $\log x$, we then have

$$g_1(p_r, a) = O\left(e^{-b\sqrt{\log p_r}}\right) + O\left(\int_{\log p_r}^{\log p_r^a} e^{-b\sqrt{y}} dy\right)$$

Substituting z for $\sqrt{y}$, we finally have

$$g_1(p_r, a) = O\left(e^{-b\sqrt{\log p_r}}\right) + O\left(\int_{\sqrt{\log p_r}}^{\sqrt{\log p_r^a}} ze^{-bz} dz\right)$$

or (note that $\int xe^{cx}dx = (cx-1)e^{cx}/c^2$)

$$g_1(p_r, a) = O\left(\sqrt{\log p_r}\ e^{-b\sqrt{\log p_r}}\right) \tag{56}$$

Let the function $g(p_r)$ be defined as

$$g(p_r) = \sqrt{\log p_r}\ e^{-b\sqrt{\log p_r}}$$

then

$$g_1(p_r, a) = O(g(p_r)) \tag{57}$$

Note that $g(p_r)$ is a function of p_r only. As p_r approaches infinity, $g_1(p_r, a)$ approaches zero. Consequently, Equation (54) can be written as

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a.$$

□

In the following Lemma, we will extend the same results for $1 \leq a < 3$

Lemma 15. *For $1 \leq a < 3$, we have*

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) + M_2(1, p_r; 1, {p_r}^a)$$

where

$$M_1(1, p_r; 1, {p_r}^a) = \log(a) + g_1(p_r, a),$$

and

$$M_2(1, p_r; 1, {p_r}^a) = \sum_{p_r \leq p_{i1} < p_{i2} < p_{i1}p_{i2} \leq {p_r}^a} \frac{1}{p_{i1}p_{i2}} = \frac{1}{2}\int_1^{a-1} \frac{\log(a-y)}{y} dy + g_2(p_r, a) + O\left(\frac{\log a}{p_r}\right),$$

where the terms of the sum $M_2(1, p_r; 1, {p_r}^a)$ are placed in descending order. Furthermore

$$g_2(p_r, a) = \frac{1}{2}\int_1^{a-1} \frac{g_1(p_r, a-y)}{y} dy + \frac{1}{2}\int_1^{a-1} \log(a-y)\frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{2}\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}., \tag{58}$$

and

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a + \frac{1}{2}\int_1^{a-1} \frac{\log(a-y)}{y} dy.$$

Proof. The terms of the partial sum $M(1, p_r; 1, {p_r}^a)$ for a in the range $1 < a < 3$ are either a reciprocal of a prime number or a reciprocal of the product of two prime numbers. Therefore, for $1 \leq a < 3$, we have

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \leq p_i \leq {p_r}^a} \frac{1}{p_i} + \sum_{p_r \leq p_{i1} < p_{i2} < p_{i1}p_{i2} \leq {p_r}^a} \frac{1}{p_{i1}p_{i2}},$$

where (by the virtue of Lemma 14)

$$M_1(1, p_r; 1, {p_r}^a) = \sum_{p_r \leq p_i \leq {p_r}^a} \frac{1}{p_i} = \log(a) + g_1(p_r, a),$$

and

$$M_2(1, p_r; 1, {p_r}^a) = \sum_{p_r \leq p_{i1} < p_{i2} < p_{i1}p_{i2} \leq {p_r}^a} \frac{1}{p_{i1}p_{i2}},$$

where p_{i1} and p_{i2} are two distinct prime numbers that are greater than or equal to p_r and the terms of sum are placed in a descending order.

Thus, $M_2(1, p_r; 1, {p_r}^a)$ can be written as

$$M_2(1, p_r; 1, {p_r}^a) = \frac{1}{2} \sum_{p_r \leq p_i \leq {p_r}^{a-1}} \frac{1}{p_i} M_1(1, p_r; 1, p_r^a/p_i) + r_2.$$

Note that for the sum $\sum_{p_r \leq p_i \leq {p_r}^{a-1}} \frac{1}{p_i} M_1(1, p_r; 1, p_r^a/p_i)$, we added the factor of half since each term of the form $1/(p_{i1}p_{i2})$ is generated twice. Furthermore, this sum includes non square-free terms (notice that, there is no repetition in any of the non square-free terms). The term r_2 was added to offset the contribution by these non square-free terms. We will show later in Lemma 18 that r_2 is given by $O(a \log p_r / p_r)$ and hence it approaches zero as p_r approaches infinity. Using Stieltjes integral, the above equation can be then written as the following integral

$$M_2(1, p_r; 1, {p_r}^a) = \frac{1}{2}\int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y} \left(\log(a-y) + g_1(p_r, a-y)\right) + r_2.$$

Recalling, for $1 \leq a < 3$, that

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) + M_2(1, p_r; 1, {p_r}^a)$$

hence (note that $d\pi({p_r}^y)/p_r^y = dy/y + dJ({p_r}^y)/p_r^y$)

$$M(1, p_r; 1, {p_r}^a) = 1 - \log(a) - g_1(p_r, a) + \frac{1}{2}\int_1^{a-1} \frac{\log(a-y)}{y} dy + g_2(p_r, a) + r_2,$$

where

$$g_2(p_r, a) = \frac{1}{2}\int_1^{a-1} \frac{g_1(p_r, a-y)}{y} dy + \frac{1}{2}\int_1^{a-1} \log(a-y)\frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{2}\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}.$$

Since, for $a \geq 1$, the function $1/y$ is a positive monotone decreasing function, thus the first integral on the right side of Equation (58) is given unconditionally by (refer to Equation (57))

$$\left|\int_1^{a-1} \frac{g_1(p_r, a-y)}{y} dy\right| = \int_1^{a-1} O(g(p_r))\frac{dy}{y}$$

or

$$\left|\int_1^{a-1} \frac{g_1(p_r, a-y)}{y} dy\right| = \log(a-1)O(g(p_r)) \tag{59}$$

By the method of integration by parts, we can write the second integral on the right side of Equation (58) as

$$\int_1^{a-1} \log(a-y)\frac{dJ({p_r}^y)}{p_r^y} = \left.\frac{J(p_r^y)\log(a-y)}{p_r^y}\right|_1^{a-1} - \int_1^{a-1} \log(a-y)J({p_r}^y)d\left(\frac{1}{p_r^y}\right) - \int_1^{a-1} \frac{J({p_r}^y)}{p_r^y} d\log(a-y)$$

where,

$$\left.\frac{J(p_r^y)\log(a-y)}{p_r^y}\right|_1^{a-1} = \log(a-1)O\left(\frac{J(p_r)}{p_r}\right)$$

and

$$\int_1^{a-1} \log(a-y)J({p_r}^y)d\left(\frac{1}{p_r^y}\right) = \log p_r \int_1^{a-1} \log(a-y)\frac{J({p_r}^y)}{p_r^y}dy$$

Since, for $a \geq 1$, $\log(a-y)/p_r^y$ is a positive monotone decreasing function, hence

$$\left|\int_1^{a-1} \log(a-y)J({p_r}^y)d\left(\frac{1}{p_r^y}\right)\right| = (a-2)\log(a-1)\ \log p_r\ O\left(\frac{J(p_r)}{p_r}\right).$$

Furthermore, since $\log(a-y)$ is a positive monotone decreasing function, thus

$$\left|\int_1^{a-1} \frac{J({p_r}^y)}{p_r^y} d\log(a-y)\right| = O\left(\frac{J(p_r)}{p_r}\right)\int_{y=1}^{a-1} d\log(a-y) = \log(a-1)\, O\left(\frac{J(p_r)}{p_r}\right)$$

Since, unconditionally and by the virtue of PNT, we have

$$O(J(p_r)/p_r) = O(p_r e^{-b\sqrt{\log x}})/p_r,$$

or

$$O(J(p_r)/p_r) = \frac{O(g(p_r))}{\sqrt{\log p_r}},$$

hence

$$\left|\int_1^{a-1} \log(a-y)\frac{dJ({p_r}^y)}{p_r^y}\right| = (a-2)\log(a-1)\sqrt{\log(p_r)}\, O(g(p_r)) \tag{60}$$

The third integral ($\int_1^{a-1} g_1(p_r, a-y)dJ({p_r}^y)/p_r^y$) on the right side of Equation (58) has two discontinuous functions, the function $J(p_r^y)$ (that has discontinuities at values of y where p_r^y is a prime) and the function $g_1(p_r, a-y)/p_r^y$ (this function may have discontinuities when ${p_r}^{a-y}$ is an integer). Therefore, we will use Lebesgue-Stieltjes integral to compute this integral [1]. The absolute value of this integral can then be written as

$$\left|\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}\right| \leq \int_1^{a-1} |g_1(p_r, a-y)| \frac{|dJ({p_r}^y)|}{p_r^y},$$

or (note that the function $J(x)$ is given by the superposition of two monotone increasing functions i.e the functions $Li(x)$ and $\pi(x)$),

$$\left|\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}\right| \leq O(g_1(p_r, a)) \int_1^{a-1} \frac{|dJ({p_r}^y)|}{p_r^y}.$$

Since $O(g_1(p_r, a)) = O(g(p_r))$ and by the virtue of Lemma 11, we then have

$$\left|\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}\right| = 2\log(a-1)\ O(g(p_r)). \tag{61}$$

Combining Equations (59), (60) and (61), we can write Equation (58) as

$$|g_2(p_r, a)| = 2(a-2)\log(a-1)\sqrt{\log p_r}\ O(g(p_r)). \tag{62}$$

As p_r approaches infinity, $g_2(p_r, a)$ approaches zero (recall that $g(p_r) = \sqrt{\log p_r}\ e^{-b\sqrt{\log p_r}}$). Thus, for $1 \leq a < 3$, we have

$$\lim_{p_r\to\infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a + \frac{1}{2}\int_1^{a-1} \frac{\log(a-y)}{y}dy$$

Therefore, as p_r approaches infinity, $M(1, p_r; 1, {p_r}^a)$ approaches a function that is dependent on only a.

□

Repeating the previous process $\lfloor a\rfloor$ times (where $\lfloor x\rfloor$ is the integer value of x) and by using the induction method, we can show that, as p_r approaches infinity, the partial sum $M(1, p_r; 1, {p_r}^a)$ approaches a function that is dependent on only a. Specifically, we first write the partial sum $M(1, p_r; 1, {p_r}^a)$ as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) + M_2(1, p_r; 1, {p_r}^a) - ... + (-1)^j M_j(1, p_r; 1, {p_r}^a) + ...+$$
$$(-1)^{\lfloor a\rfloor -1} M_{\lfloor a\rfloor -1}(1, p_r; 1, {p_r}^a) + (-1)^{\lfloor a\rfloor} M_{\lfloor a\rfloor}(1, p_r; 1, {p_r}^a), \tag{63}$$

[1]The integral $\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}$ is also a valid Riemann-Stieltjes integral if we restrict the value of a to rational numbers. For rational values of a (say $a = n/m$), if we assume that $J(p_r^y)$ and $J({p_r}^{a-y})$ have common discontinuity, then p_r^y is a prime number given by P and ${p_r}^{m(a-y)}$ is an integer given by ${p_r}^n/P^m$. This result contradicts the definition of prime numbers. Hence, $J(p_r^y)$ and $J({p_r}^{a-y})$ don't have common discontinuity when a is a rational number.

where

$$M_j(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_{i1} < p_{i2} < .. < p_{ij} < p_{i1}p_{i2}..p_{ij} \le {p_r}^a} \frac{1}{p_{i1}p_{i2}...p_{ij}},$$

$p_{i1}, p_{i2}, ..., p_{ij}$ are j distinct prime numbers greater than or equal to p_r and the terms of the sum $M_j(1, p_r; 1, {p_r}^a)$ are placed in descending order. Therefore, $M_j(1, p_r; 1, {p_r}^a)$ can be written as follows

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \sum_{p_r \le p_i \le {p_r}^{a-1}} \frac{1}{p_i} M_{j-1}(1, p_r; p_r, p_r^a/p_i) + r_j,$$

where the factor of $1/j$ was added since each term of the form $1/(p_{i1}p_{i2}...p_{ij})$ is generated j times. It should be also noted that the sum of the above equation includes non square-free terms. The term r_j was added to offset the contribution by these non square-free terms. Notice that, there is no repetition in any of the non square-free terms (this follows from the fact that each of the M_{j-1} terms is a reciprocal of a product of distinct prime factors). We will show in Lemma 18 that r_j approaches zero as p_r approaches infinity. In the following Lemma, we will provide some properties of the term $M_j(1, p_r; 1, {p_r}^a)$

Lemma 16. *For $1 \le j \le a$, $M_j(1, p_r; 1, {p_r}^a)$ can be written as follows*

$$M_j(1, p_r; 1, {p_r}^a) = h_j(a) + g_j(p_r, a) + {r'}_j,$$

where,

$$h_j(a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy$$

with $h_1(a) = \log(a)$,

$$g_j(p_r, a) = \frac{1}{j} \int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y} dy + \frac{1}{j} \int_1^{a-1} g_{j-1}(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{j} \int_1^{a-1} h_{j-1}(a-y) \frac{dJ({p_r}^y)}{p_r^y}, \tag{64}$$

with $g_1(p_r, a) = \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}$, and

$$r'_j = r_j + \frac{1}{j} \int_1^{a-1} {r'}_{j-1} \frac{d\pi({p_r}^y)}{{p_r}^y}.$$

Proof. Referring to Lemmas 14, we have

$$M_1(1, p_r; 1, {p_r}^a) = h_1(a) + g_1(p_r, a)$$

where

$$h_1(a) = \log(a)$$

and

$$g_1(p_r, a) = \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}$$

Referring to Lemmas 15, we have

$$M_2(1, p_r; 1, {p_r}^a) = h_2(a) + g_2(p_r, a) + r_2$$

where

$$h_2(a) = \frac{1}{2}\int_1^{a-1} \frac{\log(a-y)}{y} dy,$$

$$g_2(p_r, a) = \frac{1}{2}\int_1^{a-1} \frac{g_1(p_r, a-y)}{y} dy + \frac{1}{2}\int_1^{a-1} \log(a-y)\frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{2}\int_1^{a-1} g_1(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y},$$

and r_2 represents the contribution by the non square-free terms.

Using the method of induction, we write $M_{j-1}(1, p_r; 1, {p_r}^a)$ as

$$M_{j-1}(1, p_r; 1, {p_r}^a) = h_{j-1}(a) + g_{j-1}(p_r, a) + r`_{j-1}$$

then

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j}\sum_{p_r \le p_i \le {p_r}^{a-1}} \frac{1}{p_i} M_{j-1}(1, p_r; p_r, p_r^a/p_i) + r_j.$$

Using Stieltjes integral, we then have

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j}\int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y}\left(h_{j-1}(a-y) + g_{j-1}(p_r, a-y) + {r'}_{j-1}\right) + r_j. \tag{65}$$

Hence

$$M_j(1, p_r; 1, {p_r}^a) = h_j(a) + g_j(p_r, a) + {r'}_j,$$

where

$$h_j(a) = \frac{1}{j}\int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy$$

with $h_1(a) = \log(a)$,

$$g_j(p_r, a) = \frac{1}{j}\int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y} dy + \frac{1}{j}\int_1^{a-1} g_{j-1}(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y} +$$
$$\frac{1}{j}\int_1^{a-1} h_{j-1}(a-y)\frac{dJ({p_r}^y)}{p_r^y},$$

with

$$g_1(p_r, a) = \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}$$

Finally,

$${r'}_j = r_j + \frac{1}{j}\int_1^{a-1} {r'}_{j-1}\frac{d\pi({p_r}^y)}{{p_r}^y}.$$

□

Lemma 17. *For* $2 \le j \le a$*,*

$$\lim_{p_r \to \infty} M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j}\int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy = h_j(a)$$

Proof. For $j \geq 2$, $M_j(1, p_r; 1, {p_r}^a)$ is given by

$$M_j(1, p_r; 1, {p_r}^a) = h_j(a) + g_j(p_r, a) + {r'}_j,$$

Our task is to show by induction that $\lim_{p_r \to \infty} g_j(p_r, a) = 0$ and $\lim_{p_r \to \infty} {r'}_j = 0$.

To show by induction that $\lim_{p_r \to \infty} g_j(p_r, a) = 0$, we will have the following two assumptions about $g_{j-1}(p_r, a)$ and $h_{j-1}(a)$

1. For $j \geq 3$, we assume that

$$|g_{j-1}(p_r, a)| = (a-2)2^{j-2}(\log(a-1))^{j-2} \sqrt{\log(p_r)}\ O(g(p_r)). \tag{66}$$

   where $g_2(p_r, a)$ is given by Equation (62)

$$|g_2(p_r, a)| = 2(a-2)\log(a-1)\sqrt{\log p_r}\ O(g(p_r)).$$

2. For $j \geq 3$ and $a > 1$, $h_{j-1}(a)$ is a monotone increasing (or non-decreasing) function that satisfies the following inequality

$$0 \leq h_{j-1}(a) \leq (\log(a-1))^{j-1}. \tag{67}$$

   where for $a > 1$, $h_2(a) = \frac{1}{2}\int_{y=1}^{a-1} \frac{\log(a-y)}{y} dy$ is a monotone increasing function of a that satisfies the following inequality (note that $\log(a-y) \leq \log(a-1)$, where $1 \leq y \leq a$)

$$0 \leq h_2(a) \leq (\log(a-1))^2.$$

Using Equations (66) and (67) of the above two assumptions, we will then show that

$$|g_j(p_r, a)| = (a-2)2^{j-1}(\log(a-1))^{j-1}\sqrt{\log(p_r)}\ O(g(p_r)).$$

and we will also show that, for $a > 1$, $h_j(a)$ is a monotone increasing function satisfying the following inequality

$$0 \leq h_j(a) \leq (\log(a-1))^j$$

Referring to Lemma 16, we have

$$g_j(p_r, a) = \frac{1}{j}\int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y} dy + \frac{1}{j}\int_1^{a-1} h_{j-1}(a-y)\frac{dJ({p_r}^y)}{p_r^y} +$$
$$\frac{1}{j}\int_1^{a-1} g_{j-1}(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}. \tag{68}$$

Using Equations 66, the first integral on the right side of the above equation can be written as

$$\left|\int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y} dy\right| = (a-2)2^{j-2}(\log(a-1))^{j-2}\sqrt{\log(p_r)}\int_1^{a-1} \frac{O(g(p_r))}{y} dy$$

or

$$\left|\int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y} dy\right| = (a-2)2^{j-2}(\log(a-1))^{j-1}\sqrt{\log(p_r)}O(g(p_r)) \tag{69}$$

For the second integral, we can use the method of integration by parts to obtain

$$\int_1^{a-1} h_{j-1}(a-y)\frac{dJ(p_r{}^y)}{p_r^y} = h_{j-1}(a-y)\frac{J(p_r^y)}{p_r^y}\Big|_1^{a-1} - \int_1^{a-1}\left(J(p_r{}^y)h_{j-1}(a-y)d\left(\frac{1}{p_r^y}\right) + \frac{J(p_r{}^y)}{p_r^y}dh_{j-1}(a-y)\right)$$

where (refer to Equation 67 and note that $h_{j-1}(1) = 0$),

$$\left|h_{j-1}(a-y)\frac{J(p_r^y)}{p_r^y}\Big|_1^{a-1}\right| = \left|h_{j-1}(a-1)\frac{J(p_r)}{p_r}\right| \leq (\log(a-1))^{j-1}\frac{O(g(p_r))}{\sqrt{\log p_r}}, \tag{70}$$

and

$$\left|\int_1^{a-1} J(p_r{}^y)h_{j-1}(a-y)d\left(\frac{1}{p_r^y}\right)\right| \leq \int_1^{a-1} |J(p_r{}^y)||h_{j-1}(a-y)|\left|d\left(\frac{1}{p_r^y}\right)\right|,$$

or

$$\left|\int_1^{a-1} J(p_r{}^y)h_{j-1}(a-y)d\left(\frac{1}{p_r^y}\right)\right| = \log p_r\, O\left(\frac{J(p_r)}{p_r}\right)\int_1^{a-1}|h_{j-1}(a-y)|dy.$$

Since we assumed that for $x > 1$, $h_{j-1}(x)$ is a monotone increasing function and $0 \leq h_{j-1}(x) \leq (\log(x-1))^{j-1}$, hence $0 \leq h_{j-1}(a-y) \leq h_{j-1}(a) \leq (\log(a-1))^{j-1}$. We then have

$$\left|\int_1^{a-1} J(p_r{}^y)h_{j-1}(a-y)d\left(\frac{1}{p_r^y}\right)\right| = \log p_r\, O\left(\frac{J(p_r)}{p_r}\right)(\log(a-1))^{j-1}(a-2)$$

or

$$\left|\int_1^{a-1} J(p_r{}^y)h_{j-1}(a-y)d\left(\frac{1}{p_r^y}\right)\right| = (a-2)(\log(a-1))^{j-1}\log p_r\frac{O(g(p_r))}{\sqrt{\log p_r}} \tag{71}$$

Furthermore,

$$\left|\int_1^{a-1}\frac{J(p_r{}^y)}{p_r^y}dh_{j-1}(a-y)\right| \leq \int_1^{a-1}\left|\frac{J(p_r{}^y)}{p_r^y}\right||dh_{j-1}(a-y)|$$

Since $h_{j-1}(a-y)$ is a monotone decreasing function of y, thus

$$\left|\int_1^{a-1}\frac{J(p_r{}^y)}{p_r^y}dh_{j-1}(a-y)\right| = O\left(\frac{J(p_r{}^y)}{p_r^y}\right)\left|\int_1^{a-1}dh_{j-1}(a-y)\right|$$

or

$$\left|\int_1^{a-1}\frac{J(p_r{}^y)}{p_r^y}dh_{j-1}(a-y)\right| = (\log(a-1))^{j-1}\ \frac{O(g(p_r))}{\sqrt{\log p_r}} \tag{72}$$

Combining Equations (70), (71) and (72), we then have

$$\left|\int_1^{a-1} h_{j-1}(a-y)\frac{dJ(p_r{}^y)}{p_r^y}\right| = 2(a-2)(\log(a-1))^{j-1}\sqrt{\log(p_r)}\ O(g(p_r)) \tag{73}$$

The third integral $\int_1^{a-1} g_{j-1}(p_r, a-y)dJ(p_r{}^y)/p_r^y$ is given by

$$\left|\int_1^{a-1} g_{j-1}(p_r, a-y)\frac{dJ(p_r{}^y)}{p_r^y}\right| \leq \int_1^{a-1}|g_{j-1}(p_r, a-y)|\left|\frac{dJ(p_r{}^y)}{p_r^y}\right|$$

or

$$\left|\int_1^{a-1} g_{j-1}(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}\right| = O(g_{j-1}(p_r, a))\int_1^{a-1}\left|\frac{dJ({p_r}^y)}{p_r^y}\right|$$

and by the virtue of lemma 11 and Equation (66), we then have

$$\left|\int_1^{a-1} g_{j-1}(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y}\right| = (a-2)2^{j-1}(\log(a-1))^{j-1}\sqrt{\log(p_r)}\ O(g(p_r)). \tag{74}$$

Combining Equations (68), (69), (73) and (74), we then have

$$|g_j(p_r, a)| = (a-2)2^{j-1}(\log(a-1))^{j-1}\sqrt{\log(p_r)}\ O(g(p_r)).$$

Thus for any value of a, $g_j(p_r, a)$ approaches zero as p_r approaches infinity (note that $g(p_r) = \sqrt{\log p_r}\ e^{-b\sqrt{\log p_r}}$).

