ABOUT NON-CLOSEDNESS OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS SYSTEM FOR THE VISCOUS INCOMPRESSIBLE FLUID

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ABSTRACT. In this paper it is shown that the system of four equations formed by three-dimensional Navier-Stokes equations system for incompressible fluid and equation of continuity, is not closed, equation of continuity is excessive. This is because the three-dimensional Navier-Stokes equations system cannot have a bounded at infinity solutions to the Cauchy problem with a non-zero velocity field divergence.

The interest of the Navier-Stokes equations is so great that information about this question periodically appears in the newspaper news. The fact is that proven methods for analyzing partial differential equations in the case of the Navier-Stokes equations for the incompressible fluid do not work for an unknown reason. Equations remain elusively incomprehensible.

In year 2000, seven problems named as major mathematical problems of the third millennium were published on the website http://claymath.org/, one of those problems – Navier-Stokes equations. This problem is formulated by C. L. Fefferman by a range of questions regarding the solution of these equations, because until now it’s not possible to understand what properties do they have. The question about how good the set of Navier-Stokes equations describes behavior of real viscous fluids also remains open.

In the paper written by O.A. Ladyzhenskaya [1] and published in 2003 the problem of Navier-Stokes equations was formulated in the following way: «Do Navier-Stokes equations together with initial and boundary conditions give determining description of incompressible fluid dynamics or not?»

As of 2014, the situation with the problem of the Navier-Stokes equations became almost mystical. It is described in the paper by one of the leading researchers of this problem, Terence Tao [2]. In his paper, he actually comes to the conclusion that the existing methods of analysis cannot solve the problem. To date, no important results have been achieved in solving the problem of the Navier-Stokes equations.

Really mystical situation: the Navier-Stokes equations must describe real fluids, which behavior has a certain set of properties. These properties should be visible during the analysis of the equations, but this does not happen. More precisely, it happens only for the plane case of fluid motion, but not for the three-dimensional one. Suspicion occurs that the Navier-Stokes equations have something that goes unnoticed, some unique feature …

Let's consider the system of four equations formed by the three Navier-Stokes equations system

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 V_x}{\partial^2 x}+\frac{\partial^2 V_x}{\partial^2 y}+\frac{\partial^2 V_x}{\partial^2 z}\right)$$

$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V_y}{\partial^2 x}+\frac{\partial^2 V_y}{\partial^2 y}+\frac{\partial^2 V_y}{\partial^2 z}\right) \quad (1)$$

$$\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial^2 x}+\frac{\partial^2 V_z}{\partial^2 y}+\frac{\partial^2 V_z}{\partial^2 z}\right)$$

and the equation of continuity (the incompressibility condition of fluid)

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0 \quad (2)$$

Four equations include four unknown functions, and apparently, the given equations system is closed, however this impression is deceptive. Here it will be very useful to recall systems of linear algebraic equations. As is well known, in this case, the equality of the number of equations to the number of unknowns does not mean at all that the equations system is closed and uniquely solvable.

Let the three arbitrarily chosen functions $V_x=V_x(x,y,z,t)$, $V_y=V_y(x,y,z,t)$, $V_z=V_z(x,y,z,t)$ describe the velocity field. Further, it is assumed that the functions V_x, V_y, V_z are continuous together with their partial derivatives in coordinates to the third order inclusive. The second partial derivatives of these functions with respect to time and one of the coordinates are also continuous. Substituting these functions into the first equation (or any other equation, as the first equation is chosen for definiteness) of the system (1), it is possible by appropriate choice of the pressure function $P=P(x,y,z,t)$ to achieve the fulfillment of this equation. Thus, it can be said that one of the equations of the system (1) will always be satisfied for an arbitrarily specified velocity field V_x, V_y, V_z.

