# The Semi-Pascal Triangle of Maximum Deng Entropy

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**Abstract**

In D-S theory, measure the uncertainty has aroused many people's attention. Deng proposed the interesting Deng entropy that it can measure non-specificity and discord. Hence, exploring the physical meaning of Deng entropy is an essential issue. Based the maximum Deng entropy and fractal, the paper discuss the relation in them.

*Keywords:* Dempster-Shafer evidence theory, Deng entropy, Yang Hui triangle, Fractal

## 1. Introduction

Knowledge representation has been an area of intensive research since answering the question "What is it " [1]. At first, the proposition of probability provided a great convenience for the expression of knowledge. However, it is difficult to get the precise probability distribution of an indeterminacy quantity. In this case, we have to rely on the experts belief degree that each event will occur [2]. With the needs of society, there are many literature to further explore this problem, such as fuzzy sets [3], Dempster-Shafer

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(short for D-S) theory [4, 5], and rough sets [6]. D-S theory has many advantages as an important and widely used uncertain reasoning method, which was presented as an extension of the classical Probability Theory, providing a well-established structure for the representation of imprecise information [7, 8]. D-S theory assigns probabilities to the power set of events, so as to better grasp the unknown and uncertainty of the problem [9]. However, how to measure the uncertainty of D-S theory is also an open issue.

Recalled the entropy, entropy is derived from physics, which is a measure of the degree of chaos in the physical system [10]. Subsequently, various types of entropies are presented. Shannon firstly proposed the information entropy to measure the uncertainty of information [11]. Deng proposed Deng entropy to measure the uncertainty of basic probability assignment for D-S theory [12]. Tsallis pointed that multi-fractal concepts and structures are quickly acquiring importance in many active areas of research (e.g. nonlinear dynamical systems, growth models, commensurate /incommensurate structures) [13]. Later, with the use of a quantity normally scaled in multi-fractals, Tsallis proposed Tsallis entropy as generalization of Boltzmann-Gibbs statistics [13].

Fractal find the medium and bridge from partial to whole information isomorphism. Aydan proposed the fractal model of basic probability assignment [14]. In modern science, there are various fractals, such as Pascal triangle, Cantor set, and so on [15]. Based the fractal, the paper discuss the property of maximum Deng entropy which be proposed in [16], which is useful for the real application of Deng entropy.

The paper is organized as follows. The preliminaries Dempster-Shafer evidence theory and Deng entropy are briefly introduced in Section 2. Section 3 discussed the fractal of maximum Deng entropy. Finally, this paper is concluded in Section 4.

## 2. Preliminaries

In this section, the preliminaries of D-S theory [4, 5] and Deng entropy [12] will be briefly introduced.

### *2.1. D-S theory*

D-S theory offers a useful fusion tool for uncertain information. D-S theory needs weaker conditions than the Bayesian theory of probability, so it is often regarded as an extension of the Bayesian theory, which has been used to many applications, such as making-decision, complex networks, pattern classification. Besides, some new or generalized model based this theory have been proposed, for example, generalized evidence theory and D numbers. Some preliminaries in D-S theory are introduced as follows. For additional details about D-S theory, refer to [4, 5].

**Definition 2.1.** *( Frame of discernment)*
*Let $\Theta$ be the set of mutually exclusive and collectively exhaustive events $A_i$, namely*

$$\Theta = \{A_1, A_2, \cdots, A_n\} \tag{1}$$

*The power set of $\Theta$ composed of $2^N$ elements of is indicated by $2^\Theta$, namely:*

$$2^\Theta = \{\phi, \{A_1\}, \{A_2\}, \cdots, \{A_1, A_2\}, \cdots, \Theta\} \tag{2}$$

**Definition 2.2.** *( Mass Function)*
*For a frame of discernment $\Theta = \{A_1, A_2, \cdots, A_n\}$, the mass function $\mathrm{m}$ is de-*

*fined as a mapping of m from 0 to 1, namely:*

$$m : 2^{\Theta} \to [0, 1] \tag{3}$$

which satifies

$$m\left(\phi\right) = 0 \tag{4}$$

$$\sum_{A \subseteq \Theta} m\left(A\right) = 1 \tag{5}$$

In D-S theory, a mass function is also called a basic probability assignment (BPA) or a piece of evidence or belief structure. The $m(A)$ measures the belief exactly assigned to A and represents how strongly the piece of evidence supports A. If $m\left(A\right) > 0$, A is called a focal element, and the union of all focal elements is called the core of a mass function.