Next, we need to prove that for $a > 1$, if $h_{j-1}(a)$ is a monotone increasing function satisfying the inequality $0 \leq h_{j-1}(a) \leq (\log(a-1))^{j-1}$, then $h_j(a)$ is also monotone increasing function for $a > 1$ satisfying the inequality $0 \leq h_j(a) \leq (\log(a-1))^j$. To achieve this task we recall that

$$h_j(a) = \frac{1}{j}\int_1^{a-1}\frac{h_{j-1}(a-y)}{y}dy$$

Since, for $a > 1$, we assumed that $h_{j-1}(a)$ is non-negative, therefore (by the virtue of the above integral) $h_j(a)$ is a monotone increasing function for $a > 1$. Moreover, since we have assumed that $0 \leq h_{j-1}(a) \leq (\log(a-1))^{j-1}$, therefore,

$$h_j(a) \leq (\log(a-1))^{j-1}\int_1^{a-1}\frac{dy)}{y}$$

or

$$h_j(a) \leq (\log(a-1))^j$$

Finally, we need to show that, for $a > 1$, $h_2(a)$ is a monotone increasing function satisfying the inequality $0 \leq h_2(a) \leq (\log(a-1))^2$ and $|g_2(p_r, a)| = 2(a-2)\log(a-1)\sqrt{\log(p_r)}\ O(g(p_r))$. Referring to Lemma 15, we have

$$h_2(a) = \frac{1}{2}\int_1^{a-1}\frac{\log(a-y)}{y}dy \tag{75}$$

Since for $1 \leq y \leq a-1$, $\log(a-y) \geq 0$, therefore $h_2(a)$ is a monotone increasing function. Also, since $1 \leq y \leq a-1$, $0 \leq \log(a-y) \leq \log(a-1)$, thus

$$h_2(a) \leq \log(a-1)\int_1^{a-1}\frac{dy}{y} = (\log(a-1))^2$$

We have also shown in Lemma 15 that $|g_2(p_r, a)|$ has the desired value or

$$|g_2(p_r, a)| = 2(a-2)\log(a-1)\sqrt{\log p_r}\ O(g(p_r)).$$

In the next lemma, for any fixed a, we will show $\lim_{p_r \to \infty} {r'}_j = 0$. Since we have also shown that $g_j(p_r, a)$ approaches zero as p_r approaches infinity, therefore

$$\lim_{p_r \to \infty} M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy = h_j(a)$$

where $h_1(a) = \log(a)$.

□

Lemma 17 infers that for every a and as p_r approaches infinity, we have

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - h_1(a) + h_2(a) - h_3(a) + ... + (-1)^{\lfloor a \rfloor} h_{\lfloor a \rfloor}(a) = \rho(a). \tag{76}$$

It should be pointed out that the above equation implies that the partial sums $M(1, p_r; 1, {p_r}^a)$ and $M(1, p_r^y; 1, {p_r}^{ay})$ (where, p_r^y is a prime number) have the same limit as p_r approaches infinity. Hence,

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = \lim_{p_r \to \infty} M(1, p_r^y; 1, {p_r}^{ay}) = \rho(a). \tag{77}$$

Equation (77) will be used in the second step of this section to estimate the asymptotic behavior of the function $\rho(a)$ as a approaches infinity.

In Lemmas 15 and 17, we stated that the contribution by the non square-free terms (i.e r_i's) approaches zero as p_r approaches infinity. In the following lemma, we will show that, for every a and as p_r approaches infinity, the sum of the absolute contributions by these non square-free terms approaches zero.

Lemma 18.

$$\lim_{p_r \to \infty} \sum_{j=2}^{\lfloor a \rfloor} |r_j| = 0,$$

and

$$\lim_{p_r \to \infty} {r'}_j = 0,$$

where $2 \leq j \leq a$, $r_1 = {r'}_1 = 0$ *and*

$${r'}_j = r_j + \frac{1}{j} \int_1^{a-1} {r'}_{j-1} \frac{d\pi({p_r}^y)}{{p_r}^y}.$$

Furthermore,

$$\sum_{j=2}^{\lfloor a \rfloor} |r_j| = O\left(\frac{a \log(p_r)}{p_r}\right),$$

and

$$\sum_{j=2}^{\lfloor a \rfloor} |r'_j| = a 2^a (\log(a-1))^a O\left(\frac{a \log(p_r)}{p_r}\right).$$

Proof. Let S_0 be the sum of the absolute value of all the non square-free terms (within all the r_j's) that have the factor $1/p_r^2$. Therefore, S_0 can be expressed as K_0/p_r^2. Let S_1 be the sum of the remaining non square-free terms with the factor $1/(p_{r+1})^2$. Therefore, S_1 can be expressed as $K_1/(p_{r+1})^2$. Let S_2 be the sum of the remaining non square-free terms with the factor $1/(p_{r+2})^2$ where S_2 can be expressed as $K_2/(p_{r+2})^2$, and so on. Let S be sum of all the terms associated with non square-free terms. Thus, S is given by

$$|r_2|+|r_3|+...+|r_{\lfloor a \rfloor}|\leq S = \frac{1}{{p_r}^2}K_0 + \frac{1}{{p_{r+1}}^2}K_1 + ... + \frac{1}{p_{r+L}^2}K_L,$$

where p_{r+L} is the largest prime that satisfies the condition $p_{r+L}^2 \leq {p_r}^a$. Furthermore, since there is no repetition in any of the non square-free terms and there are less than p_r^a terms of the form $\mu(n)/n$ in the partial sum $M(1, p_r; 1, {p_r}^a)$, therefore

$$|K_0|, |K_1|, ..., |K_L| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{{p_r}^a},$$

and

$$|K_0|, |K_1|, ..., |K_L| = O(a \log p_r).$$

Thus,

$$\sum_{j=2}^{\lfloor a \rfloor}|r_j| \leq S = \left(\frac{1}{{p_r}^2} + \frac{1}{{p_{r+1}}^2} + ... + \frac{1}{p_{r+L}^2}\right) O(a \log p_r),$$

or

$$S = O(a \log p_r / p_r).$$

Hence, the absolute value of contribution by the non square-free terms is given by,

$$\sum_{j=2}^{\lfloor a \rfloor}|r_j| = O(a \log p_r / p_r).$$

Consequently, for every a and as p_r approaches infinity, the contribution by the non square-free terms ($\sum_{j=2}^{\lfloor a \rfloor} r_j$) approaches zero.

To show that $\lim_{p_r \to \infty} {r'}_j = 0$ by induction, we first note that $r'_1 = 0$ (refer to Lemmas 14). For $j \geq 3$, we assume that

$$r'_{j-1} = 2^{j-1}(\log(a-1))^{j-1} O\left(\frac{a \log p_r}{p_r}\right).$$

where ${r'}_2 = r_2 = O\left(a \log p_r / p_r\right) (< 2^2 (\log(a-1))^2 O\left(a \log p_r / p_r\right))$.

Referring to the definition of ${r'}_j$, we then have

$$|{r'}_j| \leq |r_j| + \frac{1}{j}\int_1^{a-1} |{r'}_{j-1}| \frac{d\pi({p_r}^y)}{{p_r}^y}.$$

Since $\int_1^{a-1} d\pi({p_r}^y)/{p_r}^y = \sum_{p_r \leq p_i \leq p_r^a} 1/p_i = \log a + O(1/p_r)$ (refer to Lemma 11), therefore

$$|{r'}_j| \leq O\left(\frac{a \log p_r}{p_r}\right) + \frac{1}{j} 2^{j-1}(\log(a-1))^{j-1} O\left(\frac{a \log p_r}{p_r}\right)(\log(a-1) + O(1/p_r)).$$

or

$$|r'_j|= 2^j (\log(a-1))^j O\left(\frac{a \log p_r}{p_r}\right)$$

Since $j \leq a$, thus for a fixed value of a, we then have

$$\lim_{p_r \to \infty} r'{}_j = 0$$

and

$$\sum_{j=2}^{\lfloor a \rfloor} |r'_j|= a2^a (\log(a-1))^a O\left(\frac{a \log p_r}{p_r}\right).$$

□

Since r_j is given by $O\left(a \log p_r/p_r\right)$ and r'_j is given by $2^a (\log(a-1))^a O\left(a \log p_r/p_r\right)$, therefore for values of a less than or equal to a fixed number, both r_j and r'_j are uniformly convergent to zero. Furthermore, referring to Lemma 17, $|g_j(p_r, a)|$ is given by $(a-2)2^{j-1}(\log(a-1))^{j-1}\sqrt{\log(p_r)}\ O(g(p_r))$. Therefor, for values of a less than or equal to a fixed number, $g_j(p_r, a)$ is also uniformly convergent to zero. Consequently for values of a less than or equal to a fixed number and by the virtue of Lemma 16, both $M_j(1, p_r; 1, {p_r}^a)$ (for $1 \leq j \leq a$) and $M(1, p_r; 1, {p_r}^a)$ are uniformly convergent to $h_j(a)$ and $\rho(a)$, respectively.

- In the second step, we will provide the first representation of the partial sum $M(1, p_r; 1, {p_r}^a)$. We will then show that the partial sum $M(1, p_r; 1, {p_r}^a)$ can be written as the sum of two components. The first one is the deterministic or regular component and it is given by $\rho(a)$ (= $\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a)$). The second one is the irregular component $R(1, p_r; 1, {p_r}^a)$ given by $M(1, p_r; 1, {p_r}^a) - \rho(a)$. We will then show that the function $\rho(a)$ is the Dickman function that has been extensively used to analyze the properties of y-smooth numbers.

  In this step, we will present the first method to compute the partial sum $M(1, p_r; 1, {p_r}^a)$ by summing the contributions of each prime number to the partial sum $M(1, p_r; 1, {p_r}^a)$ (in system analysis, this corresponds to computing the system output using its impulse response).

  In the next step (step three) of this section, we will present the second method to compute the partial sum $M_1(1, p_r; 1, {p_r}^a)$ using the results of the first step of this section. With this method, we will compute the partial sum $M_1(1, p_r; 1, {p_r}^a)$ by adding the contributions by the terms $M_1(1, p_r; 1, {p_r}^a)$ (terms with one prime factor), $M_2(1, p_r; 1, {p_r}^a)$ (terms with two prime factors) and so on (in system analysis, this corresponds to computing the system output by adding its orthogonal components).

  In the next section (section 6), we will present the third method to compute the partial sum $M(1, p_r; 1, {p_r}^a)$ using complex analysis methods (in system analysis, this corresponds to computing the system output using its frequency response).

  Comparing these three representations reveals that zeros of $\zeta(s)$ can be found arbitrary close to the line $\Re(s) = 1$.

The following lemma is the key to our first method to compute the partial sum $M(1, p_r; 1, {p_r}^a)$. With this lemma, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ in terms of the partial sums $M(1, p_i; 1, {p_r}^a/p_i)$ for $p_r \leq p_i < p_r^a$.

Lemma 19.

$$M(1,p_r;1,{p_r}^a) = 1 - \sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i) - \sum_{{p_r}^{a/2} < p_i \le {p_r}^a} \frac{1}{p_i}. \tag{78}$$

Proof. To prove this lemma, we will show that every term of the sum $M(1,p_r;1,{p_r}^a)$ is also a term of the sum on right side of Equation (78) and vice versa. We will also show that none of the terms on the right side of Equation (78) is duplicated.

To show that non of the terms on the right side of Equation (78) is duplicated, we first note that the middle sum ($\sum_{p_r \le p_i \le {p_r}^{a/2}} M(1,p_{i+1};1,{p_r}^a/p_i)/p_i$) and the last sum ($\sum_{{p_r}^{a/2} < p_i \le {p_r}^a} 1/p_i$) are void of 1. Furthermore, the middle sum and the last sum have no common terms.

In the second step, we will show that none of the terms that comprise the middle sum $\sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i)$ is duplicated. This can be verified by noting that there is no common term between the terms that comprise the middle sum. More specifically, there is no common term between the partial sum $\frac{1}{p_r} M(1,p_{r+1};1,{p_r}^a/p_r)$ and the remaining terms of the sum $\sum_{p_{r+1} \le p_i \le {p_r}^{a/2}} \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i)$ (this follows from the fact that none of the remaining terms is a reciprocal of a number with a prime factor p_r. Furthermore, none of the terms that comprise the sum $\frac{1}{p_r} M(1,p_{r+1};1,{p_r}^a/p_r)$ is duplicated). Similarly, there is no common term between the partial sum $M(1,p_{r+2};1,{p_r}^a/p_{r+1})/p_{r+1}$ and any of the remaining terms that comprise the sum $\sum_{p_{r+2} \le p_i \le {p_r}^{a/2}} M(1,p_{i+1};1,{p_r}^a/p_i)/p_i$ (this follows from the fact that none of the remaining terms is a reciprocal of a number with a prime factor p_{r+1}. Furthermore, none of the terms that comprise the sum $M(1,p_{r+2};1,{p_r}^a/p_{r+1})/p_{r+1}$ is duplicated). Following the same process for all the prime numbers $p_{r+2} \le p_i \le {p_r}^{a/2}$, we then conclude that none of the terms that comprise the sum $\sum_{p_r \le p_i \le {p_r}^{a/2}} M(1,p_{i+1};1,{p_r}^a/p_i)/p_i$ is duplicated.

Next, we will show that every term on the right side of Equation (78) is a term of the partial sum $M(1,p_r;1,{p_r}^a)$. First, we note that the term 1 is also the first term of the the partial sum $M(1,p_r;1,{p_r}^a)$. Furthermore, every term within the sum $\sum_{{p_r}^{a/2} < p_i \le {p_r}^a} \frac{-1}{p_i}$ is also a term of the partial sum $M(1,p_r;1,{p_r}^a)$. Finally, every term of the sum $\sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{-1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i)$ can be written as $(-1)^k/N$, where $p_r \le N \le {p_r}^a$ and N has k distinct prime factors that range between p_r and $p_r^{a/2}$ ($k \ge 2$).

Finally, we will show that every term of the partial sum $M(1,p_r;1,{p_r}^a)$ is also a term on the right side of Equation (78). First, we note that the term 1 and the terms of the form $-1/p_i$, where $p_r \le p_i \le {p_r}^a$, are also terms on the right side of Equation (78). Furthermore, every term of the remaining terms of the partial sum $M(1,p_r;1,{p_r}^a)$ can be written as $(-1)^k/N$ where $k \ge 2$ and N is the product of k distinct prime numbers in the range p_r and $p_r^{a/2}$. Let p_i be the lowest prime number of these k distinct prime factors, then N can be written as $(-1)^k/(p_i N_i)$ where $p_i < N_i \le p_r^a/p_i$. Hence, $(-1)^{k-1}/N_i$ is a term within the partial sum $M(1,p_{i+1};1,{p_r}^a/p_i)$. Consequently, $(-1)^k/N$ is also a term on the right side of Equation (78). □

In the following lemma, we will apply Stieltjes integral to the sums of Equation (78) in Lemma 19.

Lemma 20.

$$M(1,p_r;1,{p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^y} M(1,{p_r}^y;1,p_r^a/p_r^y) - \int_{a/2}^{a} \frac{d\pi({p_r}^y)}{p_r^y} + Q(p_r,a), \qquad (79)$$

where

$$Q(p_r,a) = \sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i^2} M(1,p_{i+1};1,{p_r}^a/p_i^2).$$

and $Q(p_r,a)$ *is unconditionally given by*

$$|Q(p_r,a)| = O(p_r^{-1}).$$

Proof. By the virtue of Lemma 1, we have

$$\sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i) = \sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i}\left(M(1,p_i;1,{p_r}^a/p_i) + \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i^2)\right).$$

Since we defined $Q(p_r,a)$ as

$$Q(p_r,a) = \sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i^2} M(1,p_{i+1};1,{p_r}^a/p_i^2),$$

thus

$$\sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i) = \sum_{p_r \le p_i \le {p_r}^{a/2}} \frac{1}{p_i} M(1,p_i;1,{p_r}^a/p_i) + Q(p_r,a).$$

To show that $Q(p_r,a)$ is given by $O(p_r^{-1})$, we will first show that

$$|M(1,p_{i+1};1,{p_r}^a/p_i^2)| \le 2.$$

The above inequality will follow if we prove the following inequality for any integer N and prime number p_r

$$\left|\sum_{n=1}^{N} \frac{\mu(n,p_r)}{n}\right| \le 2.$$

This task can be achieved by first noting that (refer to Theorem 6.5 of [11], page 128)
$\sum_{d/n} \mu(d,p_r) = 1$, if $n = 1$,
$\sum_{d/n} \mu(d,p_r) = 1$, if all the prime factors of n are less than p_r,
$\sum_{d/n} \mu(d,p_r) = 0$, if any of the prime factors of n is greater than p_r.

Adding all the terms $\sum_{d/n} \mu(d,p_r)$ for $1 \le n \le N$, we then obtain

$$0 < \sum_{n=1}^{N} \mu(n,p_r) \left\lfloor \frac{N}{n} \right\rfloor \le N, \qquad (80)$$

where $\lfloor x \rfloor$ refers to the integer value of x (note there are $\lfloor N/n \rfloor$ integers less that or equal to N that are divisible by n). Define r_n as

$$r_n = \frac{N}{n} - \left\lfloor \frac{N}{n} \right\rfloor,$$

where $0 \leq r_n < 1$. Adding the sum $\sum_{n=1}^{N} \mu(n, p_r) r_n$ to each side of Equation 80, we then have

$$\sum_{n=1}^{N} \mu(n, p_r) r_n < \sum_{n=1}^{N} \mu(n, p_r) \left\lfloor \frac{N}{n} \right\rfloor + \sum_{n=1}^{N} \mu(n, p_r) r_n \leq N + \sum_{n=1}^{N} \mu(n, p_r) r_n.$$

Since $0 \leq r_n < 1$ and $-N \leq \sum_{n=1}^{N} \mu(n, p_r) r_n \leq N$, therefore

$$-N \leq \sum_{n=1}^{N} \mu(n, p_r) \left(r_n + \left\lfloor \frac{N}{n} \right\rfloor \right) \leq 2N.$$

Thus, for every p_r we have

$$-N < \sum_{n=1}^{N} \mu(n, p_r) \frac{N}{n} \leq 2N,$$

or

$$-1 < \sum_{n=1}^{N} \frac{\mu(n, p_r)}{n} \leq 2.$$

For $N = \lfloor {p_r}^a / p_i^2 \rfloor$ and $p_r = p_{i+1}$, we then have

$$|M(1, p_{i+1}; 1, {p_r}^a / p_i^2)| \leq 2.$$

Thus

$$|Q(p_r, a)| = \left| \sum_{p_r \leq p_i \leq {p_r}^{a/2}} \frac{1}{p_i^2} M(1, p_{i+1}; 1, {p_r}^a / p_i^2) \right| \leq 2 \sum_{p_r \leq p_i \leq {p_r}^{a/2}} \frac{1}{p_i^2} = O(p_r^{-1}).$$

Using Stieltjes integral, we can write Equation (78) as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^y} M(1, {p_r}^y; 1, p_r^a / p_r^y) - \int_{a/2}^{a} \frac{d\pi({p_r}^y)}{p_r^y} + Q(p_r, a),$$

where $d\pi({p_r}^y) = d\mathrm{Li}({p_r}^y) + dJ({p_r}^y)$. □

It should pointed out that while Equations (78) and (79) of Lemmas 19 and 20 provide the value of the partial sum $M(s, p_r; 1, p_r^a)$ at $s = 1$, they can be easily modified to compute the partial sum for any value of s to the right of the line $\Re(s) = 1$ (and on RH, to the right of the line $\Re(s) = 0.5$).

In the following lemma, we will show that as p_r approaches infinity, $M(1, p_r; 1, p_r^a)$ approaches approaches the Dickman function. In other words; we will show that $\rho(a)$ is the Dickman function.

Lemma 21.

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = \rho(a)$$

where, $\rho(a)$ is the Dickman function.

Proof. As stated in the previous step, for values of a less than or equal to a fixed number, $M(1, p_r; 1, p_r^a)$ is uniformly convergent to $\rho(a)$ as p_r approaches infinity. Therefore for values of a less than or equal to a fixed number and by the virtue of Equation (77), we have uniformly as p_r approaches infinity

$$\lim_{p_r \to \infty} M(1, {p_r}^y; 1, p_r^{a-y}) = \rho\left(\frac{a-y}{y}\right) = \rho\left(\frac{a}{y} - 1\right)$$

Furthermore, for $1 \leq a_1, a_2 \leq a$, we also have uniformly

$$\lim_{p_r \to \infty} \int_{a_1}^{a_2} \frac{d\pi({p_r}^y)}{p_r^y} = \int_{a_1}^{a_2} \frac{dy}{y}.$$

Therefore, for a fixed value of a, as p_r approaches infinity, Equation (79) of Lemma 20 can be written as

$$\rho(a) = 1 - \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{a/2}^{a} \frac{dy}{y}. \tag{81}$$

In the following, we will show that $\rho(a)$ satisfies a well known first order differential equation and $\rho(a)$ is the Dickman function. This task will be achieved by using Equation (81) to compute the difference $\rho(a + \Delta a) - \rho(a)$ (where, Δa is an arbitrary small number) to obtain

$$\rho(a+\Delta a) - \rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y} - 1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{(a+\Delta a)/2}^{(a+\Delta a)} \frac{dy}{y} + \int_{a/2}^{a} \frac{dy}{y}.$$

Since the third integral of the above equation is equal to the fourth integral, therefore

$$\rho(a + \Delta a) - \rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y} - 1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy.$$

If we define $z = y/(1 + \Delta a/a)$, then we have

$$\rho(a + \Delta a) - \rho(a) = -\int_{1/(1+\Delta a/a)}^{((a+\Delta a)/2)/(1+\Delta a/a)} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy.$$

or

$$\rho(a + \Delta a) - \rho(a) = -\int_{1/(1+\Delta a/a)}^{a/2} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz + \int_1^{a/2} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz.$$

Thus,

$$\rho(a + \Delta a) - \rho(a) = -\int_{1/(1+\Delta a/a)}^{1} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz.$$

Dividing both sides of the above equation by Δa and letting Δa approach zero, we then obtain

$$\frac{d\rho(a)}{da} = -\frac{\rho(a-1)}{a}, \tag{82}$$

where $\rho(a) = 1 - \log(a)$ for $1 \leq a \leq 2$. Equation (82) is a first order delay differential equation that has been extensively analyzed in the literature [6] [9]. The function $\rho(a)$ is known as the Dickman function. □

As a approaches infinity, $\rho(a)$ can be given by the following estimate [6]

$$\rho(a) = \left(\frac{e + o(1)}{a \log a}\right)^{a}. \tag{83}$$

For sufficiently large values of a, we have $\rho(a) < a^{-a}$.

Definition 8. *The irregular component $R(1, p_r; 1, {p_r}^a)$ of the partial sum $M(1, p_r; 1, {p_r}^a)$ is given by*

$$R(1, p_r; 1, {p_r}^a) = M(1, p_r; 1, {p_r}^a) - \rho(a). \tag{84}$$

Thus, $R(1, p_r; 1, {p_r}^a)$ can be computed by subtracting Equation (81) from Equation (79) to obtain the first key theorem. This theorem provides the first equation for the value of $R(1, p_r; 1, p_r^a)$ in terms of $dJ({p_r}^y)/{p_r}^y$ with $a > 1$.

Theorem 4. *The partial sum $M(1, p_r; 1, p_r^a) = \sum_{n=1}^{\lfloor p_r^a \rfloor} \mu(n, p_r)/n$ can be expressed as*

$$M(1, p_r; 1, p_r^a) = \rho(a) + R(1, p_r; 1, {p_r}^a) \tag{85}$$

where $\rho(a)$ is Dickman function. The regular component of $M(1, p_r; 1, p_r^a)$ is given by

$$\rho(a) = \lim_{p_r \to \infty} M(1, p_r; 1, p_r^a). \tag{86}$$

and the the irregular component $R(1, p_r; 1, {p_r}^a)$ defined as $M(1, p_r; 1, p_r^a) - \rho(a)$ is given by

$$R(1, p_r; 1, {p_r}^a) = -\int_1^a \rho\left(a/y - 1\right) \frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} + Q(p_r, a) \tag{87}$$

where $Q(p_r, a)$ is unconditionally given by $O(p_r^{-1})$

It should be emphasized here that Equation (87) is valid regardless where the zeros of $\zeta(s)$ are located within the critical strip.

- In the third step, we will use Lemmas 14, 15 and 17 to drive the second equation for the the value of $R(1, p_r; 1, p_r^a)$ as a function of $dJ({p_r}^y)/{p_r}^y$ for $1 \leq a < 4$. This equation will be derived with the assumption that the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$).