Let's rotate the velocity field V_x, V_y, V_z and the pressure field P relative to the coordinate system XYZ around the Z axis, at an arbitrary angle α (hereinafter, speaking of the rotation of the velocity field, it is always assumed that this also causes the rotation of the pressure field P, the rotation of the velocity field is considered for a fixed point in time). In this case, for an arbitrarily chosen velocity field, the first equation of the system (1) will no longer be satisfied. What conditions must the velocity field satisfy so that when it is rotated the first equation continues to be satisfied? The answer is very simple: in the initial state, the velocity field must satisfy the system of two equations, namely, the first and second equations of system (1). In this case, the two equations mentioned will be fulfilled after the rotation of the velocity field relative to the Z axis by an arbitrary angle α. This is possible to prove by the analysis of the field turning process. This analysis is not complicated, but very cumbersome, so further it will be described only in general terms. Turning the field around the Z axis at a small angle $d\alpha$, we use the field continuity property and require the fulfillment of the first equation. The scheme of the velocity field rotation is shown in Figure 1. The rotation of the velocity field occurs around the Z axis counterclockwise.

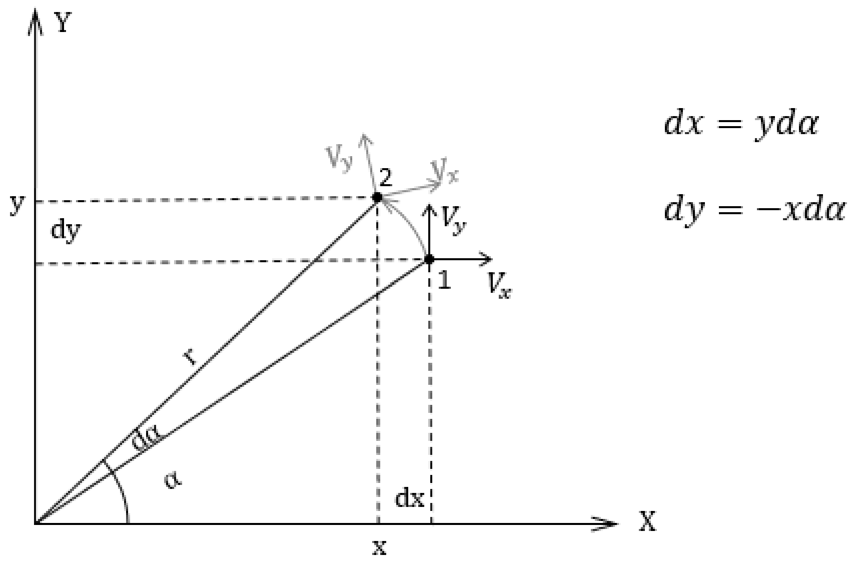

Figure 1. Velocity field rotation scheme.

After rotation of the velocity field, point 1 is in the position of point 2, for this point the values of all functions included in the first equation of system (1) are calculated. From the beginning, the velocities V_x, V_y, V_z and the derivative of the pressure $\partial P/\partial x$ are calculated. Further, the values of the functions after the velocity field rotation are taken to write with a line above, for example, $\bar{V}_x$, $\partial \bar{P}/\partial x$, etc. The functions values at point 2 before the velocity field rotation are written in ordinary letters, for example, V_x, $\partial P/\partial x$, etc. As can be clearly seen from Figure 1, the value of any function at point 2 after rotation, $\bar{f}(x, y, z)$ is determined by the value of this function at point 1 before the velocity field rotation, $f(x + dx, y + dy, z)$. In this case, it is also necessary to take into account the rotation of the vectors by the angle $d\alpha$. The following result will be obtained:

$$\begin{aligned} \bar{V}_x &= V_x + \frac{\partial V_x}{\partial x} dx + \frac{\partial V_x}{\partial y} dy - V_y d\alpha = V_x + \frac{\partial V_x}{\partial x} y d\alpha - \frac{\partial V_x}{\partial y} x d\alpha - V_y d\alpha \\ \bar{V}_y &= V_y + \frac{\partial V_y}{\partial x} dx + \frac{\partial V_y}{\partial y} dy + V_x d\alpha = V_y + \frac{\partial V_y}{\partial x} y d\alpha - \frac{\partial V_y}{\partial y} x d\alpha + V_x d\alpha \end{aligned} \quad (3)$$