**Definition 2.3.** *(Belief function)*
*The belief function (Bel) is a mapping from set* $2^{\Theta}$ *to [0,1] and satisfied:*

$$Bel\left(A\right) = \sum_{B \subseteq A} m\left(B\right) \tag{6}$$

**Definition 2.4.** *(Plausibility function)*
*The plausibility function (Pl):* $2^{\Theta} \to [0, 1]$, *and satisfied:*

$$Pl\left(A\right) = \sum_{B \cap A \neq \phi} m\left(B\right) = 1 - Bel(\bar{A}) \tag{7}$$

As can be seen from the above, $\forall A \subseteq \Theta$ , $Bel(A) < Pl(A)$, $Bel(A)$, $Pl(A)$ are respectively the lower and upper limits of A, namely [Bel(A), Pl(A)], which indicates uncertain interval for A. According to Shafers explanation, the difference between the belief and the plausibility of a proposition A expresses the ignorance of the assessment for the proposition A. From the above, it has already shown that D-S theory has more advantages than probability, besides, D-S theory can reflect the information of events.

In addition, entropy plays an essential role in measurement of uncertainty.

### 2.2. *Deng entropy*

Deng proposed an interesting entropy, namely *Deng Entropy*. *Deng Entropy* can be well explained the phenomenon that there are 32 participants in a test, besides, the 32 participants simultaneously get 100 scores. *Deng Entropy* is an generalization of Shannon entropy [12].

**Definition 2.5.** *(Deng entropy)*

*Given a BPA, Deng entropy can be defined as:*

$$H_D = -\sum_{A\subseteq\Theta} m(A) log_2 \frac{m(A))}{2^{|A|}-1} \tag{8}$$

*Through a simple transformation, Deng Entropy can be rewrite as follows:*

$$H_D = \sum_{A\subseteq\Theta} m(A) log_2(2^{|A|}-1) - \sum_{A\subseteq\Theta} m(A) log_2 m(A) \tag{9}$$

where $m$ is a BPA defined on the frame of discernment $\Theta$, and $A$ is the focal element of m, $|A|$ is the cardinality of $A$. Besides, the term $\sum m(A) \times log_2(2^{|A|}-1)$ could be interpreted as a measure of total nonspecificity in the mass function $m$, and the term $-m(A) \times log_2 m(A)$ is the measure of discord of the mass function among various focal elements.

## 3. The Semi-Pascal Triangle of Maximum Deng Entropy

### *3.1. The Maximum Deng Entropy*

Given a BPA, the maximum Deng entropy is as follows:

$$H_{M-D} = -\sum m(A) \times log_2 \frac{m(A)}{2^{|A|}-1} \tag{10}$$

if and only if

$$m(A) = \frac{2^{|A|}-1}{\sum 2^{|A|}-1} \tag{11}$$

The more details about maximum Deng entropy refer to [16]. The specific BPA of having maximum Deng entropy was showed as follows.

**Example 3.1.** *Given the event space $\Theta = A$, there is only one phenomenon, as follows:*

$$m(A) = 1$$

$$H_D = 0$$

From the above, it can be easily found that only having one element means no uncertainty, that is to say, the event is certain.

**Example 3.2.** *Given the event space $\Theta = A, B$, the BPA of having maximum Deng entropy as follows:*

$$m(A) = \frac{1}{5}, m(B) = \frac{1}{5}, m(A,B) = \frac{3}{5}$$

$$H_{M-D} = 2.3219$$

From the above, the BPA of having maximum Deng entropy is has the phenomenon that having one element has the same distribution.

**Example 3.3.** *Given the event space* $\Theta = A, B, C$*, the BPA of having maximum Deng entropy as follows:*

$$m(A) = \frac{1}{19}, m(B) = \frac{1}{19}, m(C) = \frac{1}{19}$$

$$m(A,B) = \frac{3}{19}, m(B,C) = \frac{3}{19}, m(A,C) = \frac{3}{19}, m(A,B,C) = \frac{7}{19}$$

$$H_{M-D} = 4.2474$$

It can known that the element having the same cardinality has same BPA. In the next, it can be discussed.