For $1 \le a < 2$ and referring to Lemma 14, we have

$$M(1, p_r; 1, {p_r}^a) = 1 - \log(a) - \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}$$

Hence, we have the following lemma

Lemma 22. *For $1 \le a < 2$, $R(1, p_r; 1, p_r^a)$ is given by*

$$R(1, p_r; 1, p_r^a) = -g_1(p_r, a) = -\varepsilon(1; p_r, p_r^a) = -\int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}.$$

Before we proceed with the estimate of $R(1, p_r; 1, p_r^a)$ for $a \ge 2$, we will have the following three lemmas relating to size estimate of some integrals with the term $dJ({p_r}^y)/{p_r}^y$.

Lemma 23. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \le \Re(s) \le c$ (where $1/2 \le c < 1$), then for $a \ge 2$, we have*

$$\left| \int_1^{a-1} g_1(p_r, a - y) \frac{dJ({p_r}^y)}{p_r^y} \right| = O(p_r^{-a(1-c)/2+\epsilon}).$$

where, ϵ is an arbitrary small positive number.

Proof. To compute the size of the the integral $\int_1^{a-1} g_1(p_r, a-y) dJ({p_r}^y)/p_r^y$. or the integral $\int_{y=1}^{a-1} (\int_{z=1}^{a-y} dJ({p_r}^z)/p_r^z) dJ({p_r}^y)/p_r^y$, we first note that although the function $J(x)$ is not a non-decreasing function, $J(x)$ is given by $\pi(x) - \mathrm{Li(x)}$ where both $\pi(x)$ and $\mathrm{Li}(x)$ are non-decreasing functions. Therefore, we can use theorem 21.67 of [8] for the method of integration by parts for Lebesgue-Stieljtes integrals to obtain,

$$\int_{y=1}^{a-1} \left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y} = J({p_r}^y) \frac{\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z}}{p_r^y} \Bigg|_1^{a-1} - \int_{y=1}^{a-1} J({p_r}^y) \left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z} \right) d\left(\frac{1}{p_r^y} \right) -$$

$$\int_{y=1}^{a-1} \frac{J({p_r}^y)}{p_r^y} d\left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z} \right). \tag{88}$$

By the virtue of Lemma 7 and 10 where both $|J(p_r)/p_r|$ and $|\int_1^x dJ({p_r}^y)/p_r^y|$ are given by $O({p_r}^{-(1-c)+\epsilon})$, we then have

$$\left| \frac{J({p_r}^y)}{p_r^y} \left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z} \right) \Bigg|_1^{a-1} \right| = O\left(p_r^{-2(1-c)+\epsilon} \right), \tag{89}$$

and

$$\left| \int_{y=1}^{a-1} J({p_r}^y) \left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z} \right) d\left(\frac{1}{p_r^y} \right) \right| \le \log p_r \int_{y=1}^{a-1} \left| \frac{J({p_r}^y)}{p_r^y} \right| O\left({p_r}^{-(1-c)+\epsilon} \right) dy,$$

or

$$\left| \int_{y=1}^{a-1} J({p_r}^y) \left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z} \right) d\left(\frac{1}{p_r^y} \right) \right| = O\left({p_r}^{-2(1-c)+\epsilon} \right). \tag{90}$$

For the third integral $\int_1^{a-1} \frac{J({p_r}^y)}{p_r^y} d\left(\int_1^{a-y} \frac{dJ({p_r}^z)}{p_r^z}\right)$, we have

$$\frac{d}{dy}\left(\int_1^{a-y} \frac{dJ({p_r}^z)}{p_r^z}\right) = \lim_{\Delta y\to 0} \frac{\Delta\left(\int_1^{a-y} \frac{dJ({p_r}^z)}{p_r^z}\right)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\int_1^{a-(y+\Delta y)} \frac{dJ({p_r}^z)}{p_r^z} - \int_1^{a-y} \frac{dJ({p_r}^z)}{p_r^z}}{\Delta y}$$

As it was mentioned at the end of section 2, the discontinuities of $J(x)$ are represented by a Dirac delta function as the limit of a Gaussian distribution. Thus,

$$d\left(\int_1^{a-y} \frac{dJ({p_r}^z)}{p_r^z}\right) = \frac{dJ({p_r}^{a-y})}{p_r^{a-y}} \tag{91}$$

or

$$\int_{y=1}^{a-1} \frac{J({p_r}^y)}{p_r^y} d\left(\int_{z=1}^{a-y} \frac{dJ({p_r}^z)}{p_r^z}\right) = \int_1^{a-1} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}}$$

We split the integral over the period $[1, a-1]$ into two integrals. The first integral covers the period $[1, a/2]$ and The second integral covers the period $(a/2, a-1]$

$$\int_1^{a-1} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}} = \int_1^{a/2} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}} + \int_{a/2}^{a-1} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}} \tag{92}$$

For the second integral on the right side of Equation (92), we have

$$\left|\int_{a/2}^{a-1} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}}\right| \le \int_{a/2}^{a-1} \left|\frac{J({p_r}^y)}{p_r^y}\right|\left|\frac{dJ({p_r}^{a-y})}{p_r^{a-y}}\right|$$

By the virtue of Lemmas 7 and 11 where $|J({p_r}^x)/p_r^x|$ is given by $O({p_r}^{-x(1-c)+\epsilon})$ and $\int_1^x |dJ({p_r}^y)/p_r^y|$ is given by $O(\log a)$, we then have

$$\left|\int_{a/2}^{a-1} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}}\right| = O\left(p_r^{-a(1-c)/2+\epsilon}\right).$$

The first integral on the right side of Equation (92) can be written as follows

$$\int_1^{a/2} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}} = \frac{1}{p_r^a}\int_1^{a/2} J({p_r}^y) dJ({p_r}^{a-y}).$$

By the method of integration by parts, we then have

$$\int_1^{a/2} \frac{J({p_r}^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{p_r^{a-y}} = \left.\frac{J({p_r}^y)J({p_r}^{a-y})}{p_r^a}\right|_1^{a/2} - \int_1^{a/2} \frac{J({p_r}^{a-y})}{p_r^{a-y}}\frac{dJ({p_r}^y)}{p_r^y}$$

Referring to of Lemmas 7 and 11, we then obtain

$$\left|\left.\frac{J({p_r}^y)J({p_r}^{a-y})}{p_r^a}\right|_1^{a/2}\right| = O\left(p_r^{-2(1-c)+\epsilon}\right)$$

and

$$\left|\int_1^{a/2} \frac{J({p_r}^{a-y})}{p_r^{a-y}}\frac{dJ({p_r}^y)}{p_r^y}\right| = O\left(p_r^{-a(1-c)/2+\epsilon}\right) \tag{93}$$

Combining Equations (88), (89), (90) and (93), we then have

$$\left|\int_{y=1}^{a-1}\left(\int_{z=1}^{a-y}\frac{dJ(p_r{}^z)}{p_r^z}\right)\frac{dJ(p_r{}^y)}{p_r^y}\right| = O(p_r^{-a(1-c)/2+\epsilon}).$$

or

$$\left|\int_1^{a-1} g_1(p_r, a-y)\frac{dJ(p_r{}^y)}{p_r^y}\right| = O(p_r^{-a(1-c)/2+\epsilon}).$$

□

Lemma 24. *For* $a \geq 2$

$$g_2(p_r, a) = \int_1^{a-1}\log(a-y)\frac{dJ(p_r{}^y)}{p_r^y} + \frac{1}{2}\int_1^{a-1} g_1(p_r, a-y)\frac{dJ(p_r{}^y)}{p_r^y}.$$

and If the non-trivial zeros of $\zeta(s)$ *are restricted to the strip* $1-c \leq \Re(s) \leq c$ *(where* $1/2 \leq c < 1$*), then*

$$g_2(p_r, a) = \int_1^{a-1}\log(a-y)\frac{dJ(p_r{}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

where, ϵ *is an arbitrary small positive number.*

Proof. For $a \geq 2$, by the virtue of Lemma 15, $g_2(p_r, a)$ is given by

$$g_2(p_r, a) = \frac{1}{2}\int_1^{a-1}\frac{g_1(p_r, a-y)}{y}dy + \frac{1}{2}\int_1^{a-1}\log(a-y)\frac{dJ(p_r{}^y)}{p_r^y} + \frac{1}{2}\int_1^{a-1} g_1(p_r, a-y)\frac{dJ(p_r{}^y)}{p_r^y}. \tag{94}$$

We will first compute the first term of $g_2(p_r, a)$ (i.e the integral $\int_1^{a-1} g_1(p_r, a-y)dy/y$). Since $dy/y = d\log y$, thus

$$\int_1^{a-1}\frac{g_1(p_r, a-y)}{y}dy = \int_{y=1}^{a-1}\left(\int_{z=1}^{a-y}\frac{dJ(p_r{}^z)}{p_r{}^z}\right)d\log y$$

By the method of integration by parts, we then have

$$\int_1^{a-1}\frac{g_1(p_r, a-y)}{y}dy = \log y\left(\int_{z=1}^{a-y}\frac{dJ(p_r{}^z)}{p_r{}^z}\right)\Bigg|_1^{a-1} - \int_1^{a-1}\log y\ d\left(\int_{z=1}^{a-y}\frac{dJ(p_r{}^z)}{p_r{}^z}\right),$$

where

$$\log y\left(\int_{z=1}^{a-y}\frac{dJ(p_r{}^z)}{p_r{}^z}\right)\Bigg|_1^{a-1} = 0,$$

and referring to 91, we also have

$$d\left(\int_{z=1}^{a-y}\frac{dJ(p_r{}^z)}{p_r{}^z}\right) = \frac{dJ(p_r{}^{a-y})}{p_r{}^{a-y}}.$$

Hence

$$\int_1^{a-1}\frac{g_1(p_r, a-y)}{y}dy = -\int_1^{a-1}\log y\frac{dJ(p_r{}^{a-y})}{p_r{}^{a-y}}$$

and by changing the variable y by $a-y$, we then have

$$\int_1^{a-1}\frac{g_1(p_r, a-y)}{y}dy = \int_1^{a-1}\log(a-y)\frac{dJ(p_r{}^y)}{p_r{}^y} \tag{95}$$

Combining Equations (94) and (95), we obtain

$$g_2(p_r, a) = \int_1^{a-1} \log(a-y) \frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{2} \int_1^{a-1} g_1(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y}.$$

and by the virtue of Lemmas 11 and 23, we finally have

$$g_2(p_r, a) = \int_1^{a-1} \log(a-y) \frac{dJ({p_r}^y)}{p_r^y} + O({p_r}^{-a(1-c)/2+\epsilon}).$$

□

Lemma 25. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 \le c < 1$), then for $a \ge 3$, we have*

$$\int_1^{a-1} g_2(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} = \frac{1}{2} \int_{y=1}^{a-2} \left(\int_{z=1}^{a-y-1} \left(\int_{w=1}^{a-y-z} \frac{dJ({p_r}^w)}{{p_r}^w} \right) \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y} + O({p_r}^{-2(1-c)+\epsilon}).$$

where, ϵ is an arbitrary small positive number.

Proof. Referring to Lemma 24, we have

$$\int_{y=1}^{a-1} g_2(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} =$$

$$\int_{y=1}^{a-1} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} + \frac{1}{2} \int_{z=1}^{a-y-1} g_1(p_r, a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y}$$

Since $J(p_r^x)$ is set to zero for $x < 1$, therefore the limit of the inner integral $z = a-y-1$ should be also greater or equal to 1. Hence y should not exceed $a-2$. Consequently

$$\int_{y=1}^{a-1} g_2(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} = \int_{y=1}^{a-2} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y} +$$

$$\frac{1}{2} \int_{y=1}^{a-2} \left(\int_{z=1}^{a-y-1} \left(\int_{w=1}^{a-y-z} \frac{dJ({p_r}^w)}{{p_r}^w} \right) \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y} \tag{96}$$

To compute the first integral on the right side of Equation (96), we use the method of integration by parts to obtain

$$\int_1^{a-2} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y} = \frac{J({p_r}^y)}{p_r^y} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \Bigg|_{y=1}^{a-2} -$$

$$\int_1^{a-2} J({p_r}^y) \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) d\left(\frac{1}{p_r^y}\right) - \int_1^{a-2} \frac{J({p_r}^y)}{p_r^y} d\left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right)$$

The first term on the right side of the above equation is given by

$$\frac{J({p_r}^y)}{p_r^y} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \Bigg|_{y=1}^{a-2} = -\frac{J(p_r)}{p_r} \left(\int_{z=1}^{a-2} \log(a-1-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \tag{97}$$

By the virtue of Lemmas 7 and 10 where both $|J(p_r)/p_r|$ and $|\int_1^x dJ({p_r}^y)/p_r^y|$ are given by ${p_r}^{-(1-c)+\epsilon}$, we then have

$$\left| \frac{J({p_r}^y)}{p_r^y} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \Bigg|_{y=1}^{a-2} \right| = O\left(p_r^{-2(1-c)+\epsilon}\right)$$

Furthermore,

$$\int_1^{a-2} J({p_r}^y) \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) d\left(\frac{1}{p_r^y}\right) =$$

$$-\log p_r \int_1^{a-2} \frac{J({p_r}^y)}{p_r^y} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) dy$$

and by the virtue of Lemmas 7 and 10 we then have

$$\left| \int_1^{a-2} J({p_r}^y) \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) d\left(\frac{1}{p_r^y}\right) \right| = O\left(p_r^{-2(1-c)+\epsilon}\right) \tag{98}$$

For the integral $\int_1^{a-2} \frac{J({p_r}^y)}{p_r^y} d\left(\int_{z=1}^{a-y-1} \log(a-y-z)\frac{dJ({p_r}^z)}{p_r^z}\right)$, we first need to compute the differential $d\left(\int_{z=1}^{a-y-1} \log(a-y-z)\frac{dJ({p_r}^z)}{p_r^z}\right)$ with respect to y

$$\Delta_y \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) =$$

$$\int_{z=1}^{a-y-\Delta y-1} \log(a-y-\Delta y-z) \frac{dJ({p_r}^z)}{p_r^z} - \int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z}$$

or

$$\Delta_y \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) = \int_{z=a-y-\Delta y-1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} +$$

$$\int_{z=1}^{a-y-1} \left(\log(a-y-\Delta y-z) - \log(a-y-z)\right) \frac{dJ({p_r}^z)}{p_r^z}$$

Thus,

$$\frac{d}{dy} \left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) = \log(1) \frac{dJ({p_r}^{a-y-1})}{p_r^{a-y-1}} - \int_{z=1}^{a-y-1} \frac{1}{a-z-y} \frac{dJ({p_r}^z)}{p_r^z} \tag{99}$$

and

$$\int_1^{a-2} \frac{J({p_r}^y)}{p_r^y} d\left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) = -\int_1^{a-2} \frac{J({p_r}^y)}{p_r^y} \left(\int_{z=1}^{a-y-1} \frac{1}{a-z-y} \frac{dJ({p_r}^z)}{p_r^z} \right) dy$$

Since $a-z-y \geq 1$, then for a fixed value of $a \geq 3$, we have by the virtue of Lemmas 7 and 10

$$\left| \int_1^{a-2} \frac{J({p_r}^y)}{p_r^y} d\left(\int_{z=1}^{a-y-1} \log(a-y-z) \frac{dJ({p_r}^z)}{p_r^z} \right) \right| = O\left(p_r^{-2(1-c)+\epsilon}\right) \tag{100}$$

Combining equations (96), (97), (98) and (100), we then have for $a \geq 3$

$$\int_1^{a-1} g_2(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} = \frac{1}{2} \int_{y=1}^{a-2} \left(\int_{z=1}^{a-y-1} \left(\int_{w=1}^{a-y-z} \frac{dJ({p_r}^w)}{{p_r}^w} \right) \frac{dJ({p_r}^z)}{p_r^z} \right) \frac{dJ({p_r}^y)}{p_r^y} + O(p_r^{-2(1-c)+\epsilon}).$$

□

In the following two lemmas, we will use Lemmas 15, 24 and 25 to provide an estimate for $R(1,p_r;1,p_r^a)$ for $2 \leq a < 3$ and $3 \leq a < 4$.

Lemma 26. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $2 \leq a < 3$, we have*

$$R(1,p_r;1,p_r^a) = -\int_1^a \rho(a-y)\frac{dJ(p_r{}^y)}{p_r{}^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

where, ϵ is an arbitrary small positive number.

Proof. Referring to Lemma 15, we have for $2 \leq a < 3$

$$M(1,p_r;1,p_r{}^a) = 1 - \log(a) + \frac{1}{2}\int_1^{a-1}\frac{\log(a-y)}{y}dy - g_1(p_r,a) + g_2(p_r,a) + O(p_r^{-1+\epsilon}),$$

where $O(\log(a)/p_r)$ (the contribution by non square-free terms as determined by Lemma 18) is replaced by $O(p_r^{-1+\epsilon})$, $g_1(p_r,a)$ and $g_2(p_r,a)$ are given by (refer to Lemmas 22 and 24)

$$g_1(p_r,a) = \int_1^a \frac{dJ(p_r{}^y)}{p_r{}^y}$$

$$g_2(p_r,a) = \int_1^{a-1}\log(a-y)\frac{dJ(p_r{}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

Since for $2 \leq a < 3$, $\rho(x) = 1 - \log x + \frac{1}{2}\int_1^{x-1}\frac{\log(x-z)}{z}dz$, then referring to Equation (84), $R(1,p_r;1,p_r^a)$ is given by

$$R(1,p_r;1,p_r{}^a) = -g_1(p_r,a) + g_2(p_r,a) + O(p_r^{-1+\epsilon})$$

or

$$R(1,p_r;1,p_r{}^a) = -\int_1^{a-1}\frac{dJ(p_r{}^y)}{p_r^y} - \int_{a-1}^a\frac{dJ(p_r{}^y)}{p_r^y} + \int_1^{a-1}\log(a-y)\frac{dJ(p_r{}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

Since for $2 \leq a < 3$, we have

$$\rho(a-y) = 1 - \log(a-y),$$

therefore for $2 \leq a < 3$

$$R(1,p_r;1,p_r^a) = -\int_1^{a-1}\rho(a-y)\frac{dJ(p_r{}^y)}{p_r{}^y} - \int_{a-1}^a\frac{dJ(p_r{}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

Since $\rho(a-x) = 1$ for $a-1 \leq x \leq a$, thus

$$R(1,p_r;1,p_r^a) = -\int_1^a \rho(a-y)\frac{dJ(p_r{}^y)}{p_r{}^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

□

Lemma 27. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $3 \leq a < 4$, we have*

$$R(1,p_r;1,p_r^a) = -\int_1^a \rho(a-y)\frac{dJ({p_r}^y)}{{p_r}^y} +$$

$$\frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O({p_r}^{-a(1-c)/2+\epsilon})$$

where, ϵ is an arbitrary small positive number.

Proof. We first recall that for $3 \leq a < 4$ (refer to [7], Equation (3.15)), we have

$$\rho(y) = 1 - \log(y) + \frac{1}{2}\int_1^{y-1}\frac{\log(y-z)}{z}dz$$

and

$$\rho(a-y) = 1 - \log(a-y) + \frac{1}{2}\int_1^{a-y-1}\frac{\log(a-y-z)}{z}dz$$

Referring to Equation (63) and (65), we then have for $3 \leq a < 4$

$$M(1,p_r;1,p_r^a) = \rho(a) - g_1(p_r,a) + g_2(p_r,a) - g_3(p_r,a) + O(p_r^{-1+\epsilon}),$$

where $O(p_r^{-1+\epsilon})$ is the contribution by non square-free terms as determined by Lemma 18, $g_1(p_r,a)$ and $g_2(p_r,a)$ are given by (refer to Lemmas 22 and 24)

$$g_1(p_r,a) = \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}$$

$$g_2(p_r,a) = \int_1^{a-1}\log(a-y)\frac{dJ({p_r}^y)}{p_r^y} + O({p_r}^{-a(1-c)/2+\epsilon}).$$

and referring to Equation (64), $g_3(p_r,a)$ is given by

$$g_3(p_r,a) = \frac{1}{3}\int_1^{a-1}\frac{g_2(p_r,a-y)}{y}dy + \frac{1}{3}\int_1^{a-1}g_2(p_r,a-y)\frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{3}\int_1^{a-1}h_2(a-y)\frac{dJ({p_r}^y)}{p_r^y},$$

where for $a \geq 2$ (refer to Equation (75))

$$h_2(a) = \frac{1}{2}\int_1^{a-1}\frac{\log(a-y)}{y}dy$$

For the integral $\int_1^{a-1}\frac{g_2(p_r,a-y)}{y}dy$, we refer to Lemma 24 to obtain

$$\int_1^{a-1}\frac{g_2(p_r,a-y)}{y}dy = \int_{y=1}^{a-1}\frac{\int_1^{a-y-1}\log(a-y-z)\frac{dJ({p_r}^z)}{p_r^z}}{y}dy + O({p_r}^{-a(1-c)/2+\epsilon})$$

For the integral $\int_1^{a-1} g_2(p_r,a-y)\frac{dJ({p_r}^y)}{p_r^y}$, we refer to Lemma 25 to obtain

$$\int_1^{a-1} g_2(p_r,a-y)\frac{dJ({p_r}^y)}{p_r^y} = \frac{1}{2}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O({p_r}^{-2(1-c)+\epsilon}).$$

Thus,

$$g_3(p_r, a) = \frac{1}{3}\int_1^{a-2} h_2(a-y)\frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{3}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}(\log(a-y-z))\frac{dJ({p_r}^z)}{{p_r}^z}\right)\frac{dy}{y}+$$

$$\frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon}).$$

where $J(p_r^z) = 0$ for $z < 1$ and $h_2(x) = 0$ for $x < 2$. Rearranging the second integral on the right side of the above equation, we then have

$$\frac{1}{3}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}(\log(a-y-z))\frac{dJ({p_r}^z)}{{p_r}^z}\right)\frac{dy}{y} = \frac{1}{3}\int_{z=1}^{a-2}\left(\int_{y=1}^{a-z-1}\frac{\log(a-z-y)}{y}dy\right)\frac{dJ({p_r}^z)}{{p_r}^z}$$

or

$$\frac{1}{3}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}(\log(a-y-z))\frac{dJ({p_r}^z)}{{p_r}^z}\right)\frac{dy}{y} = \frac{2}{3}\int_{z=1}^{a-2} h_2(a-z)\frac{dJ({p_r}^z)}{p_r^z}$$

Thus

$$g_3(p_r, a) = \int_1^{a-2} h_2(a-y)\frac{dJ({p_r}^y)}{p_r^y}+\frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y}+$$

$$O(p_r^{-a(1-c)/2+\epsilon}).$$

Consequently, for $3 \le a < 4$, we then have

$$M(1, p_r; 1, p_r^a) = \rho(a)+\int_1^{a-1}(-1+\log(a-y)-h_2(a-y))\frac{dJ({p_r}^y)}{p_r^y}-\int_{a-1}^{a}\frac{dJ({p_r}^y)}{{p_r}^y}+$$

$$\frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon})$$

Since for $3 \le a < 4$, we have

$$\rho(a-y) = 1 - \log(a-y) - h_2(a-y),$$

therefore for $3 \le a < 4$

$$R(1, p_r; 1, p_r^a) = -\int_1^{a-1}\rho(a-y)\frac{dJ({p_r}^y)}{{p_r}^y}-\int_{a-1}^{a}\frac{dJ({p_r}^y)}{{p_r}^y}+$$

$$\frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon})$$

Since $\rho(a-x) = 1$ for $a-1 \le x \le a$, thus

$$R(1, p_r; 1, p_r^a) = -\int_1^{a}\rho(a-y)\frac{dJ({p_r}^y)}{{p_r}^y}+$$

$$\frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon})$$

□

Combining Lemmas 22, 26 and 27, we then have the second key theorem. This theorem provides the second equation for the value of $R(1, p_r; 1, p_r^a)$ in terms of $dJ({p_r}^y)/{p_r}^y$ with $1 \leq a < 4$.

Theorem 5. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $1 \leq a < 3$, we have*

$$R(1, p_r; 1, p_r^a) = -\int_1^a \rho(a - y)\frac{dJ({p_r}^y)}{{p_r}^y} + O(p_r^{-a(1-c)/2+\epsilon}). \tag{101}$$

and for $3 \leq a < 4$, we have

$$R(1, p_r; 1, p_r^a) = -\int_1^a \rho(a-y)\frac{dJ({p_r}^y)}{{p_r}^y} + \frac{1}{6}\int_{y=1}^{a-2}\left(\int_{z=1}^{a-y-1}\left(\int_{w=1}^{a-y-z}\frac{dJ({p_r}^w)}{{p_r}^w}\right)\frac{dJ({p_r}^z)}{p_r^z}\right)\frac{dJ({p_r}^y)}{p_r^y} + O(p_r^{-a(1-c)/2+\epsilon}) \tag{102}$$

where, ϵ is an arbitrary small positive number.