$$\bar{V}_z = V_z + \frac{\partial V_z}{\partial x}dx + \frac{\partial V_z}{\partial y}dy = V_z + \frac{\partial V_z}{\partial x}yd\alpha - \frac{\partial V_z}{\partial y}xd\alpha$$

$$\frac{\partial \bar{P}}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial x^2}dx + \frac{\partial^2 P}{\partial x \partial y}dy - \frac{\partial P}{\partial y}d\alpha = \frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial x^2}yd\alpha - \frac{\partial^2 P}{\partial x \partial y}xd\alpha - \frac{\partial P}{\partial y}d\alpha \tag{3}$$

Derivatives of the velocities with respect to coordinates and time, for example $\partial\bar{V}_x/\partial x,\ \partial\bar{V}_x/\partial t,$ are calculated by differentiating formulas (3). All functions calculated in this way are substituted into the first equation of system (1). After the multiplication operation, in this equation only the members of zero and first order of smallness in $d\alpha$ are left, similar members are grouped. Demanding the fulfillment of the first equation of system (1), we obtain the condition: in the initial state, the velocity field must satisfy the first and second equation of system (1).

Physically, this result is absolutely clear. The equations system (1) is the record of the impulse conservation law for the three components of impulse. If the velocity field satisfies only the first equation of the system (1), this means that in this field the impulse conservation law is realized only for X component of impulse. When the velocity field rotates around the Z axis, the contribution of the Y component of impulse will be made in the X component. Hence it is clear that the conservation law of the X component of the impulse can be fulfilled after the field is rotated only if the conservation law of the X and Y components of impulse were fulfilled before the field was rotated.

Similarly, if the first equation of the system (1) is satisfied when the field is rotated around the Y axis, then this field will satisfy the first and third equations of system (1). And finally, if the first equation of the system (1) is satisfied when the field is rotated around to any arbitrary directional axis, then such a field will satisfy all three equations of system (1).

Two conclusions can be drawn from the above. The first conclusion: system of equations (1) is invariant to the velocity field rotation with respect to the XYZ coordinate system.

The second conclusion: the system of equations (1) can be replaced by one of the equations of this system (any), and the requirement of its implementation for an arbitrary position of the velocity field relative to the XYZ coordinate system.

So far nothing has been said about the velocity field divergence. Consider velocity field with nonzero divergence, i.e.

$$\operatorname{div} \boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \neq 0$$

Nonzero divergence of the velocity field in the incompressible fluid means that the sources (sinks) of the fluid are continuously distributed throughout the fluid volume. This will violate the mass conservation law, which from a physical point of view looks absolutely ridiculous. Consequently, there are no physical principles explaining nonzero divergence in incompressible fluid. Then, abstracting from the physical sense, mathematically, in the most general form it can be written as

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = D(x, y, z, t) \qquad (4)$$

where $D(x, y, z, t)$ - is arbitrarily given, continuous function of coordinates and time.

Let's suppose that there is a velocity field that satisfies the three equations of system (1) and equation (4). This means that the first (or any other) equation of system (1) and equation (4) must both be fulfilled when the velocity field is rotated around an arbitrarily chosen axis. The first equation of system (1) will be satisfied due to the assumptions made, but will equation (4) be fulfilled in this case?