*3.2. Further discussion*

*Table* 1 lists BPA of having maximum Deng entropy in various frame of discernment.

| Frame of Discrement | $\lvert X\rvert = 2^{\lvert 1\rvert} - 1$ | $\lvert X\rvert = 2^{\lvert 2\rvert} - 1$ | $\lvert X\rvert = 2^{\lvert 3\rvert} - 1$ | $\lvert X\rvert = 2^{\lvert 4\rvert} - 1$ |
|---|---|---|---|---|
| $\Theta = A$ | 1 | | | |
| $\Theta = A, B$ | (1/5,1/5) | 3/5 | | |
| $\Theta = A, B, C$ | (1/19,1/19,1/19) | (3/19,3/19,3/19) | 7/19 | |
| $\Theta = A, B, C, D$ | (1/68,1/68, 1/68,1/68) | (3/68,3/68,3/68, 3/68,3/68,3/68,3/68) | (7/68,7/68, 7/68,7/68) | 15/68 |

Table 1: The BPA of having Maximum Deng entropy

From the *Table* 1, it can be known that having the same cardinality has same BPA with maximum Deng entropy. Besides, the BPA can be rewrite the *Fig* .1. For $\Theta = A, B$, divide 1 equally into 5 small parts, and $m(A)$ and $m(B)$ have 1 small part, $m(A, B)$ has three small parts. Besides, analyse the Pascal triangle, it can be divided by a specific number of integers to form a fantastic fractal structure, as *Fig* .2 and *Fig* .3

| $2^{\|1\|-1}$ | $2^{\|2\|-1}$ | $2^{\|3\|-1}$ | $2^{\|4\|-1}$ | $2^{\|5\|-1}$ | $2^{\|6\|-1}$ | $2^{\|7\|-1}$ | $2^{\|8\|-1}$ | $2^{\|9\|-1}$ | $2^{\|10\|-1}$ | $2^{\|11\|-1}$ | $2^{\|12\|-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | |
| 2 | 1 | | | | | | | | | | |
| 3 | 3 | 1 | | | | | | | | | |
| 4 | 6 | 4 | 1 | | | | | | | | |
| 5 | 10 | 10 | 5 | 1 | | | | | | | |
| 6 | 15 | 20 | 15 | 6 | 1 | | | | | | |
| 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | | |
| 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | | |
| 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | | |
| 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | | |
| 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 | |
| 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 |

Figure 1: The Semi-Pascal Triangle of Maximum Deng Entropy

## 4. Conclusion

Deng entropy is an interesting, it can explain the phenomenon that all the student get the 100scores. Fractal plays an essential role in modern science. Combine fractal and Deng entropy can extend the application of Deng entropy. The paper discuss the relation between BPA and maximum Deng entropy. When BPA has the maximum Deng entropy, it presents the semi-Pascal trangle, besides, it can be divided by a specific number of integers to form a fantastic fractal structure.

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| $2^{\lvert 1\rvert-1}$ | $2^{\lvert 2\rvert-1}$ | $2^{\lvert 3\rvert-1}$ | $2^{\lvert 4\rvert-1}$ | $2^{\lvert 5\rvert-1}$ | $2^{\lvert 6\rvert-1}$ | $2^{\lvert 7\rvert-1}$ | $2^{\lvert 8\rvert-1}$ | $2^{\lvert 9\rvert-1}$ | $2^{\lvert 10\rvert-1}$ | $2^{\lvert 11\rvert-1}$ | $2^{\lvert 12\rvert-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | |
| 2 | 1 | | | | | | | | | | |
| 3 | 3 | 1 | | | | | | | | | |
| 4 | 6 | 4 | 1 | | | | | | | | |
| 5 | 10 | 10 | 5 | 1 | | | | | | | |
| 6 | 15 | 20 | 15 | 6 | 1 | | | | | | |
| 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | | |
| 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | | |
| 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | | |
| 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | | |
| 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 | |
| 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 |

Figure 2: The fractal structure of Maximum Deng Entropy

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| $2^{\|1\|-1}$ | $2^{\|2\|-1}$ | $2^{\|3\|-1}$ | $2^{\|4\|-1}$ | $2^{\|5\|-1}$ | $2^{\|6\|-1}$ | $2^{\|7\|-1}$ | $2^{\|8\|-1}$ | $2^{\|9\|-1}$ | $2^{\|10\|-1}$ | $2^{\|11\|-1}$ | $2^{\|12\|-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | |
| 2 | 1 | | | | | | | | | | |
| 3 | 3 | 1 | | | | | | | | | |
| 4 | 6 | 4 | 1 | | | | | | | | |
| 5 | 10 | 10 | 5 | 1 | | | | | | | |
| 6 | 15 | 20 | 15 | 6 | 1 | | | | | | |
| 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | | |
| 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | | |
| 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | | |
| 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | | |
| 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 | |
| 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 |

Figure 3: The fractal structure of Maximum Deng Entropy


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