Comparing Equation (102) of theorem 5 with Equation (87) of theorem 4, we notice that $R(1, p_r; 1, p_r^a)$ is represented in terms of $\int_1^a \rho(a-y)dJ({p_r}^y)/{p_r}^y$ in Equation (102) while it is represented in terms of $\int_1^a \rho(a/y-1)dJ({p_r}^y)/{p_r}^y$ in Equation (87). This difference in $R(1, p_r; 1, p_r^a)$ representation will be exploited in our analysis of the zeta function non-trivial zeros. In section 6, we will present a third method for computing $R(1, p_r; 1, p_r^a)$ that is based on complex analysis. With this method, we will drive a replica for Equation (102) with a tighter bound on the estimation of the triple integral on the right side of Equation (102). In section 7, the equation derived through complex analysis (in section 6) will be compared with Equation (87) to show that non-trivial zeros can be found arbitrary close to the line $\Re(s) = 1$. Before we do so, we need to expand our method to compute $M(s, p_r; 1, N)$ where s is a complex number in the region where the series $M(s, p_r)$ is convergent. We will achieve this task in the next step of this section.

- In the fourth step, we will extend the concept of regular and irregular components of $M(1, p_r; 1, x)$ to right side of the line $\Re(z) = 0.5$ in the complex plain where the series $M(s, p_r)$ is convergent.

Toward this task, we first note that the partial sum $M(s, p_r; 1, x)$ is given by the sum $\sum_{n=1}^{\lfloor x \rfloor} \mu(n, p_r)/n^s$ and therefore it can be written as follows

$$M(s, p_r; 1, x) = M(s, p_r; 1, \lfloor x \rfloor) = 1 + \int_{x=p_r}^{x} \frac{x}{x^s} dM(1, p_r; 1, x),$$

or

$$M(s, p_r; 1, p_r^a) = M(s, p_r; 1, \lfloor p_r^a \rfloor) = 1 + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dM(1, p_r; 1, p_r^y).$$

Since $M(1, p_r; 1, p_r^y) = \rho(y) + R(1, p_r; 1, p_r^y)$, thus

$$M(s, p_r; 1, p_r^a) = 1 + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} d\rho(y) + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dR(1, p_r; 1, {p_r}^y). \tag{103}$$

Therefore, for any s, the partial sum $M(s, p_r; 1, p_r^a)$ has two components. The first one is the deterministic or regular component given by $1+\int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} d\rho(y)$. The second one is the irregular component given by $\int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dR(1, p_r; 1, {p_r}^y)$. Therefore, if we define α as

$$\alpha = (s-1)\log p_r$$

and the regular component of $M(s, p_r; 1, p_r^a)$ as $F(\alpha, a)$, then

$$F(\alpha, a) = 1 + \int_1^a \frac{{p_r}^x}{{p_r}^{sx}} d\rho(x) = 1 + \int_1^a {p_r}^{(1-s)x} \rho'(x) dx,$$

or,

$$F(\alpha, a) = 1 + \int_1^a e^{-\alpha x} \rho'(x) dx,$$

Definition 9. *For the region of convergence of the series $M(s, p_r)$, the regular component of the partial sum $M(s, p_r; 1, p_r^a)$ is defined as*

$$F(\alpha, a) = 1 + \int_1^a e^{-\alpha x} \rho'(x) dx, \tag{104}$$

while the irregular component of the partial sum $M(s, p_r; 1, p_r^a)$ is defined as

$$R(s, p_r; 1, p_r^a) = M(s, p_r; 1, p_r^a) - F(\alpha, a). \tag{105}$$

Notice that for $s = 1$, we have $\alpha = 0$ and $F(0, a) = \rho(a)$. We also notice that the regular component exists for any value of s with $\Re(s) > 0$. This is expected since the regular components of both the prime counting function and $M(s, p_r; 1, p_r^a)$ are not determined by the location of the non-trivial zeros within the critical strip.

We now define $F(\alpha)$ as

$$F(\alpha) = \lim_{a\to\infty} F(\alpha, a) = 1 + \int_1^\infty e^{-\alpha x} \rho'(x) dx. \tag{106}$$

Thus, for $\Re(s) \geq 1$, α is a complex variable in the complex plane to the right of the line $\Re(s) = 1$ (which corresponds to the line $\Re(\alpha) = 0$). Hence, the integral $\int_1^\infty e^{-\alpha x} \rho'(x) dx$ is the Laplace transform of the function $\rho^{'}(x)$ and it is given by $F(\alpha) - 1$ (where $F(\alpha)$ is the regular component of the series $M(s, p_r)$, i.e. $M(s, p_r; 1, \infty)$). Since the Laplace transform of $\rho(x)$ multiplied by s is given by $e^{-E_1(s)}$ [10] (refer to page 569) [9] and the Laplace transform of $\rho^{'}(x)$ is given by $s\mathcal{L}(\rho(x)) - \rho(0)$, therefore

$$F(\alpha) = e^{-E_1(\alpha)}.$$

Definition 10. *For the region of convergence of the series $M(s, p_r)$, the regular component of the series $M(s, p_r)$ is defined as*

$$F(\alpha) = e^{-E_1(\alpha)}.$$

where $\alpha = (s-1)\log p_r$. The irregular component of the series $M(s, p_r)$ is defined as

$$R(s, p_r) = M(s, p_r) - F(\alpha)$$

In Theorem 3, we have shown that

$$M(s, p_r) = e^{-E_1(\alpha)-\varepsilon(p_r,s)+\delta(p_r,s)}, \tag{107}$$

where $\varepsilon(s;p_r) = \int_{p_r}^{\infty} dJ(x)/x^s$ and $J(x) = \pi(x) - \mathrm{Li}(x)$. Using the definitions presented in this step, we can rephrase Theorem 3 as follows

Theorem 6. *For the region of convergence of the series $M(s,p_r)$, $M(s,p_r)$ can be expressed as*

$$M(s, p_r) = \lim_{a\to\infty} M(s, p_r; 1, {p_r}^a) = F(\alpha) + R(s, p_r) \tag{108}$$

where $\alpha = (s-1)\log p_r$ and $F(\alpha)$ is the regular component of $M(s,p_r)$ given by

$$F(\alpha) = e^{-E_1(\alpha)}. \tag{109}$$

Furthermore, $R(s,p_r)$ is the irregular component of $M(s,p_r)$ and it is given by

$$R(s, p_r) = \lim_{a\to\infty} R(s, p_r; 1, {p_r}^a) = e^{-E_1(\alpha)}(e^{-\varepsilon(s;p_r)+\delta(s;p_r)} - 1). \tag{110}$$

It should be emphasized here that the regular component $F(\alpha)$ is the value of $M(s,p_r)$ due to $\mathrm{Li}(x)$ component of the prime counting function $\pi(x)$. The irregular component $R(s,p_r)$ is given by $\lim_{a\to\infty} R(s,p_r;1,{p_r}^a) = \lim_{a\to\infty} M(s,p_r;1,{p_r}^a) - \lim_{a\to\infty} F(\alpha,{p_r}^a)$. It should be also pointed out that for $s=1$, the irregular component $R(1,p_r) = F(0)(e^{-\varepsilon(1;p_r)+\delta(1;p_r)} - 1)$ is zero for every p_r (note that $R(1,p_r;1,{p_r}^a)$ may deviate from zero but it ultimately approaches zero as a approaches ∞). For $s \neq 1$, the irregular component $R(s,p_r) = F(\alpha)(e^{-\varepsilon(s;p_r)+\delta(s;p_r)} - 1)$ may have values different from zero although it approaches zero as p_r approaches infinity

In the following section, we will use Theorem 6 and the complex analysis to obtain an alternative representation for $R(1,p_r;1,p_r^a)$. This representation will then be compared with Equation (87) of Theorem 4 to show that non-trivial zeros of the Riemann zeta function can be found arbitrary close to the line $\Re(s) = 1$

6 Computing the irregular component of $M(1, p_r; 1, p_r^a)$ using complex analysis.

In the second step of the previous section, we have used integration methods to compute the irregular component of $M(1,p_r;1,p_r^a)$ for values of $a > 1$. Referring to Equation (87) of Theorem 4, we have

$$R(1,p_r;1,{p_r}^a) = -\int_1^a \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} + Q(p_r,a)$$

It should be noted here that $R(1,p_r;1,{p_r}^y)$ is zero for $y<1$, hence for our analysis to compute $R(1,p_r;1,{p_r}^y)$, $J(p_r^y)$ is set to 0 for $y<1$. In the third step of the previous section, Theorem 5 provides a second representation of $R(1,p_r;1,{p_r}^a)$ in terms of $\int_1^a \rho(a-y)dJ({p_r}^y)/{p_r}^y$ instead of the term $\int_1^a \rho(a/y-1)dJ({p_r}^y)/{p_r}^y$. In this section, we will use Equation (110) of Theorem 6 and the complex analysis to find another representation of $R(1,p_r;1,{p_r}^y)$ in terms of $\int_1^a \rho(a-y)dJ({p_r}^y)/{p_r}^y$. The complex analysis will allow us to find a tight bound for the O

term associated with the term $\int_1^a \rho(a-y)dJ({p_r}^y)/{p_r}^y$. Toward this end, we recall Equation (110) of Theorem 6 for the value of $R(s,p_r)$ (or the irregular component of the series $M(s,p_r)$)

$$R(s,p_r;1,\infty) = e^{-E_1(\alpha)}(e^{-\varepsilon(s;p_r)+\delta(s;p_r)}-1),$$

To compute $R(s,p_r;1,p_r^a)$, we recall Equation (103) that establishes the connection between $R(s,p_r;1,p_r^a)$ and $R(1,p_r;1,p_r^a)$

$$R(s,p_r;1,{p_r}^a) = \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dR(1,p_r;1,{p_r}^y).$$

Recall that $\alpha = (s-1)\log p_r$, hence

$$R(s,p_r;1,{p_r}^a) = \int_{y=1}^{a} e^{-\alpha y} \frac{dR(1,p_r;1,{p_r}^y)}{dy} dy,$$

or

$$R(s,p_r) = \lim_{a\to\infty} R(s,p_r;1,p_r^a) = \int_{y=1}^{\infty} e^{-\alpha y} \frac{dR(1,p_r;1,{p_r}^y)}{dy} dy. \tag{111}$$

Thus, $R(s,p_r)$ is the Laplace transform of the derivative of the partial sum $R(1,p_r;1,{p_r}^a)$. Consequently, our complex analysis representation of the partial sum $R(1,p_r;1,{p_r}^a)$ is given by the following theorem

Theorem 7. *If $\zeta(s)$ is void of non-trivial zeros in the vicinity of the line $\Re(s)=1$, then the partial sum $R(1,p_r;1,{p_r}^a)$ can be written as*

$$R(1,p_r;1,{p_r}^a) = \int_1^a \mathcal{L}^{-1}(R(\alpha,p_r))(y)dy,$$

or

$$R(1,p_r;1,{p_r}^a) = \int_1^a \mathcal{L}^{-1}\left(e^{-E_1(\alpha)}(e^{-\varepsilon(\alpha;p_r)+\delta(\alpha;p_r)}-1)\right)(y)dy,$$

where $\alpha = (s-1)\log p_r$.

Proof. Referring to Equation (111), $R(s,p_r)$ or $R(\alpha,p_r)$ is given by

$$R(\alpha,p_r) = \int_{y=1}^{\infty} e^{-\alpha y} \frac{dR(1,p_r;1,{p_r}^y)}{dy} dy.$$

Hence, if $M(s,p_r)$ is analytic at $s=1$, then

$$\frac{dR(1,p_r;1,{p_r}^y)}{dy} = \mathcal{L}^{-1}(R(\alpha,p_r))(y).$$

Hence (recall that $R(1,p_r;1,{p_r}^a)=0$ for $a<1$)

$$R(1,p_r;1,{p_r}^a) = \int_1^a \mathcal{L}^{-1}(R(\alpha,p_r))(y)dy,$$

or

$$R(1,p_r;1,{p_r}^a) = \int_1^a \mathcal{L}^{-1}\left(e^{-E_1(\alpha)}(e^{-\varepsilon(s;p_r)+\delta(s;p_r)}-1)\right)(y)dy,$$

or

$$R(1,p_r;1,{p_r}^a) = \int_1^a \mathcal{L}^{-1}\left(e^{-E_1(\alpha)}(e^{-\varepsilon(\alpha;p_r)+\delta(\alpha;p_r)}-1)\right)(y)dy,$$

where $\alpha = (s-1)\log p_r$. □

For the remaining of this section, our efforts will be centered around computing the integral $\int_1^a \mathcal{L}^{-1}(e^{-E_1(\alpha)}(e^{-\varepsilon(\alpha;p_r)+\delta(\alpha;p_r)}-1))(y)dy$ (this task may be simplified by removing the term $\delta(\alpha;p_r)$. This term corresponds to an absolutely convergent series for $\Re(s) > 0.5$ and therefore it has no impact on the region of convergence of the series $M(s,p_r)$). We will start this task by the following definition and lemma for computing the terms $\varepsilon(\alpha;p_r)$ and $\delta(\alpha;p_r)$.

Definition 11. *We define the function $f_\varepsilon(y)$ for $y \geq 1$ as follows*

$$f_\varepsilon(y) = \frac{dJ(p_r^y)/p_r^y}{dy}. \tag{112}$$

where $J(p_r^y)$ is set to 0 for $y < 1$ and $f_\varepsilon(y) = 0$ for $y < 1$.

Lemma 28. *If $\zeta(s)$ is void of non-trivial zeros in the vicinity of the line $\Re(s) = 1$, then*

$$\mathcal{L}^{-1}(\varepsilon(\alpha;p_r))(y) = \frac{dJ(p_r^y)/p_r^y}{dy} = f_\varepsilon(y). \tag{113}$$

We also have uncoditionally

$$\mathcal{L}^{-1}(\delta(\alpha;p_r))(y) = -\sum_{i=r}^{\infty}\left(\frac{\delta(y-2)}{2{p_i}^2} + \frac{\delta(y-3)}{3{p_i}^3} + \frac{\delta(y-4)}{4{p_i}^4}...\right) \tag{114}$$

where, $J(p_r^y)$ is set to 0 for $y < 1$.

Proof. Referring to Definition 5

$$\varepsilon(s,p_r) = \int_1^\infty \frac{dJ(p_r^y)}{(p_r^y)^s},$$

or

$$\varepsilon(s,p_r) = \int_1^\infty \frac{p_r^y}{p_r^{sy}}\frac{dJ(p_r^y)}{p_r^y}.$$

Since $\alpha = (s-1)\log p_r$, thus

$$\varepsilon(s,p_r) = \varepsilon(\alpha,p_r) = \int_{y=1}^\infty e^{-\alpha y}\frac{1}{p_r^y}\frac{dJ(p_r^y)}{dy}dy.$$

Furthermore, since we have assumed that $\zeta(s)$ is void of non-trivial zeros in the vicinity of the line $\Re(s) = 1$, therefore

$$\varepsilon(\alpha;p_r) = \mathcal{L}\left(\frac{dJ(p_r^y)/p_r^y}{dy}\right)(\alpha) \tag{115}$$

and

$$\mathcal{L}^{-1}(\varepsilon(\alpha;p_r))(y) = \frac{dJ(p_r^y)/p_r^y}{dy} = f_\varepsilon(y).$$

Referring to Definition 2

$$\delta(s;p_r) = \sum_{i=r}^{\infty}\left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}}...\right),$$

or

$$\delta(s;p_r) = \sum_{i=r}^{\infty}\left(-\frac{1}{2{p_i}^2}\frac{{p_i}^2}{{p_i}^{2s}} - \frac{1}{3{p_i}^3}\frac{{p_i}^3}{{p_i}^{3s}} - \frac{1}{4{p_i}^4}\frac{{p_i}^4}{{p_i}^{4s}}...\right).$$

However,

$$\frac{{p_i}^n}{{p_i}^{ns}} = \frac{e^{n\log p_i}}{e^{ns\log p_i}}.$$

Since $\alpha = (s-1)\log p_r$, thus

$$\frac{{p_i}^n}{{p_i}^{ns}} = e^{-n\alpha}.$$

Thus

$$\delta(\alpha;p_r) = -\sum_{i=r}^{\infty}\left(\frac{e^{-2\alpha}}{2{p_i}^2} + \frac{e^{-3\alpha}}{3{p_i}^3} + \frac{e^{-4\alpha}}{4{p_i}^4}...\right).$$

Since $\mathcal{L}^{-1}(e^{a\alpha})(y) = \delta(y+a)$, hence

$$\mathcal{L}^{-1}(\delta(\alpha;p_r))(y) = -\sum_{i=r}^{\infty}\left(\frac{\delta(y-2)}{2{p_i}^2} + \frac{\delta(y-3)}{3{p_i}^3} + \frac{\delta(y-4)}{4{p_i}^4}...\right).$$

□

Lemma 29. *If $\zeta(s)$ is void of non-trivial zeros in the vicinity of the line $\Re(s) = 1$, then*

$$R(1,p_r;1,{p_r}^a) = -\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha;p_r)\right)(y)dy +$$
$$\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right)\right)(y)\,dy +$$
$$\int_1^a \left((\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\delta^m(\alpha;p_r)\right)\right)(y)\,dy +$$
$$\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\delta^m(\alpha;p_r)\right)\right)(y)\,dy \qquad (116)$$

Proof. Referring to Theorem 7 where $R(1,p_r;1,{p_r}^a)$ is given by

$$R(1,p_r;1,{p_r}^a) = \int_1^a \mathcal{L}^{-1}\left(e^{-E_1(\alpha)}e^{-\varepsilon(\alpha;p_r)+\delta(\alpha;p_r)} - e^{-E_1(\alpha)}\right)(y)dy,$$

and recalling that multiplication in the transform domain corresponds to convolution in the function domain, therefore

$$R(1,p_r;1,{p_r}^a) = \int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}e^{-\varepsilon(\alpha;p_r)} * \mathcal{L}^{-1}e^{\delta(\alpha;p_r)} - \mathcal{L}^{-1}e^{-E_1(\alpha)}\right)(y)\,dy.$$

where

$$e^{-\varepsilon(\alpha;p_r)} = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r),$$

and

$$e^{\delta(\alpha;p_r)} = 1 + \sum_{m=1}^{\infty}\frac{1}{m!}\delta^m(\alpha;p_r).$$

Since $\mathcal{L}(\delta(y)) = 1$ and $\delta(y) * f(y) = f(y)$, therefore

$$R(1, p_r; 1, {p_r}^a) = -\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \varepsilon(\alpha; p_r)\right)(y)dy +$$

$$\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k!}\varepsilon^k(\alpha; p_r)\right)\right)(y)\,dy +$$

$$\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty} \frac{1}{m!}\delta^m(\alpha; p_r)\right)\right)(y)\,dy +$$

$$\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\varepsilon^k(\alpha; p_r)\right) * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty} \frac{1}{m!}\delta^m(\alpha; p_r)\right)\right)(y)\,dy$$

□

In the following lemmas we will compute the four integrals on the right side of Equation (116). In the next section (section 7), we will show that for our present work, we only need to compute Equation (116) for $a < 4$. This will simplify our effort to compute the second integral of Equation (116). For $a < 4$, we only need to compute this integral for $k = 2$ and 3. For $k \geq 4$, the second integral is zero (this follows from the fact that for $k \geq 4$, we have the convolution$(f_\epsilon * f_\epsilon * f_\epsilon * f_\epsilon)(y)$ where $f_\epsilon(y) = 0$ for $y < 1$ and $(f_\epsilon * f_\epsilon * f_\epsilon * f_\epsilon)(y) = 0$ for $y < 4$). The next lemma deals with the computation of the first integral on the right side of Equation (116)

Lemma 30.

$$\int_{y=1}^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha; p_r)\right)(y)\,dy = \int_{x=1}^a \rho(a - x)\frac{dJ(p_r^x)}{p_r^x} \tag{117}$$

Proof. Since $\mathcal{L}^{-1}e^{-E_1(\alpha)} = \rho'(y) + \delta(y)$ and $\mathcal{L}^{-1}\varepsilon(\alpha; p_r) = f_\varepsilon$, therefore

$$\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha; p_r)\right)(y) = \left((\rho' + \delta) * f_\varepsilon\right)(y)$$

Since $f_\varepsilon(y)$ and $\rho'(y)$ are zero for $y < 1$, hence

$$\left((\rho' + \delta) * f_\varepsilon\right)(y) = \int_{x=1}^{y-1} \rho'(y - x)f_\varepsilon(x)dx + f_\varepsilon(y)$$

Consequently (note that $f_\varepsilon(y)$ and $\rho'(y)$ are zero for $y < 1$ and $\left(\rho' * f_\varepsilon\right)(y)$ is zero for $y < 2$),

$$\int_{y=1}^a \left((\rho' + \delta) * f_\varepsilon\right)(y)dy = \int_{y=2}^a \left(\int_{x=1}^{y-1} \rho'(y - x)f_\varepsilon(x)dx\right)dy + \int_{y=1}^a f_\varepsilon(y)(y)dy,$$

Since the limit of integration for the second integral on the right side of the above equation are fixed numbers, thus it can be written as

$$\int_{y=1}^a \left((\rho' + \delta) * f_\varepsilon\right)(y)dy = \int_{y=2}^a \left(\int_{x=1}^{y-1} \rho'(y - x)f_\varepsilon(x)dx\right)dy + \int_{x=1}^a f_\varepsilon(x)(x)dx. \tag{118}$$

The next step is to change the order of integration for the double integral. For the double integral, the limit of the inner integral is given by $1 \leq x \leq y - 1$ and the limit for the outer integral is given by $2 \leq y \leq a$. To change the order of integration of this double integral, we need to cover the same region of integration. This can be achieved by setting the limit of the inner integral as $x+1 \le y \le a$ and the limit of the outer integral as $1 \le x \le a-1$. Therefore,

$$\int_{y=2}^{a} \left(\int_{x=1}^{y-1} \rho'(y-x) f_\varepsilon(x) dx \right) dy = \int_{x=1}^{a-1} f_\varepsilon(x) \left(\int_{y=x+1}^{a} \rho'(y-x) dy \right) dx,$$

or

$$\int_{y=1}^{a} \left((\rho' + \delta) * f_\varepsilon \right)(y) dy = \int_{x=1}^{a-1} f_\varepsilon(x) \left(1 + \int_{y=x+1}^{a} \rho'(y-x) dy \right) dx + \int_{x=a-1}^{a} f_\varepsilon(x).$$

Since $f_\varepsilon(x) = \frac{dJ(p_r^x)/p_r^x}{dx}$, therefore

$$\int_{y=1}^{a} \left((\rho' + \delta) * f_\varepsilon \right)(y) dy = \int_{x=1}^{a-1} \frac{dJ(p_r^x)}{p_r^x} \left(1 + \int_{y=x+1}^{a} \rho'(y-x) dy \right) + \int_{x=a-1}^{a} \frac{dJ(p_r^x)}{p_r^x}. \quad (119)$$

Since $\rho(z) = 1 + \int_1^z \rho'(x) dx$, thus $\rho(a-x) = 1 + \int_{x+1}^{a} \rho'(y-x) dy$. Moreover, since $\rho(a-x) = 1$ for $a-1 \le x \le a$, thus

$$\int_{y=1}^{a} \left((\rho' + \delta) * f_\varepsilon \right)(y) dy = \int_{y=1}^{a} \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right)(y)\, dy = \int_{x=1}^{a} \rho(a-x) \frac{dJ(p_r^x)}{p_r^x}.$$

□

To analyze the last three integrals on the right side of Equation (116), we first need to compute the convolution integral $\int (f_\varepsilon * f_\varepsilon)(x) dx$

Lemma 31.

$$\int_{-\infty}^{\infty} (f_\varepsilon * f_\varepsilon)(x) dx = \int_{x=2}^{\infty} (f_\varepsilon * f_\varepsilon)(x) dx = \left(\int_{1}^{\infty} \frac{dJ(p_r^x)}{p_r^x} \right)^2. \quad (120)$$

Proof. By the virtue of Lemma 28 and recalling that convolution in the function domain corresponds to multiplication in the transform domain, hence

$$\mathcal{L}(f_\varepsilon(x) * f_\varepsilon(x))(\alpha) = \varepsilon^2(\alpha; p_r) = \left(\int_{x=1}^{\infty} e^{-\alpha x} \frac{dJ(p_r^x)}{p_r^x} \right)^2,$$

where $J(p_r^x)$ is set to 0 for $x < 1$. By recalling that if $F(s)$ is the Laplace transform of $f(t)$ then $F(s)/s$ is the Laplace transform of $\int_0^t f(x) dx$, hence

$$\mathcal{L} \left(\int_{x=2}^{y} (f_\varepsilon * f_\varepsilon)(x) dx \right)(\alpha) = \frac{1}{\alpha} \left(\int_{x=1}^{\infty} e^{-\alpha x} \frac{dJ(p_r^x)}{p_r^x} \right)^2.$$

Using the final value theorem (which states that $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$), we then have

$$\int_{x=2}^{\infty} (f_\varepsilon * f_\varepsilon)(x) dx = \left(\int_{x=1}^{\infty} \frac{dJ(p_r^x)}{p_r^x} \right)^2.$$

□

To analyze the second integral on the right side of Equation (116) for $a < 4$, we need to compute the integral $\int_{x=2}^{y} (f_\varepsilon * f_\varepsilon)(x) dx$. We will start this task by the following definition.