The left side of equation (4) is the first invariant of the strain rate tensor, I_1. For the point (x, y, z) given in the velocity field, the value of the invariant I_1 does not depend on the choice of coordinate system. Therefore, for the point in question, the divergence value will not change after the velocity field rotation. But this point will occupy a new position (x_1, y_1, z_1) in the XYZ coordinate system. It can be said that the divergence field will rotate as a whole with respect to the XYZ coordinate system together with the velocity field. But then, at the point in question (x, y, z), the divergence value will change. In this case, the right side of the equation, the function $D(x, y, z, t)$, will not change. This function is initially given, it is tied to the XYZ coordinate system and is not involved in the rotation. This shows that the implementation of equation (4) is possible only in one case, if

$$D(x, y, z, t) = const$$

(this constant may be a function only of time, but in this case, it doesn't matter). Otherwise, equation (4) will not have the property of invariance to the velocity field rotation. Therefore, it cannot be simultaneously performed for all equations of system (1). In this case, in the system of equations (1) only one equation can be performed.

The conclusion that $D = const$ can be illustrated. The Figure 2 schematically shows the case of the velocity field (the divergence field) rotation. Color density characterizes the magnitude of the velocity field divergence. Left part (a), the initial position of the velocity field. Upper part is the divergence of the velocity field div $\boldsymbol{V}$. Lower part is the divergence distribution defined by the function $D(x, y, z)$, equation (4) is satisfied. Right side (b), the velocity field is rotated by 180 °, equation (4) is not satisfied. It is clearly seen that equation (4) can be satisfied only if $D(x, y, z) = const$. Here you can argue that the function D may have a radial symmetry. However, the choice of a different position of the rotation axis makes this type of symmetry impossible.

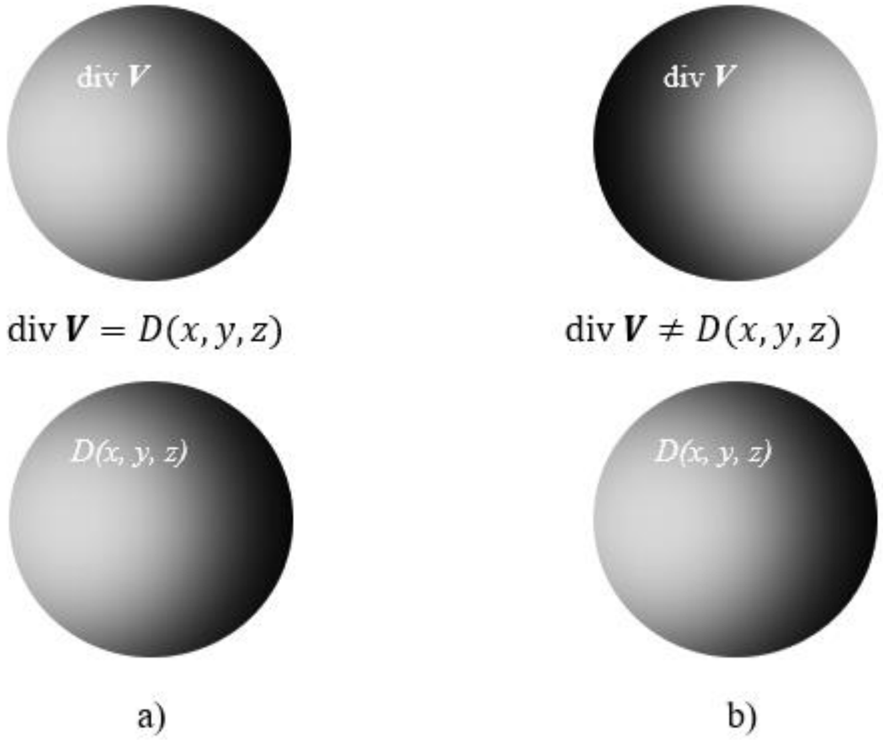

Figure 2. Explanatory diagram.

