On the Pythagoras' and De Gua's theorems in geometric algebra

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This small article is intended to be a contribution to the LinkedIn group "*Pre-University Geometric Algebra*". The main idea is to show that in geometric algebra we have the Pythagoras' and De Gua's theorems without a metric defined. This allows us to generalize these theorems to any dimension and any signature.

Keywords: *the Pythagoras' theorem, De Gua's theorem, geometric algebra, metric, bivector*

The geometric product

In geometric algebra, we define a non-commutative product of two vectors with the properties of associativity and distributivity, which can be decomposed into the symmetric and anti-symmetric parts

$$ab = \frac{ab+ba}{2} + \frac{ab-ba}{2} = S + A,$$

where we can define that vectors are orthogonal if

$$S = \frac{ab+ba}{2} = 0 \Rightarrow ab = -ba,$$

which means that orthogonal vectors anti-commute. Likewise, we can define that vectors are parallel if

$$A = \frac{ab-ba}{2} = 0 \Rightarrow ab = ba,$$

which means that parallel vectors commute. These definitions are in accordance with the usual definitions in algebras. For example, we could define that two vectors *a* and *b* are parallel if $a = \lambda b$, where λ is a real number, but it is obvious that these vectors commute in geometric algebra, since real numbers commute with vectors.

Now we can show that products $a^2 = aa$ commute with al vectors. One can say that this is obvious, since a^2 is a real (or a complex) number. However, we do not need such an interpretation (that is, we do not need to introduce a metric, yet). Obviously, a^2 commutes with the vector *a*. Consider a vector *b*, which is orthogonal to the vector *a*. Then we have

$$a^2 b = aab = -aba = baa = ba^2 ,$$

which means that the commutativity here follows from the geometric product properties. Now we can show that this means that a^2 commutes with all vectors, but the pleasure is left to the reader.

Orthogonal vectors

Consider two orthogonal vectors in any dimension and of any signature. We have

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab - ab + b^2 = a^2 + b^2 ,$$

which means that the Pythagoras' theorem is valid. Let us look at two 2D examples

$$\Re^2 : \quad e_1^2 = e_2^2 = 1 \Rightarrow (e_1 + e_2)^2 = 1 + 1 + e_1 e_2 + e_2 e_1 = 2 = e_1^2 + e_2^2 ,$$

$$\Re^{1,1} : \quad e_1^2 = -e_2^2 = 1 \Rightarrow (e_1 + e_2)^2 = 1 - 1 + e_1 e_2 + e_2 e_1 = 0 = e_1^2 + e_2^2 .$$

Note that the commutativity properties of geometric product play a central role here. Simply stated, with the geometric product we have the Pythagoras' theorem in any vector space we can imagine. Moreover, we have this important result without definition of a metric.

De Gua's theorem

Now we can show how to get De Gua's theorem easily. First, note that the anti-symmetric part of geometric product of two vectors is a bivector, which we can write as

$$A = \frac{ab - ba}{2} \equiv a \wedge b ,$$

where $\wedge$ stands for the outer (wedge) product. It is not difficult to show that the magnitude of a bivector is proportional to the area of the parallelogram defined by the vectors *a* and *b*. Namely, decomposing the vector *b* into the prats parallel and orthogonal to the vector *a*, we can write

$$A = a \wedge b = a \wedge \left(b_{\parallel} + b_{\perp}\right) = a \wedge b_{\perp} = ab_{\perp},$$

whence, using $|b_{\perp}| = |b||\sin\alpha|$, we get the parallelogram area formula. Defining the reverse involution

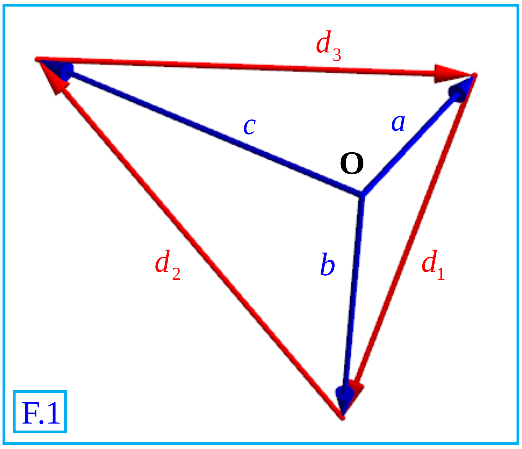
$$A^{\dagger} = b_{\perp} a,$$

we have

$$AA^{\dagger} = ab_{\perp}b_{\perp}a = a^2 b_{\perp}^2,$$

which we can interpret as the square of the area of the parallelogram defined by the vectors *a* and *b*, but we have to define the square of a vector to be a positive real number (metric) first. Here, we will proceed without a metric, in order to get formulae that are more general.

Consider three orthogonal vectors *a*, *b*, and *c* (F.1) with the initial point O, whose end points span a triangle. We can write



F.1

$$a + d_1 - b = 0,$$

$$b + d_2 - c = 0,$$

$$c + d_3 - a = 0,$$

whence follows that $d_1 + d_2 + d_3 = 0$. Now we can define the bivector $B = d_1 \wedge d_2$ whose magnitude is double of the red triangle area. Therefore, $BB^{\dagger}/4$ gives the squared area of the red triangle. Ignoring the factor 4, we can calculate

$$B = d_1 \wedge d_2 = (b-a) \wedge (c-b) = b \wedge c + c \wedge a - b \wedge b + a \wedge b = bc + ca + ab,$$

whence follows that

$$BB^{\dagger} = (bc + ca + ab)(cb + ac + ba) = \cdots = a^2b^2 + a^2c^2 + b^2c^2.$$

The details of the calculation are left to the reader; however, note that the result follows from the fact that orthogonal vectors anti-commute.

Finally, there are two important facts that we should stress here. First, note that the result is independent of a signature. Second, generalizations to higher dimension are straightforward; however, we should formulate a problem in terms of hyper-volumes.

Literature

[1] Josipović, Miroslav: *Multiplication of vectors and structure of 3D Euclidean space*, http://vixra.org/abs/1609.0024

[2] Josipović, Miroslav: *Geometric Multiplication of Vectors - An Introduction to Geometric Algebra in Physics*, Birkhäuser, 2019 (it is to be printed soon)

[3] http://www.cut-the-knot.org/pythagoras/