Definition 12.

$$h_2(y) = \int_{x=2}^{y} (f_{\varepsilon 1} * f_{\varepsilon 2})(x)dx,$$

and in general,

$$h_n(y) = \int_{x=2}^{y} (f_{\varepsilon 1} * f_{\varepsilon 2} ... f_{\varepsilon n})(x)dx,$$

where, $f_\varepsilon = f_{\varepsilon 1} = f_{\varepsilon 2} = ... = f_{\varepsilon n}$*.*

Let $g(x)$ be defined as follows

$$g(x) = f_\varepsilon(x) \quad for \quad 1 \le x \le a$$

$$g(x) = 0 \qquad otherwise$$

then

$$\mathcal{L}(g(x))(\alpha) = \int_{x=1}^{\infty} e^{-\alpha x} g(x)dx = \int_{x=1}^{a} e^{-\alpha x} \frac{1}{p_r^x} \frac{dJ(p_r^x)}{dx} dx = \int_{x=1}^{a} e^{-\alpha x} f_\varepsilon(x)dx$$

Therefore, be the virtue of Lemma 31, for $y \ge 2a$ we have

$$\int_{x=2}^{y \ge 2a} (g * g)(x)dx = \int_{x=2}^{\infty} (g * g)(x)dx = \left(\int_{1}^{a} \frac{dJ(p_r^x)}{p_r^x} \right)^2,$$

We also note that

$$\int_{x=2}^{a} (f_\varepsilon * f_\varepsilon)(x)dx = \int_{x=2}^{a} (g * g)(x)dx,$$

and

$$\int_{x=2}^{a} (f_\varepsilon * f_\varepsilon)(x)dx = \int_{x=2}^{\infty} (g * g)(x)dx - \int_{x=a}^{2a} (g * g)(x)dx,$$

Lemma 32. *If the non-trivial zeros of* $\zeta(s)$ *are restricted to the strip* $1 - c \le \Re(s) \le c$ *(where* $1/2 \le c < 1$*), then*

$$h_2(a) = \int_{x=2}^{a} (f_\varepsilon * f_\varepsilon)(x)dx = O\left(p_r{}^{-(1-c)\min(2,a/2)+\epsilon}\right) \tag{121}$$

where, ϵ *is an arbitrary small positive number.*

Proof. First, we dissect the function $f_\varepsilon(x)$ into two functions $g_1(x)$ and $g_2(x)$ where

$$g_1(x) = f_\varepsilon(x) \quad for \quad 1 \le x \le a/2$$

$$g_1(x) = 0 \qquad otherwise$$

$$g_2(x) = f_\varepsilon(x) \quad for \quad a/2 < x \le a$$

$$g_2(x) = 0 \qquad otherwise$$

Therefore,

$$\int_{x=2}^{a} (f_\varepsilon * f_\varepsilon)(x)dx = \int_{x=2}^{a} (g_1 * g_1)(x)dx + \int_{x=2}^{a} (g_1 * g_2)(x)dx + \int_{x=2}^{a} (g_2 * g_1)(x)dx + \int_{x=2}^{a} (g_2 * g_2)(x)dx$$

Since $(g_1 * g_1)(x) = 0$ for $x > a$, therefore

$$\int_{x=2}^{a} (g_1 * g_1)(x)dx = \int_{x=2}^{\infty} (g_1 * g_1)(x)dx = \left(\int_{1}^{a/2} \frac{dJ(p_r^x)}{p_r^x} \right)^2 .$$

and by the virtue of Lemma 9 we then have

$$\int_{x=2}^{a} (g_1 * g_1)(x)dx = O\left({p_r}^{-2(1-c)} / \log^2 p_r \right) . \tag{122}$$

Also, since $(g_2 * g_2)(x) = 0$ for $x \leq a$, therefore

$$\int_{x=2}^{a} (g_2 * g_2)(x)dx = 0. \tag{123}$$

Furthermore, $\int_{x=2}^{a} (g_1 * g_2)(x)dx = \int_{x=2}^{a} (g_2 * g_1)(x)dx$, where

$$\int_{x=2}^{a} (g_1 * g_2)(x)dx = \int_{x=a/2}^{a} \left(\int_{\tau=1}^{a/2} g_1(\tau) g_2(x-\tau) d\tau \right) dx,$$

note that for the inter integral $g_1(\tau) = 0$ outside the interval $[1, a/2]$ and for the outer integral $(g_1 * g_2)(y) = 0$ outside the interval $[a/2, a]$. Changing the order of integration, we then have

$$\int_{x=2}^{a} (g_1 * g_2)(x)dx = \int_{\tau=1}^{a/2} g_1(\tau) \left(\int_{x=a/2}^{a} g_2(x-\tau)dx \right) d\tau.$$

Since $g_2(z) = 0$ for $z \leq a/2$, therefore by the virtue of Lemma 9 we then have

$$\left| \int_{x=a/2}^{a} g_2(x-\tau)dx \right| = O\left({p_r}^{(c-1)a/2} \log p_r^{a/2} \right) .$$

Furthermore

$$\left| \int_{x=2}^{a} (g_1 * g_2)(x)dx \right| \leq \int_{\tau=1}^{a/2} |g_1(\tau)| \left| \int_{x=a/2}^{a} g_2(x-\tau)dx \right| d\tau,$$

thus

$$\left| \int_{x=2}^{a} (g_1 * g_2)(x)dx \right| = O\left({p_r}^{-a(1-c)/2} \log p_r^{a/2} \right) \int_{\tau=1}^{a/2} |g_1(\tau)| \, d\tau.$$

By the virtue of Lemma 11, we then have

$$\left| \int_{x=2}^{a} (g_1 * g_2)(x)dx \right| = O\left({p_r}^{-a(1-c)/2} \log p_r^{a/2} \right) O(\log a) \tag{124}$$

Combining Equations (122), (124) and (123), we then have

$$\int_{x=2}^{a} \left(f_\varepsilon * f_\varepsilon \right)(x)dx = O\left({p_r}^{-(1-c)\min(2,a/2)+\epsilon} \right)$$

□

Lemma 33. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 \le c < 1$), then*

$$\int_{y=2}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r)\right)(y)\,dy = O\left({p_r}^{-(1-c)\min(2,a/2)+\epsilon}\right) \tag{125}$$

Proof. Since $\mathcal{L}^{-1}(e^{-E_1(\alpha)})(y) = \rho'(y) + \delta(y)$ and $\mathcal{L}^{-1}\varepsilon(\alpha;p_r)(y) = f_\varepsilon(y)$, therefore

$$\int_{y=1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r)\right)(y)\,dy = \int_{y=1}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy$$

We will first compute the integral $\int_{y=1}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy$ by using the final limit theorem and recalling that if $F(s)$ is the Laplace transform of $f(t)$ then $F(s)/s$ is the Laplace transform of $\int_0^t f(x)dx$. Thus

$$\int_{y=1}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \lim_{\alpha\to 0}\left(e^{-E_1(\alpha)}\varepsilon(\alpha;p_r)\varepsilon(\alpha;p_r)\right),$$

or

$$\int_{y=2}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \left(\int_1^\infty \frac{dJ(p_r^x)}{p_r^x}\right)^2$$

Note that the integral lower limit of y was set to 2 since $f_\varepsilon(y) = 0$ for $y \le 1$. If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$, then by the virtue of Lemma 9 we then have

$$\int_{y=2}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = O\left({p_r}^{-2(1-c)}\log^2 p_r\right). \tag{126}$$

To compute the size of the integral $\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy$, we will first analyze the size of $\int_{y=a}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy$. The size of $\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy$ will then be given by

$$\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{y=2}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y) - \int_{y=a}^{\infty} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y).$$

Toward this end, we first write

$$\left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=2}^{y} \rho^{'}(y-x)(f_\varepsilon * f_\varepsilon)(x)dx + (f_\varepsilon * f_\varepsilon)(y)dy$$

Thus

$$\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{y=2}^{a}\left(\int_{x=2}^{y} \rho^{'}(y-x)(f_\varepsilon * f_\varepsilon)(x)dx + (f_\varepsilon * f_\varepsilon)(y)\right)dy,$$

or

$$\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{y=2}^{a}\left(\int_{x=2}^{y} \rho^{'}(y-x)(f_\varepsilon * f_\varepsilon)(x)dx\right)dy + \int_{y=2}^{a}(f_\varepsilon * f_\varepsilon)(y)dy,$$

Changing the the order of integration and noting that $\rho^{'}(z) = 0$ for $z < 1$, we then have

$$\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=2}^{a}(f_\varepsilon * f_\varepsilon)(x)\left(\int_{y=x+1}^{a} \rho^{'}(y-x)dy\right)dx + \int_{x=2}^{a}(f_\varepsilon * f_\varepsilon)(x)dx,$$

or

$$\int_{y=2}^{a} \left((\rho'+\delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=2}^{a}(f_\varepsilon * f_\varepsilon)(x)\left(1 + \int_{y=x+1}^{a} \rho^{'}(y-x)dy\right)dx.$$

Hence
$$\int_{y=2}^{a} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=2}^{a} \rho(a-x)(f_\varepsilon * f_\varepsilon)(x)dx.$$
Similarly, we can also show that
$$\int_{y=2}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=2}^{\infty} \rho(a-x)(f_\varepsilon * f_\varepsilon)(x)dx.$$
Thus
$$\int_{y=a}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=a}^{\infty} \rho(a-x)(f_\varepsilon * f_\varepsilon)(x)dx.$$
Using integration by parts, we can write the above integral as
$$\int_{y=a}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \rho(0)h_2(a) - \int_{x=a}^{\infty} h_2(x)d\rho(a-x).$$
Since the function $\rho(x)$ is a positive monotone decreasing function where $\rho(x) \leq 1$ and since $h_2(x) = O\left({p_r}^{-(1-c)\min(2,a/2)+\epsilon}\right)$ for $x \geq a$, hence [2]
$$\left|\int_{y=a}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy\right| = O\left({p_r}^{-(1-c)\min(2,a/2)+\epsilon}\right) \tag{127}$$
Finally, we have
$$\int_{y=2}^{a} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{y=2}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy - \int_{y=a}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy,$$
and referring to Equations (126) and (127), we then have
$$\left|\int_{y=2}^{a} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy\right| = O\left({p_r}^{-(1-c)\min(2,a/2)+\epsilon}\right)$$
or
$$\int_{y=2}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r)\right)(y)\,dy = O\left({p_r}^{-(1-c)\min(2,a/2)+\epsilon}\right)$$
□

In the following lemma, we will analyze the second integral of Equation (116) for $k = 3$

Lemma 34. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $a < 4$*
$$\int_{y=1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r)\right)(y)\,dy = O\left({p_r}^{-a(1-c)/3+\epsilon}\right) \tag{128}$$
where, ϵ is an arbitrary small positive number.

[2] Note that $\int_{y=a}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{x=a}^{\infty} \rho(a-x)(f_\varepsilon * f_\varepsilon)(x)dx$. Since $\rho(x) = 0$ for $x < 0$, thus $\rho(a-x) = 0$ for $x > a$ and the integral $\int_{y=a}^{\infty} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon\right)(y)dy$ is equal to zero. However, using this result instead of that given by Equation (127) will not affect our final estimate of $R(1, p_r; 1, {p_r}^a)$.

Proof. Since $\mathcal{L}^{-1}(e^{-E_1(\alpha)})(y) = \rho'(y) + \delta(y)$ and $\mathcal{L}^{-1}\varepsilon(\alpha; p_r)(y) = f_\varepsilon(y)$, therefore

$$\int_{y=1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha; p_r) * \mathcal{L}^{-1}\varepsilon(\alpha; p_r) * \mathcal{L}^{-1}\varepsilon(\alpha; p_r)\right)(y)\, dy = \int_{y=1}^{a} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)$$

and

$$\int_{y=1}^{a} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)dy = \int_{y=1}^{a} \left(\rho' * f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)dy + \int_{y=1}^{a} \left(f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)dy$$

We will first compute the integral $\int_{y=1}^{a} \left(f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)dy$. As it was the case with Lemma 32, we dissect the function $f_\varepsilon(x)$ into three functions $g_1(x)$, $g_2(x)$ and $g_3(x)$ where

$$g_1(x) = f_\varepsilon(x) \quad for \quad 1 \le x \le a/3$$

$$g_1(x) = 0 \qquad otherwise$$

$$g_2(x) = f_\varepsilon(x) \quad for \quad a/3 < x \le 2a/3$$

$$g_2(x) = 0 \qquad otherwise$$

and

$$g_3(x) = f_\varepsilon(x) \quad for \quad 2a/3 < x \le a$$

$$g_3(x) = 0 \qquad otherwise$$

Hence,

$$\int_{x=1}^{a} \left(f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(x)dx = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \int_{=1}^{a} \left(g_i * g_j * g_k\right)(x)dx \tag{129}$$

We will first compute $\int_{x=1}^{a}(g_1 * g_1 * g_1)(x)dx$. Since $(g_1 * g_1)(x) = 0$ for $x > 2a/3$, therefore

$$\int_{x=1}^{a}(g_1 * g_1)(x)dx = \int_{x=1}^{\infty}(g_1 * g_1)(x)dx = \left(\int_{1}^{a/3} \frac{dJ(p_r^x)}{p_r^x}\right)^2.$$

and by the virtue of Lemma 9, we then have

$$\int_{x=1}^{a}(g_1 * g_1)(x)dx = O\left({p_r}^{2(c-1)} \log^2 p_r\right).$$

Furthermore, since $(g_1 * g_1)(x) = 0$ for $x > 2a/3$ and $g_1(x) = 0$ for $x > a/3$, hence

$$\int_{x=1}^{a}(g_1 * (g_1 * g_1))(x)dx = \int_{x=1}^{\infty}(g_1 * g_1 * g_1)(x)dx$$

As it was the case with Lemma 31, by the virtue of Lemma 28 and recalling that convolution in the function domain corresponds to multiplication in the transform domain,

$$\mathcal{L}(g_1 * g_1 * g_1)(x) = \left(\int_{x=1}^{a/3} e^{-\alpha x}\frac{dJ(p_r^x)}{p_r^x}\right)^3.$$

By recalling that if $F(s)$ is the Laplace transform of $f(t)$ then $F(s)/s$ is the Laplace transform of $\int_0^t f(x)dx$, we then have

$$\mathcal{L}\left(\int_{x=1}^{y}(g_1*(g_1*g_1))(x)dx\right)(\alpha)=\frac{1}{\alpha}\left(\int_1^{a/3}e^{-\alpha x}\frac{dJ(p_r^x)}{p_r^x}\right)^3.$$

Using the final value theorem (which states that $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$), we then have

$$\int_{x=1}^{a}(g_1*(g_1*g_1))(x)dx=\int_{x=1}^{\infty}(g_1*(g_1*g_1))(x)dx=\left(\int_{x=1}^{a/3}\frac{dJ(p_r^x)}{p_r^x}\right)^3.$$

and by the virtue of Lemma 9, we then have

$$\int_{x=1}^{a}(g_1*g_1*g_1)(x)dx=O\left({p_r}^{3(c-1)}\log^3 p_r\right) \tag{130}$$

It should be noted that the integral $\int_{x=1}^{a}(g_1*g_1*g_1)(x)dx$ is equal to zero for $a<3$. This follows from the fact that $g_1(y)=0$ for $y<1$.

To compute the remaining terms of Equation (129), we first need to compute the convolution $(g_i*g_j)(y)$ for $y\le a$, where i = 1,2 or 3 and j = 1,2 or 3. Let $g_{ij}(y)$ be defined for $y\le a$ as

$$g_{ij}(y)=(g_i*g_j)(y)=\int_{\tau=1}^{a}g_i(\tau)g_j(y-\tau)d\tau$$

where $i=1,2$ or 3 and $j=1,2$ or 3. Note that for $y\le a$, both $g_{23}(y)$ and $g_{33}(y)$ are equal to zero. Our task is then to compute the integral $\int_{x=1}^{a}(g_{11}*g_k)(x)dx$ for $k=2$ and 3 (note that that we have computed earlier this integral for $k=1$) and the integrals $\int_{x=1}^{a}(g_{12}*g_2)(x)dx$ and $\int_{x=1}^{a}(g_{22}*g_2)(x)dx$ (note that $\int_{x=1}^{a}(g_{12}*g_1)(x)dx=\int_{x=1}^{a}(g_{11}*g_2)(x)dx$ and $\int_{x=1}^{a}(g_{12}*g_3)(x)dx=0$ since $g_{23}(y)=0$ for $y\le a$. Furthermore, $\int_{x=1}^{a}(g_{13}*g_1)(x)dx=\int_{x=1}^{a}(g_{11}*g_3)(x)dx$, while $\int_{x=1}^{a}(g_{13}*g_2)(x)dx=\int_{x=1}^{a}(g_{23}*g_1)(x)dx=0$ and $\int_{x=1}^{a}(g_{13}*g_3)(x)dx=\int_{x=1}^{a}(g_{33}*g_1)(x)dx=0$. Moreover, $\int_{x=1}^{a}(g_{22}*g_1)(x)dx=\int_{x=1}^{a}(g_{12}*g_2)(x)dx$ and $\int_{x=1}^{a}(g_{22}*g_3)(x)dx=\int_{x=1}^{a}(g_{23}*g_2)(x)dx=0$).

We will first compute $g_{11}(y)$,

$$g_{11}(y)=(g_1*g_1)(y)=\int_{\tau=1}^{a/3}g_1(\tau)g_1(y-\tau)d\tau$$

note that since $g_1(x)=0$ for $x<1$ and $x>a/3$, hence the product $g_1(\tau)g_1(y-\tau)=0$ for $y<2$ and $y>2a/3$. For $2\le y\le a/3+1$, $g_{11}(y)$ is given by [3]

$$g_{11}(y)=\int_{\tau=1}^{y-1}g_1(\tau)g_1(y-\tau)d\tau$$

[3]The convolution $g_{11}(y)$ can be depicted as the integral within the overlap of two windows. This overlap falls within the interval $[1,a/3]$. The first window is fixed with a starting point at $\tau=1$ and ending point at $\tau=a/3$. The second window is a sliding window with a leading edge at $y-1$and a lagging edge at $y-a/3$. Initially, for values of y less than 2, there is no overlap between the two windows. For values of y between 2 and $1+a/3$, we will have an overlap between the starting point of the fixed window (i.e $\tau=1$) and the leading edge of the sliding window (i.e $y-1$). At $y=1+a/3$, the overlap covers the entire fixed window. For values of y between $1+a/3$ and $2a/3$, we will have an overlap between the lagging edge of the sliding window (i.e $y-a/3$) and the ending point of the fixed window (i.e $\tau=a/3$).

and by the virtue of Equation (30), we then have

$$g_{11}(y) = \int_{\tau=1}^{y-1} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} - \frac{1}{\tau} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} - \frac{1}{y-\tau} \right) d\tau$$

For $a/3+1 \leq y \leq 2a/3$, $g_{11}(y)$ is given by

$$g_{11}(y) = \int_{\tau=y-a/3}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} - \frac{1}{\tau} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} - \frac{1}{y-\tau} \right) d\tau$$

Hence, for $2 \leq y \leq a/3+1$, $g_{11}(y)$, we have

$$|g_{11}(y)| \leq \int_{\tau=1}^{y-1} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} + \frac{1}{\tau} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} + \frac{1}{y-\tau} \right) d\tau \tag{131}$$

and for $a/3+1 \leq y \leq 2a/3$, $g_{11}(y)$, we have

$$|g_{11}(y)| \leq \int_{\tau=y-a/3}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} + \frac{1}{\tau} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} + \frac{1}{y-\tau} \right) d\tau \tag{132}$$

For the above two integrals (Equations (131) and (132)), $1 \leq \tau \leq a/3$ and $1 \leq y-\tau \leq a/3$, hence, for $a \geq 1$, $1/\tau$ and $1/(y-\tau)$ are both less than or equal to three. Thus

$$\int_{y=1}^{a} |g_{11}(y)| \leq 9 \int_{y=1}^{a} \int_{\tau=1}^{a/3} d\tau dy + 6 \int_{y=1}^{a} \int_{\tau=1}^{a/3} \sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} d\tau dy +$$

$$\int\limits_{y=1}^{a} \int\limits_{\substack{\tau=1 \\ 1\leq y-\tau \leq a/3}}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} \right) d\tau \, dy$$

by changing the order of integration for the third double integral on the right side of the above equation, we then have

$$\int\limits_{y=1}^{a} \int\limits_{\substack{\tau=1 \\ 1\leq y-\tau \leq a/3}}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} \right) d\tau dy \leq$$

$$\int\limits_{\tau=1}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} \right) \left(\int\limits_{\substack{y=1 \\ 1\leq y-\tau \leq a/3}}^{a} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} \right) dy \right) d\tau$$

By the virtue of the Mertens' theorem where $\sum_{p_k=p_r}^{p_r^a} 1/p_k = \log a + O(1/\log p_r)$, we then have

$$\int\limits_{y=1}^{a} \int\limits_{\substack{\tau=1 \\ 1\leq y-\tau \leq a/3}}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} \right) d\tau dy \leq (\log(a/3) + O(1/\log p_r)) \int\limits_{\tau=1}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} \right) d\tau$$

or

$$\int\limits_{y=1}^{a} \int\limits_{\substack{\tau=1 \\ 1\leq y-\tau \leq a/3}}^{a/3} \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{\tau} - p_i)}{p_r^{\tau}} \right) \left(\sum_{i=1}^{\infty} \frac{\delta({p_r}^{y-\tau} - p_i)}{p_r^{y-\tau}} \right) d\tau dy \leq (\log(a/3) + O(1/\log p_r))^2$$

Consequently,

$$\int_{y=1}^{a} |g_{11}(y)| \leq 3a^2 + 6\int_{y=1}^{a}\left(\sum_{p_i=p_r}^{p_r^{a/3}} \frac{1}{p_i}\right) dy + \left(\log(a/3) + O(1/\log p_r)\right)^2 . \tag{133}$$

or

$$\int_{y=1}^{a} |g_{11}(y)| \leq 9a^2 \tag{134}$$

Following the same steps to show that $\int_{y=1}^{a}|g_{11}(y)|dy \leq a^2$, we can also show that $\int_{y=1}^{a}|g_{12}(y)|dy$, $\int_{y=1}^{a}|g_{22}(y)|dy$ and $\int_{y=1}^{a}|g_{13}(y)|dy$ are less than or equal to $9a^2$.