So div $\boldsymbol{V} = const$, this does not mean at all that there necessarily exist solutions of system (1) for which div $\boldsymbol{V} = const$. This means that they may exist, since the method used here does not allow such solutions to be cut off. However, there is another way to do that. As mentioned above, the constant divergence in the incompressible fluid is a continuous distribution of the sources (sinks) of the fluid itself by volume of the fluid. Moreover, the distribution density of sources is constant throughout the volume of the fluid. Applying the Cauchy problem, the presence of a velocity field with a constant divergence will lead to an unlimited increase in velocities (or any one velocity) with increasing distance from the origin. This means that all solutions of the system (1) that are bounded at infinity will have zero velocity field divergence. Thus, the solutions of the three-dimensional Cauchy problem for an incompressible fluid will be all solutions of system (1) bounded at infinity. All such solutions will automatically satisfy continuity equation (2), i.e. the continuity equation turns out to be unnecessary.

An interesting result can be obtained when trying to solve some simplest boundary problem for the Navier-Stokes equations using a velocity field with constant divergence. So, for example, one-dimensional problem of fluid motion between flat walls in the case of a divergence-free velocity field becomes two-dimensional in case of constant divergence. In the set of equations obtained in this case, internal contradictions arise, with the result that the solution of this problem simply does not exist.

The system of equations (1) consists of three equations, containing four unknown functions, hence it is not closed. Solution for this equations system can be performed according to the following scheme, for example. You can arbitrarily choose one of the velocities, for example $V_x(x, y, z, t)$, choosing a function that tends to zero at infinity and fades with the time and has integrals that are bounded for any points of time t

$$\iiint_{-\infty}^{+\infty} |V_x| dxdydz$$

$$\iiint_{-\infty}^{+\infty} V_x^2 dxdydz$$

The boundedness of the first integral means the boundedness of the fluid impulse (momentum) associated with the velocity V_x. The boundedness of the second integral means the boundedness of the kinetic energy. The value of the integral

$$\iiint_{-\infty}^{+\infty} V_x dxdydz$$

doesn't have to depend from time, this is a requirement of the impulse conservation law. Perhaps there are some other restrictions on the function V_x, at the moment it does not matter.

Substituting V_x to equations system (1), we will get a closed system of three equations for three unknown functions V_y, V_z and P. Since V_x was chosen arbitrarily, it is clear that there are infinitely many solutions. Basically, it cannot be argued that absolutely all solutions obtained in this way will be limited at infinity. But it is also obvious that there will be solutions, and there will be infinitely many of them. A physically adequate system of equations (1) describing a dissipative process cannot respond to a localized and energetically limited effect by an unlimited increase in velocities at infinity. All said does not exclude a local unlimited growth of velocities (blowup), but with a limitation on the impulse and kinetic energy of the entire mass of the fluid.

So, if the equations system (1) has infinitely many solutions, is there the only solution for three-dimensional Cauchy problem for a given initial velocity field? Maybe not, the loss of determinism is quite possible. This question remains open in this paper, it requires additional research.

All mentioned above about non-closedness of equations system is relative also for two-dimensional case of fluid flow. Two-dimensional system of equations is non-closed as well, equation of continuity is excessive. However there is no such arbitrariness as in three-dimensional case. Let's look at the two-dimensional system of equations:

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial^2 x} + \frac{\partial^2 V_x}{\partial^2 y}\right)$$

$$\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial^2 x} + \frac{\partial^2 V_y}{\partial^2 y}\right) \qquad (5)$$

This system of equations contain three unknown functions and it seems that it's possible to choose one of velocities while second velocity and pressure can be found from system of equations (5). In this case, this can not be done, the obstacle is the equation of continuity

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \qquad (6)$$

All solutions of equations system (5) bounded at infinity satisfy the equation of continuity (6), but not all solutions of continuity equation (6) will satisfy the equations system (5). If arbitrarily choose one of velocities we can find the second velocity from equation of continuity (with accuracy of function of only one coordinate and time). Then we obtain the solution of the continuity equation (6), i.e. we find the velocity field. And it is not necessary that this velocity field will satisfy the system of equations (5). This equations system, for given initial conditions, has only one solution, as shown in [3].

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