To compute the integral $\int_{x=1}^{a}(g_2 * g_1 * g_1)(x)dx$, we first write it as follows

$$\int_{x=1}^{a}(g_2 * g_1 * g_1)(x)dx = \int_{x=1}^{a}(g_{11} * g_2)(x)dx = \int_{x=1}^{a}\int_{\tau=1}^{a} g_{11}(\tau)g_2(x-\tau)d\tau dx$$

or

$$\int_{x=1}^{a}(g_{11} * g_2)(x)dx = \int_{\tau=1}^{a} g_{11}(\tau)\left(\int_{x=1}^{a} g_2(x-\tau)dx\right) d\tau$$

Since $g_2(y) = \frac{dJ(p_r^y)/p_r^y}{dy}$ for $a/3 < y \leq 2a/3$ and $g_2(y) = 0$ for $y < a/3$, therefore by the virtue of Lemma 10 we then have

$$\left|\int_{x=1}^{a} g_2(x-\tau)dx\right| = O\left({p_r}^{-(1-c)a/3+\epsilon}\right)$$

Since $\int_{y=1}^{a}|g_{11}(y)| \leq a^2$ and

$$\left|\int_{x=1}^{a}(g_{11} * g_2)(x)dx\right| \leq \int_{\tau=1}^{a}|g_{11}(\tau)|\left|\int_{x=1}^{a} g_2(x-\tau)dx\right| d\tau,$$

therefore for a fixed value of a, we then have

$$\left|\int_{x=1}^{a}(g_{11} * g_2)(x)dx\right| = O\left({p_r}^{-a(1-c)/3+\epsilon}\right) \tag{135}$$

Similarly, the integral $\int_{x=1}^{a}(g_3 * g_1 * g_1)(x)dx$ can be written as

$$\int_{x=1}^{a}(g_{11} * g_3)(x)dx = \int_{\tau=1}^{a} g_{11}(\tau)\left(\int_{x=1}^{a} g_3(x-\tau)dx\right) d\tau$$

Since $g_3(y) = \frac{dJ(p_r^y)/p_r^y}{dy}$ for $2a/3 < y \leq a$ and $g_3(y) = 0$ for $y < 2a/3$, then by the virtue of Lemma 10, we then have

$$\left|\int_{x=1}^{a} g_3(x-\tau)dx\right| = O\left({p_r}^{-(1-c)2a/3+\epsilon}\right)$$

Thus,

$$\left|\int_{x=1}^{a}(g_{11} * g_3)(x)dx\right| \leq \int_{\tau=1}^{a}|g_{11}(\tau)|\left|\int_{x=1}^{a} g_3(x-\tau)dx\right| d\tau = O\left({p_r}^{-2a(1-c)/3+\epsilon}\right) \tag{136}$$

Following the same steps to derive Equations (135) and (136), we can also show that

$$\left|\int_{x=1}^{a}(g_{12} * g_2)(x)dx\right| \ , \ \left|\int_{x=1}^{a}(g_{22} * g_2)(x)dx\right| = O\left({p_r}^{-a(1-c)/3+\epsilon}\right) \tag{137}$$

Combining Equations (129), (130), (135), (136), (137) and noting that $g_{23}(y)$ and $g_{33}(y)$ are both equal to zero, we then have

$$\int_{x=1}^{a} \left(f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(x)dx = O\left({p_r}^{-a(1-c)/3+\epsilon}\right) \tag{138}$$

Furthermore, since $f_\varepsilon(x)$ and $\rho'(x)$ are zero for $x < 1$, hence for $a < 4$ we have

$$\int_{y=1}^{a} \left(\rho' * f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)dx = 0 \tag{139}$$

Combining Equations (138) and (139), we then have

$$\int_{y=1}^{a} \left((\rho' + \delta) * f_\varepsilon * f_\varepsilon * f_\varepsilon\right)(y)dx = O\left({p_r}^{-a(1-c)/3+\epsilon}\right)$$

or

$$\int_{y=1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r) * \mathcal{L}^{-1}\varepsilon(\alpha;p_r)\right)(y)\,dy = O\left({p_r}^{-a(1-c)/3+\epsilon}\right)$$

□

The next lemma deals with the remaining terms of the second integral of Equation (116) (i.e terms with $k \geq 4$).

Lemma 35. *For $a < 4$ and $k \geq 4$,*

$$\int_{1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\varepsilon^k(\alpha;p_r)\right)\right)(y)\,dy = 0$$

Proof. Since $\mathcal{L}^{-1}(e^{-E_1(\alpha)})(y) = \rho'(y) + \delta(y)$ and $\mathcal{L}^{-1}(\varepsilon(\alpha;p_r))(y) = f_\varepsilon(y)$, therefore

$$\int_{1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\varepsilon^k(\alpha;p_r)\right)\right)(y)\,dy = \int_{y=1}^{a} \left((\rho' + \delta) * f_{\varepsilon 1} * f_{\varepsilon 2} * \ldots * f_{\varepsilon k}\right)(y)dy$$

where $f_\varepsilon(x) = f_{\varepsilon 1}(x) = f_{\varepsilon 2}(x) = \ldots = f_{\varepsilon k}(x)$. Furthermore, since $f_\varepsilon(x) = 0$ for $x < 1$ and $\rho'(x) = 0$ for $x < 1$, hence for $a < 4$ and $k \geq 4$,

$$\int_{1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\varepsilon^k(\alpha;p_r)\right)\right)(y)\,dy = 0$$

□

Combining Lemmas 33, 34 and 35, we can then have the following lemma for the second integral of Equation (116)

Lemma 36. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $a < 4$*

$$\int_{1}^{a} \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right)\right)\,dy = O\left({p_r}^{-a(1-c)/3+\epsilon}\right)$$

where, ϵ is an arbitrary small positive number.

The following two lemmas deals with the third and fourth integrals of Equation (116)

Lemma 37. *Unconditionally, we have*

$$\int_1^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty} \frac{1}{m!}\delta^m(\alpha;p_r)\right)\right) dy = O\left({p_r}^{-1}\right)$$

Proof. Referring to Equation (114), we have

$$\mathcal{L}^{-1}\delta(\alpha;p_r)(y) = -\sum_{i=r}^{\infty}\left(\frac{\delta(y-2)}{2{p_i}^2} + \frac{\delta(y-3)}{3{p_i}^3} + \frac{\delta(y-4)}{4{p_i}^4}...\right).$$

Thus,

$$\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y) = -\sum_{i=r}^{\infty}\left(\frac{\rho'(y-2)}{2{p_i}^2} + \frac{\rho'(y-3)}{3{p_i}^3} + \frac{\rho'(y-4)}{4{p_i}^4}...\right) -$$

$$\sum_{i=r}^{\infty}\left(\frac{\delta(y-2)}{2{p_i}^2} + \frac{\delta(y-3)}{3{p_i}^3} + \frac{\delta(y-4)}{4{p_i}^4}...\right)$$

where $\mathcal{L}^{-1}(e^{-E_1(\alpha)})(y) = \rho'(y) + \delta(y)$ and the convolution of two Dirac delta functions is also a Dirac delta function. More specifically, the convolution $\delta(x-a) * \delta(x-b)$ is given by $\delta(x-a-b)$. Since $|\int_1^\infty \rho'(x)dx| = 1$ and $\int_{-\infty}^\infty \delta(x)dx = 1$, thus

$$\left|\int_{y=1}^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| \le 2\sum_{i=r}^{\infty}\left(\frac{1}{2{p_i}^2} + \frac{1}{3{p_i}^3} + \frac{1}{4{p_i}^4}...\right)$$

Since $\sum_{i=r}^\infty \frac{1}{{p_i}^2} < 1/p_{r-1}$ and $\sum_{i=r}^\infty \frac{1}{{p_i}^{n+1}} < 1/(p_{r-1})^{n-1}$, therefore

$$\left|\int_{y=1}^a \left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| < \frac{2}{p_r}$$

Let $m_1(y) = (\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\delta(\alpha;p_r))(y)$. Let $m_2(y) = (\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\delta(\alpha;p_r) * \mathcal{L}^{-1}\delta(\alpha;p_r))(y)$ and so on. Thus

$$\left|\int_{y=1}^a m_1(y)dy\right| < \frac{2}{p_r}$$

Furthermore,

$$\int_{y=1}^a m_2(y)dy = \int_{y=1}^a \left(m_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy = \int_{y=1}^a\int_{\tau=1}^\infty m_1(y-\tau)\mathcal{L}^{-1}(\delta(\alpha;p_r))(\tau)d\tau\, dy$$

thus

$$\left|\int_{y=1}^a \left(m_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| \le \int_{\tau=1}^\infty \left|\mathcal{L}^{-1}(\delta(\alpha;p_r))(\tau)\right| \left|\int_{y=1}^a m_1(y-\tau)dy\right| d\tau$$

or

$$\left|\int_{y=1}^a \left(m_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| < \frac{2}{p_r}\int_{\tau=1}^\infty \left|\mathcal{L}^{-1}(\delta(\alpha;p_r))(\tau)\right| d\tau$$

Since

$$\int_{\tau=1}^\infty \left|\mathcal{L}^{-1}\delta(\alpha;p_r)(\tau)\right| d\tau \le \int_{y=1}^\infty \left(\sum_{i=r}^{\infty}\left(\frac{\delta(y-2)}{2{p_i}^2} + \frac{\delta(y-3)}{3{p_i}^3} + \frac{\delta(y-4)}{4{p_i}^4}...\right)\right) dy < \frac{1}{p_r},$$

thus

$$\left|\int_{y=1}^{a}\left(m_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| < \frac{2}{p_r^2},$$

or

$$\left|\int_{y=1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\delta(\alpha;p_r) * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| < \frac{2}{p_r^2}.$$

Repeating these steps $m-1$ times (i.e. the steps to derive $\int_{y=1}^{a} m_2(y)dy$), we then have

$$\int_{y=1}^{a} m_m(y)dy = \int_{y=1}^{a}\left(m_{m-1} * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy < \frac{2}{p_r^m}.$$

Consequently

$$\int_{1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\delta^m(\alpha;p_r)\right)\right) dy = O\left({p_r}^{-1}\right).$$

□

Lemma 38. *For a fixed a, we have unconditionally*

$$\int_{1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\delta^m(\alpha;p_r)\right)\right) dy = O\left({p_r}^{-1+\epsilon}\right).$$

where, ϵ is an arbitrary small positive number.

Proof. The proof of this lemma follows similar steps to those presented in the proof of Lemma 37. Details of this proof are presented in Appendix 2. □

Combining Lemmas 29, 30, 36, 37 and 38, we then have the following third key theorem

Theorem 8. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $a < 3$*

$$R(1,p_r;1,{p_r}^a) = -\int_{x=1}^{a}\rho(a-x)\frac{dJ(p_r^x)}{p_r^x} + O\left({p_r}^{-a(1-c)/2+\epsilon}\right), \tag{140}$$

and for $a < 4$

$$R(1,p_r;1,{p_r}^a) = -\int_{x=1}^{a}\rho(a-x)\frac{dJ(p_r^x)}{p_r^x} + O\left({p_r}^{-a(1-c)/3+\epsilon}\right). \tag{141}$$

where, ϵ is an arbitrary small positive number.

Equations (140) and (141) (of Theorem 8) and Equation (87) (of Theorem 4) provide two different representations for the term $R(1,p_r;1,{p_r}^a)$. Our analysis to examine the validity of the Riemann Hypothesis (and in general, the location of the non-trivial zeros) will be based on analyzing the difference between these two representations. Before we proceed with this task, we will first analyze more properties of the integral $\int_{y=1}^{\infty}(dJ(p_r^y)/p_r^y)$. If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then by the virtue of Lemma 10 we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = O\left({p_r}^{-(1-c)} \log p_r\right).$$

Furthermore, referring to Appendix 3, if one or more of the Riemann zeta function zeros are located on the line $\Re(s) = c$, then for any $z \geq 1$, there are infinitely many prime numbers p's such that

$$\left|\int_{y=z}^{\infty} \frac{dJ(p^y)}{p^y}\right| = \Omega\left(p^{(-(1-c)-\epsilon)z}\right).$$

where ϵ can be made arbitrary small by choosing p sufficiently large. Therefore, for sufficiently large N and for some constant k, there are infinitely many prime numbers p_r's (that are greater than N) such that

$$\left|\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-(1-c)-\epsilon} > 0.$$

Moreover, for any positive number h, we also have

$$\int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = O\left((1+h){p_r}^{-h(1-c)}{p_r}^{-(1-c)} \log p_r\right) = O\left({p_r}^{-h(1-c)}{p_r}^{-(1-c)} \log p_r\right).$$

Thus,

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y} + O\left({p_r}^{-h(1-c)}{p_r}^{-(1-c)} \log p_r\right).$$

Therefore, for any arbitrary small h, we can always find infinitely many p_r's so that the integral $\int_{y=1}^{\infty}(dJ(p_r^y)/p_r^y)$ is determined by values of y in the vicinity of one. In other words; we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y} + \int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y}.$$

where,

$$\left|\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-(1-c)-\epsilon} > 0,$$

and

$$\left|\int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| < k_1{p_r}^{-h(1-c)}{p_r}^{-(1-c)} \log p_r,$$

for some constant k_1. Therefore, for any h and for sufficiently large p_r, there are infinitely many p_r satisfying the following equation

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = (1+\delta_1) \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y}, \tag{142}$$

where δ_1 is given by $O({p_r}^{-h(1-c)})$ and it can be made arbitrary close to zero by choosing p_r

sufficiently large.

It should be noted that the above analysis for the integral $\int_{y=1}^{\infty}(dJ(p_r^y)/p_r^y)$ can be extended to the integral $\int_{y=1}^{\infty}(g(y)dJ(p_r^y)/p_r^y)$ where $g(y)$ is a differentiable function for $y \geq 1$ with the function $g(x)$ and its derivative $g'(x)$ are non-vanishing in the vicinity of $y = 1$ and both the function and its derivative don't grow faster than $e^{\delta y}$ or decay faster than $e^{-\delta y}$ for any $\delta > 0$ (for example, $g(y)$ or $-g(y)$ is given by $1, y, y^2, ..., y^n, 1/y, 1/y^2, .., 1/y^n, (\log y)^n$). For the integral $\int_{y=1}^{\infty}(g(y)dJ(p_r^y)/p_r^y)$, we then have

$$\left|\int_{y=1}^{\infty} g(y)\frac{dJ(p_r^y)}{p_r^y}\right| = O\left({p_r}^{-(1-c)+\epsilon}\right),$$

and if one or more of the Riemann zeta function zeros are located on the line $\Re(s) = c$, then for any $z \geq 1$, there are infinitely many p's (refer to Appendix 3),

$$\left|\int_{y=z}^{\infty} g(y)\frac{dJ(p^y)}{p^y}\right| = \Omega\left((p^{(-(1-c)-\epsilon)z}\right),$$

where ϵ can be made arbitrary small by choosing p sufficiently large. Therefore, for sufficiently large N and for some constant k, there are infinitely many prime numbers p_r's (that are greater than N) such that

$$\left|\int_{y=1}^{\infty} g(y)\frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-(1-c)-\epsilon} > 0.$$

After analyzing the integral $\int_{y=1}^{\infty}(dJ(p_r^y)/p_r^y)$, we now turn our attention in the next section to the analysis of two representations of the term $R(1, p_r; 1, {p_r}^a)$ in Theorems 4 and 8.

7 The two representations of $R(1, p_r; 1, {p_r}^a)$ and the location of $\zeta(s)$ non-trivial zeros.

The first representation of $R(1, p_r; 1, {p_r}^a)$ is based on Equation (87) of Theorem where we have unconditionally

$$R(1, p_r; 1, {p_r}^a) = -\int_1^a \rho\left(a/y - 1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1, {p_r}^y; 1, p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} + O(p_r^{-1})$$

The second representation of the term $R(1, p_r; 1, {p_r}^a)$ for $a < 4$ is based on Equation (141) of Theorem 8

$$R(1, p_r; 1, {p_r}^a) = -\int_{x=1}^{a} \rho(a-x)\frac{dJ(p_r^x)}{p_r^x} + O\left({p_r}^{-a(1-c)/3+\epsilon}\right).$$

Consequently, we have the following theorem

Theorem 9. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $a < 4$*

$$\left(-\int_1^a \rho\left(a/y - 1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1, {p_r}^y; 1, p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}\right) + \int_{x=1}^{a} \rho(a-x)\frac{dJ(p_r^x)}{p_r^x} = O\left({p_r}^{-a(1-c)/3+\epsilon}\right). \tag{143}$$

where ϵ can be made arbitrary small positive number by choosing p_r sufficiently large.

For the remaining of the paper, we will analyze Equation (143) to examine which part of the critical strip is void of not-trivial zeros (in other words; use Equation (143) to determine the value of c). The difference $\int_{x=1}^{a}(\rho(a-x)dJ(p_r^x)/p_r^x) - \int_1^a(\rho(a/y-1)dJ(p_r{}^y)/p_r^y)$ in Equation (143) can be written as $\int_{x=1}^{a} g(x)dJ(p_r^x)/p_r^x)$ where $g(x)$ a differentiable function for $y \geq 1$ and both $g(x)$ and $g'(x)$ are non-vanishing functions in the vicinity of $y = 1$ that don't grow faster than $e^{\delta y}$ or decay faster than $e^{-\delta y}$ for any $\delta > 0$ (it should be pointed here that the function $\rho(y)$ has the asymptotic representation of y^{-y}. However for $a < 4$, both $\rho(a-y)$ and $\rho(a/y-y)$ are represented by logarithmic functions as shown in Lemma 39). Thus referring to Appendix 3, for any $z \geq 1$, there are infinitely many p's, we have

$$\left|\int_{y=z}^{\infty} (\rho(a-y) - \rho(a/y-1)) \frac{dJ(p_r^y)}{p^y}\right| = \Omega\left(p^{(-(1-c)-\epsilon)z}\right),$$

Hence, there are infinitely many prime numbers p_r's such that

$$\left|\int_{x=1}^{a} \rho(a-x)\frac{dJ(p_r^y)}{p_r^y} - \int_1^a \rho(a/y-1)\frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-(1-c)-\epsilon}$$

for some positive constant k. Therefore, If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$, then the the integral $I_R = \int_1^{a/2}(R(1, p_r{}^y; 1, p_r^{a-y})d\pi(p_r^y)/p_r^y)$ has to equal the sum $S_J = \int_{x=1}^{a} \rho(a-x)(dJ(p_r^x)/p_r^x) - \int_1^a(\rho(a/y-1)dJ(p_r{}^y)/p_r^y)$ within a margin of $O\left({p_r}^{-a(1-c)/3+\epsilon}\right)$. Our task will then be focused on computing the integral $I_R = \int_1^{a/2}(R(1, p_r{}^y; 1, p_r^{a-y})d\pi(p_r^y)/p_r^y)$ at different values of a and comparing the result with the sum $S_J = \int_{x=1}^{a}(\rho(a-x)dJ(p_r^x)/p_r^x) - \int_1^a(\rho(a/y-1)dJ(p_r{}^y)/p_r^y)$

In the following, we will compute the integral $I_R = \int_1^{a/2}(R(1, p_r{}^y; 1, p_r^{a-y})d\pi(p_r^y)/p_r^y)$ and the sum $S_J = \int_{x=1}^{a}(\rho(a-x)dJ(p_r^x)/p_r^x) - \int_1^a(\rho(a/y-1)dJ(p_r{}^y)/p_r^y)$ for values of a in the range $3 < a < 4$.

Lemma 39. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $3 < a < 4$, the sum $S_J = \int_{y=1}^{a}(\rho(a-y)dJ(p_r^y)/p_r^y) - \int_1^a(\rho(a/y-1)dJ(p_r{}^y)/p_r^y)$ is given by*

$$S_J = -\int_1^{a/2} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_1^{a/3}\left(\int_{(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y} +$$
$$\int_{y=a/3}^{a-2}\left(\int_{v=2}^{a-y} \log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y} - \int_{a/2}^{a-1} \log(a-y)\frac{dJ(p_r^y)}{p_r^y}$$

Proof. For the interval $3 < a \leq 4$, the representation of the functions $\rho(a-y)$ and $\rho((a-y)/y)$ is dependent on the value of y. For values of y in the range $1 \leq y \leq a/3$, we have [7]

$$\rho(a-y) = 1 - \log(a-y) + \int_2^{a-y} \log(v-1)\frac{dv}{v},$$

and

$$\rho\left(\frac{1}{y}(a-y)\right) = 1 - \log(a-y) + \log y + \int_2^{(a-y)/y} \log(v-1)\frac{dv}{v},$$

Thus, for values of y in the range $1 \le y \le a/3$, we have

$$-\int_1^{a/3} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} + \int_1^{a/3} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} =$$

$$-\int_1^{a/3} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_{y=1}^{a/3} \left(\int_{v=(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y}. \quad (144)$$

For values of y in the range $a/3 \le y \le a-2$, we have

$$\rho(a-y) = 1 - \log(a-y) + \int_2^{a-y} \log(v-1)\frac{dv}{v},$$

and

$$\rho\left(\frac{1}{y}(a-y)\right) = 1 + \log y - \log(a-y).$$

Thus, for values of y in the range $a/3 \le y \le a-2$, we have

$$-\int_{a/3}^{a-2} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} + \int_{a/3}^{a-2} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} =$$

$$-\int_{a/3}^{a-2} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_{a/3}^{a-2} \left(\int_2^{a-y} \log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y}. \quad (145)$$

Similarly, for values of y in the range $a-2 \le y \le a/2$, we have

$$-\int_{a-2}^{a/2} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} + \int_{a-2}^{a/2} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = -\int_{a-2}^{a/2} \log y \frac{dJ(p_r^y)}{p_r^y}. \quad (146)$$

For values of y in the range $a/2 \le y \le a-1$, we have

$$\int_{a/2}^{a-1} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = \int_{a/2}^{a-1} (1-\log(a-y))\frac{dJ(p_r^y)}{p_r^y}.$$

while for values of y in the range $a-1 \le y \le a$, we have

$$\int_{a-1}^{a} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = \int_{a-1}^{a} \frac{dJ(p_r^y)}{p_r^y}.$$

Thus, for values of y in the range $a/2 \le y \le a$, we have

$$\int_{a/2}^{a} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = -\int_{a/2}^{a-1} \log(a-y)\frac{dJ(p_r^y)}{p_r^y} + \int_{a/2}^{a} \frac{dJ(p_r^y)}{p_r^y}. \quad (147)$$

For values of y in the range $a/2 \le y \le a$, we also have

$$\int_{a/2}^{a} \rho(a/y-1)\frac{dJ(p_r^y)}{p_r^y} = \int_{a/2}^{a} \frac{dJ(p_r^y)}{p_r^y}. \quad (148)$$

Combining Equations (144), (145), (146), (147) and (149), we then get desired result. □

The next lemma deals with the term I_R

Lemma 40. *If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$), then for $3 < a < 4$, the integral $I_R = \int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) d\pi(p^y)/p^y$ is given by*

$$\int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2} \rho\left(\frac{a}{y}-2\right) \log y \frac{dJ(p_r^y)}{p_r^y} - \int_{a/2}^{a-1} \log(a-y) \frac{dJ(p_r^y)}{p_r^y} +$$

$$\int_{z=1}^{a/2} \left(\int_{y=1}^{z} \log y \ \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} dy \right) \frac{dJ(p_r^z)}{p_r^z} +$$

$$\int_{z=a/2}^{a-1} \left(\int_{y=1}^{a-z} \log y \ \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} dy \right) \frac{dJ(p_r^z)}{p_r^z} + O({p_r}^{-a(1-c)/3+\epsilon}). \qquad (149)$$

where $\rho'\left(\frac{a-z}{y}-1\right) = \frac{d}{dy}\rho\left(\frac{a-z}{y}-1\right)$ and ϵ can be made arbitrary small positive number by choosing p_r sufficiently large.

Proof. To compute the integral $I_R = \int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) d\pi(p^y)/p^y$, we first note that for $a < 4$, the value of $a-y$ is less than 3. Referring to Equation (141) of Theorem 8 or Equation (101) of Theorem 5, we have for $b < 3$

$$R(1, p; 1, p^b) = -\int_1^b \rho(b-x) \frac{dJ(p^x)}{p^x} + O(p^{-b(c-1)/2+\epsilon}).$$

Therefore, for $a < 4$ and $1 \leq y \leq a/2$, we have

$$R(1, {p_r}^y; 1, p_r^{a-y}) = R(1, {p_r}^y; 1, (p_r^y)^{\frac{a-y}{y}}) = -\int_{x=1}^{\frac{a-y}{y}} \rho\left(\frac{a-y}{y}-x\right) \frac{dJ((p_r^y)^x)}{(p_r^y)^x} + H.$$

where by the virtue of Lemma 22,

$$H = 0 \qquad when \quad 1 \leq \frac{a-y}{y} < 2$$

and by the virtue of Theorem 5 or theorem 8

$$H = O((p_r^y)^{-\frac{a-y}{y}(c-1)/2+\epsilon}). \qquad when \quad 2 \leq \frac{a-y}{y} < 3$$

However, the inequality $2 \leq \frac{a-y}{y} < 3$ is equivalent to the following inequality

$$\frac{a}{4} < y \leq \frac{a}{3}$$

or

$$\frac{2a}{3} \leq a-y < \frac{3a}{4}$$

and

$$\frac{a}{3y}(c-1) \leq \frac{a-y}{y}(c-1)/2 < \frac{3a}{8y}(c-1)$$

Consequently,

$$H = O({p_r}^{-a(c-1)/3+\epsilon}). \qquad when \quad 2 \leq \frac{a-y}{y} < 3$$

and for $a < 4$ and $1 \leq y \leq a/2$, we have

$$R(1, {p_r}^y; 1, p_r^{a-y}) = -\int_{x=1}^{\frac{a-y}{y}} \rho\left(\frac{a-y}{y} - x\right) \frac{dJ((p_r^y)^x)}{(p_r^y)^x} + O({p_r}^{-a(c-1)/3+\epsilon}).$$

Defining $z = yx$, we then have

$$R(1, {p_r}^y; 1, p_r^{a-y}) = -\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z} + O({p_r}^{-a(c-1)/3+\epsilon}),$$

and

$$\int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} = -\int_{y=1}^{a/2} \left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z} + O({p_r}^{-a(1-c)/3+\epsilon})\right) \frac{d\pi(p_r^y)}{p_r^y}.$$

Since $d\pi(p_r^y) \geq 0$ and p_r^y is a monotone increasing and strictly positive function of y, therefore

$$\int_1^{a/2} O({p_r}^{-a(1-c)/3+\epsilon}) \frac{d\pi(p_r^y)}{p_r^y} = O({p_r}^{-a(c-1)/3+\epsilon}) \int_1^{a/2} \frac{d\pi(p_r^y)}{p_r^y}$$

and by the virtue of Lemma 11, we then have

$$\int_1^{a/2} O({p_r}^{-a(1-c)/3+\epsilon}) \frac{d\pi(p_r^y)}{p_r^y} = O({p_r}^{-a(1-c)/3+\epsilon}).$$

Therefore

$$\int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2} \left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z}\right) \frac{d\pi(p_r^y)}{p_r^y} + O({p_r}^{-a(1-c)/3+\epsilon}).$$

By noting that $d\pi(p_r^y)/p_r^y = d\log y + dJ(p_r^z)/p_r^z$ and referring to Appendix 4 (where we showed that $\int_1^{a/2} \left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z}\right) \frac{dJ(p_r^y)}{p_r^y} = O({p_r}^{-a(1-c)/2+\epsilon})$), we then have

$$\int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2} \left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z}\right) d\log y + O({p_r}^{-a(1-c)/3+\epsilon}).$$

Using the method of integration by parts, we then have

$$\int_1^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} = \int_1^{a/2} \log y \; d\left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z}\right) + O({p_r}^{-a(1-c)/3+\epsilon}).$$

The change in the integral $\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z}$ due to the change in y by Δy is given by

$$\Delta\left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right) \frac{dJ(p_r^z)}{p_r^z}\right) =$$

$$\int_{z=y+\Delta y}^{a-y-\Delta y} \rho\left(\frac{a-z}{y+\Delta y} - 1\right) \frac{dJ({p_r}^z)}{{p_r}^z} - \int_{z=y}^{a-y} \rho\left(\frac{a-z}{y} - 1\right) \frac{dJ(p_r^z)}{p_r^z},$$

or

$$\Delta\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)=-\int_{z=y}^{y+\Delta y}\rho\left(\frac{a-z}{y}-1\right)\frac{dJ(p_r^z)}{p_r^z}-$$

$$\int_{z=a-y-\Delta y}^{a-y}\rho\left(\frac{a-z}{y}-1\right)\frac{dJ({p_r}^{a-z})}{{p_r}^{a-z}}+\int_{z=y}^{a-y}\left(\rho\left(\frac{a-z}{y+\Delta y}-1\right)-\rho\left(\frac{a-z}{y}-1\right)\right)\frac{dJ(p_r^z)}{p_r^z},$$

where

$$\rho\left(\frac{a-z}{y+\Delta y}-1\right)-\rho\left(\frac{a-z}{y}-1\right)=\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\Delta y.$$

Consequently

$$d\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)=-\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}-\rho(0)\frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}+$$

$$dy\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}, \tag{150}$$

and

$$\int_1^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}=-\int_1^{a/2}\rho\left(\frac{a}{y}-2\right)\log y\frac{dJ(p_r^y)}{p_r^y}-\int_1^{a/2}\log z\frac{dJ({p_r}^{a-z})}{{p_r}^{a-z}}+$$

$$\int_1^{a/2}\log y\left(\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}\right)dy+O({p_r}^{-a(1-c)/3+\epsilon}),$$

or

$$\int_1^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}=-\int_1^{a/2}\rho\left(\frac{a}{y}-2\right)\log y\frac{dJ(p_r^y)}{p_r^y}-\int_{a/2}^{a-1}\log(a-y)\frac{dJ(p_r^y)}{p_r^y}+$$

$$\int_1^{a/2}\log y\left(\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}\right)dy+O({p_r}^{-a(1-c)/3+\epsilon}).$$

For the third integral on the right side of above equation, we rearrange the double integral as follows

$$\int_{y=1}^{a/2}\log y\left(\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}\right)dy=$$

$$\int_{z=1}^{a/2}\left(\int_{y=1}^{z}\log y\ \rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}dy\right)\frac{dJ(p_r^z)}{p_r^z}+\int_{z=a/2}^{a-1}\left(\int_{y=1}^{a-z}\log y\ \rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}dy\right)\frac{dJ(p_r^z)}{p_r^z}.$$

Consequently,

$$\int_1^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}=-\int_1^{a/2}\rho\left(\frac{a}{y}-2\right)\log y\frac{dJ(p_r^y)}{p_r^y}-\int_{a/2}^{a-1}\log(a-y)\frac{dJ(p_r^y)}{p_r^y}+$$

$$\int_{z=1}^{a/2}\left(\int_{y=1}^{z}\log y\ \rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}dy\right)\frac{dJ(p_r^z)}{p_r^z}+$$

$$\int_{z=a/2}^{a-1}\left(\int_{y=1}^{a-z}\log y\ \rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}dy\right)\frac{dJ(p_r^z)}{p_r^z}+O({p_r}^{-a(1-c)/3+\epsilon}).$$

where $\rho'\left(\frac{a-z}{y}-1\right)=\frac{d}{dy}\rho\left(\frac{a-z}{y}-1\right)$ □

Thus, for $3 < a < 4$, the difference between $S_J = \int_{y=1}^{a}(\rho(a-y)dJ(p_r^y)/p_r^y) - \int_1^a(\rho(a/y-1)dJ({p_r}^y)/p_r^y)$ and $I_R = \int_1^{a/2}(R(1,{p_r}^y;1,p_r^{a-y})d\pi(p_r^y)/p_r^y)$ can be computed by combining Equations (144), (145), (146), (147) and (149) to get

$$S_J - I_R = -\int_1^{a/2} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_1^{a/3}\left(\int_{(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y} +$$
$$\int_{y=a/3}^{a-2}\left(\int_{v=2}^{a-y}\log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y} + \int_1^{a/2}\rho\left(\frac{a}{y}-2\right)\log y\frac{dJ(p_r^y)}{p_r^y} -$$
$$\int_{y=1}^{a/2}\left(\int_{v=1}^{y}\log v\ \rho'\left(\frac{a-y}{v}-1\right)\frac{a-y}{v^2}dv\right)\frac{dJ(p_r^y)}{p_r^y} -$$
$$\int_{y=a/2}^{a-1}\left(\int_{v=1}^{a-y}\log v\ \rho'\left(\frac{a-y}{v}-1\right)\frac{a-y}{v^2}dv\right)\frac{dJ(p_r^y)}{p_r^y} + O({p_r}^{-a(1-c)/3+\epsilon}), \qquad (151)$$

where $\rho'\left(\frac{a-y}{v}-1\right) = \frac{d}{dv}\rho\left(\frac{a-y}{v}-1\right)$.

Since for $a/3 \le y \le a-2$, the integral $\int_{v=2}^{a-y}(\log(v-1)dv/v)$ is a differentiable function that grows no faster than $p_r^{\epsilon y}$ (for any $\epsilon > 0$), hence

$$\int_{y=a/3}^{a-2}\left(\int_{v=2}^{a-y}\log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y} = O\left({p_r}^{-a(1-c)/3+\epsilon}\right).$$

Similarly for $a/2 \le y \le a-1$, the integral $\int_{v=1}^{a-y}\log v\ \rho'\left(\frac{a-y}{v}-1\right)\frac{a-y}{v^2}dv$ is a differentiable function that grows no faster than $p_r^{\epsilon y}$ (for any $\epsilon > 0$),. Therefore,

$$\int_{y=a/2}^{a-1}\left(\int_{v=1}^{a-y}\log v\ \rho'\left(\frac{a-y}{v}-1\right)\frac{a-y}{v^2}dv\right)\frac{dJ(p_r^y)}{p_r^y} = O\left({p_r}^{-a(1-c)/2+\epsilon}\right).$$

Furthermore, the functions $-\log y$ and $\rho(a/y-2)\log y$ are differentiable functions that grow no faster than $p_r^{\epsilon y}$ (for any $\epsilon > 0$), therefore,

$$-\int_{a/3}^{a/2}\log y\frac{dJ(p_r^y)}{p_r^y} + \int_{a/3}^{a/2}\rho\left(\frac{a}{y}-2\right)\log y\frac{dJ(p_r^y)}{p_r^y} = O\left({p_r}^{-a/6+\epsilon}\right).$$

However, by the virtue of Theorem 9, we have $S_J - I_R = O\left({p_r}^{-a(1-c)/3+\epsilon}\right)$. Thus, Equation (151) can written as follows,

$$O\left({p_r}^{-a(1-c)/3+\epsilon}\right) = -\int_1^{a/3}\log y\frac{dJ(p_r^y)}{p_r^y} + \int_1^{a/3}\rho\left(\frac{a}{y}-2\right)\log y\frac{dJ(p_r^y)}{p_r^y} +$$
$$\int_1^{a/3}\left(\int_{(a-y)/y}^{a-y}\log(v-1)\frac{dv}{v}\right)\frac{dJ(p_r^y)}{p_r^y} -$$
$$\int_{y=1}^{a/3}\left(\int_{v=1}^{y}\log v\ \rho'\left(\frac{a-y}{v}-1\right)\frac{a-y}{v^2}dv\right)\frac{dJ(p_r^y)}{p_r^y}. \qquad (152)$$

For $1 \leq y \leq a/3$, let

$$g_1(y) = \left(-1 + \rho\left(\frac{a}{y} - 2\right)\right) \log y,$$

$$g_2(y) = \int_{v=(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v},$$

and

$$g_3(y) = -\int_{v=1}^{y} \log v \ \rho'\left(\frac{a-y}{v} - 1\right)\frac{a-y}{v^2}dv,$$

where $\rho'\left(\frac{a-y}{v} - 1\right) = \frac{d}{dv}\rho\left(\frac{a-y}{v} - 1\right)$.

Therefore, Equation (152) can be written as

$$O\left({p_r}^{-a(1-c)/3+\epsilon}\right) = \int_{1}^{a/3} (g_1(y) + g_2(y) + g_3(y))\frac{dJ(p_r^y)}{p_r^y}.$$

Without loss of generality, we can define $g_1(y)$ for $y > a/3$ as $g_1(y) = c_1 + d_1/y$, where $g_1(a/3) = c_1 + 3d_1/a$ and $g_1^{'}(a/3) = -9d_1/a^2$. Also, for $y > a/3$, we set $g_2(y) = c_2 + d_2/y$, where $g_2(a/3) = c_2 + 3d_2/a$ and $g_2^{'}(a/3) = -9d_2/a^2$. Similarly, for $y > a/3$, we let $g_3(y) = c_3 + d_3/y$, where $g_3(a/3) = c_3 + 3d_3/a$ and $g_3^{'}(a/3) = -9d_3/a^2$. With these definitions of $g_1(y)$, $g_2(y)$ and $g_3(y)$ for $y > a/3$ (where the functions $g_1(y)$, $g_2(y)$ and $g_3(y)$ are bounded, differentiable and monotone increasing or decreasing depending on the sign of d_1, d_2 and d_3), we have

$$O\left({p_r}^{-a(1-c)/3+\epsilon}\right) = \int_{a/3}^{\infty} (g_1(y) + g_2(y) + g_3(y))\frac{dJ(p_r^y)}{p_r^y}.$$

Combining the above two equations, we then have

$$O\left({p_r}^{-a(1-c)/3+\epsilon}\right) = \int_{1}^{\infty} (g_1(y) + g_2(y) + g_3(y))\frac{dJ(p_r^y)}{p_r^y}. \tag{153}$$

For $1 \leq y \leq a/3$, the derivative of the functions $g_1(y)$, $g_2(y)$ and $g_3(y)$ are given by (note that for $1 \leq y \leq a/3$ and $1 \leq v \leq y$, $\rho(a/y - 2) = 1 - \log(a/y - 2)$ and $\rho(\frac{a-y}{v} - 1) = 1 - \log(\frac{a-y}{v} - 1)$),

$$\frac{dg_1(y)}{dy} = 2\left(\frac{1}{a-2y} + \frac{1}{y}\right)\log y - \frac{1}{y}\log(a-2y),$$

$$\frac{dg_2(y)}{dy} = \frac{1}{a-y}\log(a-1-y) - \frac{y}{(a-y)}\log(a-2y) + \frac{y}{(a-y)}\log y,$$

and

$$\frac{dg_3(y)}{dy} = -\frac{(a-y)^2}{y^3(a-2y)}\log y.$$

Therefore at $y = 1$, $g(1) = 0$ while $g'(1)$ is negative (notice that at $y = 1$, $\log(y) = 0$ and $g'(1)$ is given by $-\log(a-2)$). Therefore for $a > 3$, both $g(y)$ and $g'(y)$ are are non-vanishing in the vicinity of $y = 1$ and both the function and its derivative don't grow faster than $e^{\delta y}$ or decay faster than $e^{-\delta y}$ for any $\delta > 0$). Thus, referring to appendix 3, if one or more of the Riemann zeta function zeros are located on the line $\Re(s) = c$, we then have for infinitely many p_r's

$$\left|\int_{y=z}^{\infty} (g_1(y) + g_2(y) + g_3(y))\frac{dJ(p_r^y)}{p_r^y}\right| = \Omega\left(p_r^{(c-1-\epsilon)z}\right)$$

Therefore, If the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$) and one or more of these zeros are located on the line $\Re(s) = c$, then for sufficiently large N, there are infinitely many prime numbers p_r (where $p_r \geq N$) satisfying the following equation

$$O\left({p_r}^{-a(1-c)/3+\epsilon}\right) = \Omega\left(p_r^{-(1-c)-\epsilon}\right) \tag{154}$$

This result leads to our main theorem

Theorem 10 (Main Theorem). *Non-trivial zeros of the Riemann zeta function $\zeta(s)$ can be found arbitrary close to the line $\Re(s) = 1$*

Proof. Our previous analysis shows that if the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$) with one or more of these zeros are located on the line $\Re(s) = c$, then Equation (154) will follow, i.e.

$$O\left({p_r}^{-a(1-c)/3+\epsilon}\right) = \Omega\left(p_r^{-(1-c)-\epsilon}\right)$$

However for $3 < a < 4$, $a(1-c)/3 > 1-c$. This contradicts Equation (154). This contradiction infers that non-trivial zeros of the Riemann zeta function $\zeta(s)$ can be found arbitrary close to the line $\Re(s) = 1$.

□

Theorem 10 infers the following important corollary

Corollary 2. *Not all of the non-trivial zeros of the Riemann zeta function $\zeta(s)$ are on the critical line $Re(s) = \frac{1}{2}$*

Moreover, Equation (154) can be used to estimate where the distribution of the prime numbers deviates or starts to deviate from what has been predicted by the Riemann hypotheses. As mentioned earlier, we don't expect to have inconsistent results with RH for values of a less than 3. Hence, we need to set a greater than 3. In the following, we will set a equal to $4-\delta$ with $c = 0.5$ (where δ is an arbitrary small number). For $a = 4-\delta$ and $c = 0.5$, the left side of Equation (154) is less than $k_1 {p_r^{-2/3+\epsilon}}$ for some constant k_1 while the right side of the equation is greater than $k_2 {p_r}^{-1/2-\epsilon}$ for some constant k_2. Therefore, to contradict Equation (154), we need to set p_r greater than p_{r1} where

$$k_2 {p_{r1}^{-1/2}} > k_1 {p_{r1}^{-2/3}},$$

or

$$k_2 p_{r1}^{1/6} > k_1. \tag{155}$$

Equation (155) infers that there are infinitely many prime numbers $p_r > p_{r1}$ where $I_R = \int_1^{a/2}(R(1, {p_r}^y; 1, p_r^{a-y})d\pi(p_r^y)/p_r^y)$ and the sum $S_J = \int_{x=1}^{a} \rho(a-x)(dJ(p_r^x)/p_r^x) - \int_{x=1}^{a}(\rho(a/x-1)dJ({p_r}^x)/p_r^x)$ are not the same within a margin of $O({p_r^{-1/6+\epsilon}})$ where $a = 4-\delta$. Consequently, there are infinitely many prime numbers greater than p_{r1}^4 that do not follow the distribution predicted by the Riemann hypothesis. In other words; if we set N such than

$$N > \left(\frac{k_1}{k_2}\right)^{24},$$

then there are are infinitely many prime numbers greater than N that don't follow what has been predicted by the Riemann Hypothesis.

Appendix 1

Using Lebesgue-Stieltjes integral, we can write the sum $\sum_{i=r1}^{r2} \frac{1}{{p_i}^s}$ as the following integral

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^s}$$

or

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}} \frac{d\mathrm{Li}(x)}{x^s} + \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^s}.$$

Hence

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \varepsilon(s; p_{r1}, p_{r2}).$$

For $\Re(s) \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^{\infty} \frac{e^{-tz}}{t} dt$ (where $\Re(z) \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ for $\Re(z) < 1$, we first write the integral as follows

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx - i \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx.$$

The first integral on the right side $\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx$ can be computed by using the substitution $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy + \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy.$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy - \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy -$$

$$\int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy + \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}(1-\cos(t(\log p_{r1})z_1))}{z_1} dz_1 -$$

$$\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}(1-\cos(t(\log p_{r2})z_2))}{z_2} dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2$$

By the virtue of the following identity [1] (refer to page 230)

$$\int_0^1 \frac{e^{at}(1-\cos(bt))}{t}dt = \frac{1}{2}\log(1+b^2/a^2) + \mathrm{Li}(a) + \Re[E_1(-a+ib)],$$

where $a > 0$, we then obtain the following

$$\begin{aligned}\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = &\Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) - \\ &\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) - \\ &\int_{\epsilon/\log p_{r1}}^1 \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1}dz_1 + \int_{\epsilon/\log p_{r2}}^1 \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2}dz_2\end{aligned}$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_2$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\begin{aligned}\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = &\Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) - \\ &\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) + \\ &\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2}dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1}dw_1 + \\ &\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.\end{aligned}$$

Using the following identity [1] (refer to page 230)

$$\int_0^a \frac{e^t-1}{t}dt = \mathrm{Ei}(a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = \Re[E_1((s-1)\log p_{r1})] - \Re[E_1((s-1)\log p_{r2})]$$

Similarly, using the identity [1] (refer to page 230)

$$\int_0^1 \frac{e^{at}\sin(bt)}{t}dt = \pi - \arctan(b/a) + \Im[E_1(-a+ib)],$$

where $a > 0$, we can show that for $\sigma < 1$, we have

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\sin(t\log x)}{\log x}dx = \Im[E_1((s-1)\log p_{r1})] - \Im[E_1((s-1)\log p_{r2})].$$

Therefore, for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon(s;p_{r1},p_{r2})$$

where, $\varepsilon(s;p_{r1},p_{r2}) = \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^s}$.

Appendix 2

To unconditionally show that

$$\int_1^a \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \varepsilon^k(\alpha; p_r) \right) * \mathcal{L}^{-1} \left(\sum_{m=1}^{\infty} \frac{1}{m!} \delta^m(\alpha; p_r) \right) \right) dy = O\left({p_r}^{-1+\epsilon} \right).$$

we will first unconditionally show that

$$\left| \int_1^a \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \varepsilon^k(\alpha; p_r) \right) \right) (y) dy \right| < e^{2\log a + O(1/p_r)}.$$

For $k = 1$, by referring to Lemma 30, we have

$$\int_{y=1}^a \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y)\, dy = \int_{x=1}^a \rho(a - x) \frac{dJ(p_r^x)}{p_r^x}$$

Since $0 < \rho(y) \leq 1$ and referring to Lemma 11, we then unconditionally have

$$\left| \int_{y=1}^a \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y)\, dy \right| < 2\log a + O(1/p_r)$$

Let $k_1(y) = (\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r))(y)$. Let $k_2(y) = (\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) * \mathcal{L}^{-1} \varepsilon(\alpha; p_r))(y)$ and so on. Thus

$$\left| \int_{y=1}^a k_1(y) dy \right| < 2\log a + O(1/p_r)$$

Furthermore,

$$\int_{y=1}^a k_2(y) dy = \int_{y=1}^a \left(k_1 * \mathcal{L}^{-1} \delta(\alpha; p_r) \right) (y) dy = \int_{y=1}^a \int_{\tau=1}^{\infty} k_1(y - \tau) (\mathcal{L}^{-1} \varepsilon(\alpha; p_r))(\tau) d\tau\, dy$$

thus, by changing the order of integration, we then have

$$\left| \int_{y=1}^a \left(k_1 * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y) dy \right| \leq \int_{\tau=1}^{\infty} \left| (\mathcal{L}^{-1} \varepsilon(\alpha; p_r))(\tau) \right| \left| \int_{y=1}^a k_1(y - \tau) dy \right| d\tau$$

or

$$\left| \int_{y=1}^a \left(k_1 * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y) dy \right| < (2\log a + O(1/p_r)) \int_{\tau=1}^{\infty} \left| \mathcal{L}^{-1} \varepsilon(\alpha; p_r)(\tau) \right| d\tau$$

and by the virtue of Lemma 11, we then have

$$\left| \int_{y=1}^a \left(m_1 * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y) dy \right| < (2\log a + O(1/p_r))^2,$$

or

$$\left| \int_{y=1}^a \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y) dy \right| < (2\log a + O(1/p_r))^2.$$

Repeating these steps $k - 1$ times (i.e. the steps to derive $\int_{y=1}^a k_2(y) dy$), we then have

$$\left| \int_{y=1}^a k_k(y) dy \right| = \left| \int_{y=1}^a \left(k_{k-1} * \mathcal{L}^{-1} \varepsilon(\alpha; p_r) \right) (y) dy \right| < (2\log a + O(1/p_r))^k.$$

Consequently,

$$\left| \int_1^a \left(\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \varepsilon^k(\alpha; p_r) \right) \right) (y) dy \right| < e^{2\log a + O(1/p_r)}. \tag{156}$$

Let $(km)_1(y) = (\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}(\sum_1^\infty \frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)) * \mathcal{L}^{-1}\delta(\alpha;p_r))(y)$. Let $(km)_2(y) = (\mathcal{L}^{-1}e^{-E_1(\alpha)}\mathcal{L}^{-1}(\sum_1^\infty \frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)) * \mathcal{L}^{-1}\delta(\alpha;p_r) * \mathcal{L}^{-1}\delta(\alpha;p_r))(y)$ and so on. Thus

$$\int_{y=1}^{a}(km)_1(y)dy = \int_{y=1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy$$

and by the virtue of Equation(156), we then have

$$\left|\int_{y=1}^{a}(km)_1(y)dy\right| < e^{2\log a + O(1/p_r)}\int_{\tau=1}^{\infty}\left|(\mathcal{L}^{-1}\delta(\alpha;p_r))(\tau)\right|d\tau$$

Since

$$\left|\int_{\tau=1}^{\infty}\mathcal{L}^{-1}\delta(\alpha;p_r)(\tau)d\tau\right| \le \int_{y=1}^{\infty}\left(\sum_{i=r}^{\infty}\left(\frac{\delta(y-2)}{2{p_i}^2} + \frac{\delta(y-3)}{3{p_i}^3} + \frac{\delta(y-4)}{4{p_i}^4}...\right)\right)dy < \frac{1}{p_r},$$

thus

$$\left|\int_{y=1}^{a}(km)_1(y)dy\right| < \frac{e^{2\log a + O(1/p_r)}}{p_r}$$

or

$$\int_{y=1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy < \frac{e^{2\log a + O(1/p_r)}}{p_r}$$

Similarly,

$$\int_{y=1}^{a}(km)_2(y)dy = \int_{y=1}^{a}\left((km)_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy.$$

where

$$\left|\int_{y=1}^{a}\left((km)_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| \le \int_{\tau=1}^{\infty}\left|(\mathcal{L}^{-1}\delta(\alpha;p_r))(\tau)\right|\left|\int_{y=1}^{a}(km)_1(y-\tau)dy\right|d\tau$$

or

$$\left|\int_{y=1}^{a}\left((km)_1 * \mathcal{L}^{-1}\delta(\alpha;p_r)\right)(y)dy\right| < \frac{e^{2\log a + O(1/p_r)}}{p_r}\int_{\tau=1}^{\infty}\left|(\mathcal{L}^{-1}\delta(\alpha;p_r))(\tau)\right|d\tau$$

Thus

$$\left|\int_{y=1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\left(\frac{1}{2!}\delta(\alpha;p_r)^2\right)\right)(y)dy\right| < \frac{e^{2\log a + O(1/p_r)}}{2!\,p_r^2}$$

Repeating these steps m times, we then have

$$\left|\int_{y=1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\left(\frac{1}{m!}\delta(\alpha;p_r)^m\right)\right)(y)dy\right| < \frac{e^{2\log a + O(1/p_r)}}{m!\,p_r^m}$$

Consequently, for a fixed a, we then have

$$\int_{1}^{a}\left(\mathcal{L}^{-1}e^{-E_1(\alpha)} * \mathcal{L}^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k(\alpha;p_r)\right) * \mathcal{L}^{-1}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\delta^m(\alpha;p_r)\right)\right)dy = O\left({p_r}^{-1+\epsilon}\right).$$

Appendix 3

On RH, we will show that for infinitely many prime numbers p_r's, we have

$$\left|\int_1^\infty \frac{dJ(p_r^y)}{p_r^y}\right| = \Omega\left(p_r^{-1/2-\epsilon}\right).$$

In other words; there are infinitely many prime numbers p_r that satisfy the following

$$\left|\int_1^\infty \frac{dJ(p_r^y)}{p_r^y}\right| > kp_r^{-1/2-\epsilon}$$

for some constant k. We will also show that, on RH and for infinitely many prime numbers p_r's, we have

$$\left|\int_1^\infty g(y)\frac{dJ(p_r^y)}{p_r^y}\right| = \Omega\left(p_r^{-1/2-\epsilon}\right),$$

where $g(y)$ a differentiable function for $y \geq 1$ with the function $g(x)$ and its derivative $g'(x)$ are non-vanishing in the vicinity of $y = 1$ and both the function and its derivative don't grow faster than $e^{\delta y}$ or decay faster than $e^{-\delta y}$ for any $\delta > 0$ (for example, for $y \geq 1$, $g(y) = 1, y, y^2$, ..., $y^n, 1/y, 1/y^2, .., 1/y^n$, $(\log y)^n$). There are a variety of theorems (that are based on Paley-Wiener theorems) that establish the relationship between the decay properties of a function with its Fourier, Laplace or Mellin transform (within its region of convergence). Our analysis is similar to Landau approach that establishes the relationship between the decay (or growth) rate of a Riemann integrable function and the region over which its Mellin transform is analytic [12].

Toward this end, we first write $J(x) = \pi(x) - \mathrm{Li}(x)$ as (refer to Lemma 5)

$$J(x) = \pi(x) - \mathrm{Li}(x) = -\sum_{n=2}^{\lfloor \log x/\log 2\rfloor} \frac{\pi(x^{1/n})}{n} + \frac{\psi(x) - x}{\log x} + P(x),$$

where,

$$P(x) = \int_2^x \frac{\psi(u) - u}{u\log^2 u}du + \frac{2}{\log 2} - \mathrm{Li}(2),$$

Hence, on RH, we have (refer to Lemma 6)

$$J(x) = \frac{\psi(x) - x}{\log x} + \int_2^x \frac{\psi(u) - u}{u\log^2 u}du - \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right),$$

or

$$J(x) = \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + \int_2^x \left(\frac{1}{u\log^2 u}\sum_\rho \frac{u^\rho}{\rho}\right)du - \mathrm{Li}(x^{1/2}) + O\left(x^{1/3}\right),$$

and

$$J(p_r^y) = \frac{1}{y\log p_r}\sum_\rho \frac{e^{y\rho\log p_r}}{\rho} + \int_{\frac{\log 2}{\log p_r}}^y \left(\frac{1}{z^2\log p_r}\sum_\rho \frac{e^{z\rho\log p_r}}{\rho}\right)dz - \mathrm{Li}(p_r^{y/2}) + O\left(p_r^{y/3}\right)$$

Let

$$J_1(p_r^y) = \frac{1}{y\log p_r}\sum_\rho \frac{e^{y\rho\log p_r}}{\rho}$$

and

$$J_2(p_r^y) = \int_{\frac{\log 2}{\log p_r}}^{y} \left(\frac{1}{z^2 \log p_r} \sum_{\rho} \frac{e^{z\rho \log p_r}}{\rho} \right) dz$$

then,

$$J(p_r^y) = J_1(p_r^y) + J_2(p_r^y) - \mathrm{Li}(p_r^{y/2}) + O\left(p_r^{y/3}\right)$$

In the following, we will show that, on RH, the function $f(y) = |\int_{z=y}^{\infty} dJ(p_r^z)/p_r^z|$ grows faster than $p_r^{(-1/2-\epsilon)y}$ by showing that the Laplace transform of integral $\int_y^{\infty} dJ(p_r^z)/p_r^z$ is analytic function for $\sigma > -(1/2)\log p_r$ with singularities at $(-1/2 + i\beta_i)\log p_r$ (that correspond to the zeros of the zeta function at $\rho_i = 1/2 + i\beta_i$). Thus, the value of $|\int_{z=y}^{\infty} dJ(p_r^z)/p_r^z|$ grows faster than $p_r^{(-1/2-\epsilon)y}$ due to the presence of these singularities at $(-1/2 + i\beta_i)\log p_r$. In other words; if the value of $|\int_{z=y}^{\infty} dJ(p_r^z)/p_r^z|$ grows at a rate slower than $p_r^{(-1/2-\epsilon)y}$ then the Laplace transform of the integral $\int_y^{\infty} dJ(p_r^z)/p_r^z$ will be analytic at $\sigma = -(1/2)\log p_r$. This contradicts our earlier assertion that the Laplace transform function has singularities at $(-1/2+i\beta_i)\log p_r$ (or the Laplace transform integral diverges for $\sigma = -(1/2)\log p_r$).

To compute the Laplace transform (and its singularities) of the integral $\int_y^{\infty} dJ(p_r^z)/p_r^z$, we have,

$$\int_y^{\infty} \frac{dJ_1(p_r^z)}{p_r^z} = \frac{J_1(p_r^y)}{p_r^y} - \int_y^{\infty} J_1(p_r^z) dp_r^{-z}$$

Therefore,

$$\int_y^{\infty} \frac{dJ_1(p_r^z)}{p_r^z} = \frac{1}{y p_r^y \log p_r} \sum_{\rho} \frac{(p_r^y)^{\rho}}{\rho} + \int_y^{\infty} \left(\frac{1}{z p_r^z} \sum_{\rho} \frac{(p_r^z)^{\rho}}{\rho} \right) dz$$

As mentioned earlier, the sum $\sum_{\rho}(x^{\rho}/\rho)$ is conditionally convergent and it should be performed over the non-trivial zeros with $|\beta_i| \leq T$ as T approaches infinity. Furthermore, referring to lemma 2 of reference [16], the sum is $\sum_{\rho}(x^{\rho-1}/\rho)$ is uniformly convergent. Hence, the integral and the sum in the above equation can be interchanged. In other words; the integral on the right side of the above equation can be performed term by term. Therefore, on RH, we have

$$\int_y^{\infty} \frac{dJ_1(p_r^z)}{p_r^z} = \frac{1}{y \log p_r} \sum_{\rho} \frac{e^{y(-1/2+\beta_i)\log p_r}}{\rho_i} + \sum_{\rho} \left(\int_y^{\infty} \frac{e^{z(-1/2+\beta_i)\log p_r}}{z\rho_i} dz \right) \tag{157}$$

Furthermore,

$$\int_y^{\infty} \frac{dJ_2(p_r^z)}{p_r^z} = \int_y^{\infty} \frac{1}{p_r^z} \frac{d}{dz} \left(\int_{\frac{\log 2}{\log p_r}}^{z} \frac{1}{w^2 \log p_r} \sum_{\rho} \frac{e^{w\rho_i \log p_r}}{\rho_i} dw \right) dz$$

Since

$$\frac{d}{dz} \left(\int_{\frac{\log 2}{\log p_r}}^{z} \frac{1}{w^2 \log p_r} \sum_{\rho} \frac{e^{w\rho_i \log p_r}}{\rho_i} dw \right) = \frac{1}{z^2 \log p_r} \sum_{\rho} \frac{e^{z\rho_i \log p_r}}{\rho_i},$$

thus, on RH, we have

$$\int_y^{\infty} \frac{dJ_2(p_r^z)}{p_r^z} = \frac{1}{\log p_r} \sum_{\rho} \left(\int_y^{\infty} \frac{e^{z(-1/2+\beta_i)\log p_r}}{z^2 \rho_i} dz \right). \tag{158}$$

Furthermore,

$$\int_y^\infty \frac{d\mathrm{Li}(p_r^{z/2}))}{p_r^z} = \int_y^\infty \frac{1}{z} e^{-(z/2)\log p_r} dz. \tag{159}$$

Moreover, using the method of integration by parts, we then have

$$\int_y^\infty \frac{dO\left(p_r^{z/3}\right)}{p_r^z} = O\left(p_r^{-2y/3}\right) \tag{160}$$

Combining Equations (157), (158), (159) and (161) we then have

$$\int_y^\infty \frac{dJp_r^z)}{p_r^z} =$$

$$\frac{1}{\log p_r}\sum_{\rho_i}\left(\frac{e^{y(-1/2+\beta_i)\log p_r}}{y\rho_i} + \log p_r \int_y^\infty \frac{e^{z(-1/2+\beta_i)\log p_r}}{z\rho_i}dz + \int_y^\infty \frac{e^{z(-1/2+\beta_i)\log p_r}}{z^2\rho_i}dz\right) -$$

$$\int_y^\infty \frac{1}{z}e^{-(z/2)\log p_r}dz + O\left(p_r^{-2y/3}\right). \tag{161}$$

To compute the Laplace transform of the above integral, we note that the Laplace transform of the function e^{at} is given $1/(s-a)$ with a pole (or singularity) at $s = a$. We also note the Laplace transform of the function $e^{at}f(t)$ is given by $F(s-a)$ where $F(s)$ is the Laplace transform of $f(t)$. In other words; multiplication of a function $f(t)$ by e^{at} will shift the poles or singularities of its Laplace transform $F(s)$ by a. Furthermore, the Laplace transform of the integral $\int_y^\infty f(t)dt$ is given by $F(0)/s - F(s)/s$ (note that $\int_y^\infty f(t)dt = \int_1^\infty f(t)dt - \int_1^y f(t)dt$. The Laplace transform of the integral $\int_1^y f(t)dt$ is given by $F(s)/s$. Furthermore, by the virtue of the final value theorem, the integral $\int_1^\infty f(t)dt$ is given by $F(0)$ and its Laplace transform is then given by $F(0)/s$). Consequently, the Laplace transform of the integral $\int_y^\infty f(t)dt$ has a removable singularities at $s = 0$ and its singularities are the same as the singularities of $F(s)$. Using these Laplace transform properties, one may then conclude that, on RH, all the singularities of the Laplace transform of the integral $\int_y^\infty dJ(p_r^z)/p_r^z$ in Equation (161) are on the line $\sigma = -\frac{1}{2}\log p_r$. Thus for some constant k, the function $f(y) = |\int_{z=y}^\infty dJ(p_r^z)/p_r^z|$ grows faster than $ke^{(-0.5\log p_r - \epsilon)y}$. Hence,

$$\sup_{y\geq 1}\left|\int_{z=y}^\infty dJ(p_r^z)/p_r^z\right| - ke^{(-0.5\log p_r-\epsilon)y} > 0$$

In other words; for any $y \geq 1$, there are infinitely many prime numbers p's such that

$$\left|\int_{z=y}^\infty \frac{dJ(p^z)}{p^z}\right| = \Omega\left(p^{(-\frac{1}{2}-\epsilon)y}\right).$$

Hence, there are infinitely many prime numbers p_r's such that

$$\left|\int_1^\infty \frac{dJ(p_r^z)}{p_r^z}\right| > kp_r^{-\frac{1}{2}-\epsilon}.$$

where k is a constant. In other words; for infinitely many prime numbers p_r's, we have

$$\left|\int_1^\infty \frac{dJ(p_r^z)}{p_r^z}\right| = \Omega\left(p_r^{-\frac{1}{2}-\epsilon}\right).$$

Similar analysis can be applied to show that if the Laplace transform of a function $g(z)$ is analytic for $\sigma > 0$ with singularities on the line $\Re(s) = 0$ (this includes functions that are differentiable where the function and its derivative are non-vanishing in the vicinity of $y = 1$ and both the function and its derivative don't grow faster than $e^{\delta y}$ or decay faster than $e^{-\delta y}$ for any $\delta > 0$), then on RH, there are infinitely many prime numbers p_r such that

$$\left|\int_1^{\infty} g(z)\frac{dJ(p_r^z)}{p_r^z}\right| = \Omega\left(p_r^{-\frac{1}{2}-\epsilon}\right).$$

In general, if the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1 - c \leq \Re(s) \leq c$ (where $1/2 \leq c < 1$) with one or more of the zeros located on the the line $\Re(s) = c$, then there are infinitely many prime numbers p_r's such that

$$\left|\int_1^{\infty} g(z)\frac{dJ(p_r^z)}{p_r^z}\right| = \Omega\left(p_r^{-(1-c)-\epsilon}\right).$$

This result can be proven by following the same steps we employed to find the singularities of the Laplace transform of the integral $\int_{z=y}^{\infty} g(z)dJ(p_r^z)/p_r^z$ (refer to Equation 161). More specifically, referring to Lemma 8, we have

$$\int_{z=y}^{\infty} g(z)\frac{dJ(p_r^z)}{p_r^z} = \int_{z=y}^{\infty} g(z)\left(\frac{dJ_1(p_r^z)}{p_r^z} + \frac{dJ_2(p_r^z)}{p_r^z} + \frac{d\mathrm{Li}(p_r^{z/2}))}{p_r^z} + \frac{dO\left(p_r^{\max(1/3,c/2)}\right)}{p_r^z}\right).$$

where

$$\int_y^{\infty} g(z)\frac{dJ_1(p_r^z)}{p_r^z} = \frac{g(y)}{y\log p_r}\sum_{\rho}\frac{e^{-y(1-\rho_i)\log p_r}}{\rho_i} + \sum_{\rho}\left(\int_y^{\infty}\frac{g(z)}{z}\frac{e^{-z(1-\rho_i)\log p_r}}{\rho_i}dz\right) -$$

$$\frac{1}{\log p_r}\sum_{\rho}\left(\int_y^{\infty}\frac{g'(z)}{z}\frac{e^{-z(1-\rho_i)\log p_r}}{\rho_i}dz\right)$$

$$\int_y^{\infty} g(z)\frac{dJ_2(p_r^z)}{p_r^z} = \frac{1}{\log p_r}\sum_{\rho}\left(\int_y^{\infty}\frac{g(x)}{z^2}\frac{e^{-z(1-\rho_i)\log p_r}}{\rho_i}dz\right)$$

and

$$\int_y^{\infty} g(z)\frac{d\mathrm{Li}(p_r^{z/2}))}{p_r^z} = \int_y^{\infty}\frac{g(z)}{z}e^{-(z/2)\log p_r}dz.$$

Thus, one or more of the singularities of the Laplace transform of the above three integrals are on the line $\Re(s) = -(1-c)\log p_r$. Thus for some constant k, $|\int_{z=y}^{\infty}(g(z)(dJ_1(p_r^z)/p_r^z + dJ_2(p_r^z)/p_r^z + dJ_3(p_r^z)/p_r^z)|$ grows faster than $ke^{(-(1-c)\log p_r - \epsilon)y}$. Since

$$\int_y^{\infty} g(z)\frac{dO\left(p_r^{\max(1/3,c/2)}\right)}{p_r^z} = O\left(p_r^{1-\max(1/3,c/2)}\right)$$

therefore there are infinitely many prime numbers p_r's such that

$$\left|\int_1^{\infty} g(z)\frac{dJ(p_r^z)}{p_r^z}\right| = \Omega\left(p_r^{-(1-c)-\epsilon}\right).$$

Same results are attained if the not-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 < c < 1$) with (some) zeros of $\zeta(s)$ are located to the right of the line $\Re(s) = c - \delta$ (where δ is an arbitrary small positive number).

Appendix 4

In this appendix, we will compute the size of the the integral $\int_1^{a/2}\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)\frac{dJ(p_r^y)}{p_r^y}$ when the non-trivial zeros of $\zeta(s)$ are restricted to the strip $1-c \le \Re(s) \le c$ (where $1/2 \le c < 1$). First we note that although the function $J(x)$ is not a non-decreasing function, $J(x)$ is given by $\pi(x) - \mathrm{Li(x)}$ where both $\pi(x)$ and $\mathrm{Li}(x)$ are non-decreasing functions. Therefore, we can use theorem 21.67 of [8] for the method of integration by parts for Lebesgue-Stieljtes integrals to obtain,

$$\int_1^{a/2}\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)\frac{1}{p_r^y}dJ(p_r^y) = \left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)\frac{J(p_r^y)}{p_r^y}\Bigg|_1^{a/2}$$
$$-\int_1^{a/2}J(p_r^z)\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)d\left(\frac{1}{p_r^y}\right)$$
$$-\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}d\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right) \tag{162}$$

where

$$\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)\frac{J(p_r^y)}{p_r^y}\Bigg|_{y=1}^{a/2} = -\left(\int_{z=1}^{a-1}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)\frac{J(p_r)}{p_r}$$

Since the function $\rho\left((a-y)/y - z/y\right)$ is positive, bounded and differential over the range $y \le z \le a-y$ ($y \ge 1$), hence

$$\left|\int_{z=1}^{a-1}\rho\left((a-y)/y - z/y\right)dJ(p_r^z)/p_r^z\right| = O(p_r^{-(1-c)+\epsilon}),$$

or

$$\left|\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)\frac{J(p_r^y)}{p_r^y}\Bigg|_{y=1}^{a/2}\right| = O(p_r^{-2(1-c)+\epsilon}).$$

We also have

$$\left|\int_{z=y}^{a-y}\rho\left((a-y)/y - z/y\right)dJ(p_r^z)/p_r^z\right| = O(p_r^{-(1-c)+\epsilon}),$$

thus

$$\left|\int_1^{a/2}J(p_r^y)\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)d\left(\frac{1}{p_r^y}\right)\right| = O(p_r^{-2(1-c)+\epsilon}).$$

To compute the size of the last term $\int_1^{a/2} J(p_r^z)\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)d\left(\frac{1}{p_r^y}\right)$, we first refer to Equation (150)

$$d\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)=-\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}-\rho(0)\frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}+$$
$$dy\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z},$$

we then have

$$\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}d\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)=-\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}$$
$$-\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}+\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\left(\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}\right)dy. \tag{163}$$

To compute the first integral $\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}$, we use the method of integration by parts to obtain

$$\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}=\left(\frac{J(p_r^y)}{p_r^y}\right)^2\rho\left(\frac{a}{y}-2\right)\Bigg|_1^{a/2}-\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}-$$
$$\int_1^{a/2}J^2(p_r^y)d\left(\frac{\rho(a/y-2)}{p_r^{2y}}\right), \tag{164}$$

where by the virtue of Lemma 9,

$$\left|\left(\frac{J(p_r^y)}{p_r^y}\right)^2\rho\left(\frac{a}{y}-2\right)\Bigg|_1^{a/2}\right|=O(p_r^{-2(1-c)+\epsilon}),$$

and

$$\left|\int_1^{a/2}J^2(p_r^y)d\left(\frac{\rho(a/y-2)}{p_r^{2y}}\right)\right|=O(p_r^{-2(1-c)+\epsilon}).$$

Thus by rearranging the terms of Equation (164), we then have

$$2\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}=O(p_r^{-2(1-c)+\epsilon}),$$

or,

$$\left|\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\rho\left(\frac{a}{y}-2\right)\frac{dJ(p_r^y)}{p_r^y}\right|=O(p_r^{-2(1-c)+\epsilon}).$$

The second integral on the right side of Equation (163) can be written as

$$\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}=\frac{1}{p_r^a}\int_1^{a/2}J(p_r^y)dJ({p_r}^{a-y}),$$

or

$$\int_1^{a/2}\frac{J(p_r^y)}{p_r^y}\frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}=\frac{1}{p_r^a}\left(J(p_r^y)J({p_r}^{a-y})\Big|_1^{a/2}-\int_1^{a/2}J({p_r}^{a-y})dJ(p_r^y)\right),$$

hence

$$\int_1^{a/2} \frac{J(p_r^y)}{p_r^y} \frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}} = \left(\frac{J(p_r^y)}{p_r^y} \frac{J({p_r}^{a-y})}{p_r^{a-y}} \bigg|_1^{a/2} - \int_1^{a/2} \frac{J(p_r^{a-y})}{p_r^{a-y}} \frac{dJ({p_r}^{y})}{p_r^y} \right),$$

and the virtue of Lemmas 9 and 11, we then have

$$\left| \int_1^{a/2} \frac{J(p_r^y)}{p_r^y} \frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}} \right| = O(p_r^{-a(1-c)/2+\epsilon}).$$

For the third integral on the right side of Equation (163)

$$\left| \int_1^{a/2} \frac{J(p_r^y)}{p_r^y} \left(\int_{z=y}^{a-y} \rho' \left(\frac{a-z}{y} - 1 \right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z} \right) dy \right| \leq$$

$$\int_1^{a/2} \left| \frac{J(p_r^y)}{p_r^y} \right| \left| \left(\int_{z=y}^{a-y} \rho' \left(\frac{a-z}{y} - 1 \right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z} \right) \right| dy.$$

Since the function $\rho' \left(\frac{a-z}{y} - 1 \right) \frac{a-z}{y^2}$ is positive, bounded and differentiable over the range $y \leq z \leq a - y$ ($y \geq 1$), thus $\left| \int_{z=y}^{a-y} \rho' \left(\frac{a-z}{y} - 1 \right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z} \right| = O({p_r}^{-(1-c)+\epsilon})$. Therefore, for $a < 4$

$$\left| \int_1^{a/2} \frac{J(p_r^y)}{p_r^y} \left(\int_{z=y}^{a-y} \rho' \left(\frac{a-z}{y} - 1 \right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z} \right) dy \right| = O(p_r^{-a(1-c)/2+\epsilon}).$$

Consequently, for $a < 4$

$$\left| \int_1^{a/2} \left(\int_{z=y}^{a-y} \rho \left(\frac{a-y}{y} - \frac{z}{y} \right) \frac{dJ(p_r^z)}{p_r^z} \right) \frac{dJ(p_r^y)}{p_r^y} \right| = O(p_r^{-a(1-c)/2+\epsilon}).$$

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