

A Factorial Identity Resulting from the Orthogonality Relation of the Associated Laguerre Polynomials

Spiros Konstantogiannis

spiroskonstantogiannis@gmail.com

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Abstract

Plugging the closed-form expression of the associated Laguerre polynomials into their orthogonality relation, the latter reduces to a factorial identity that takes a simple, non-trivial form for even-degree polynomials.

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1. Introduction

The associated Laguerre polynomials $L'_\lambda(x)$ are λ -degree polynomial solutions to the associated Laguerre differential equation

$$xy''(x) + (\nu + 1 - x)y'(x) + \lambda y(x) = 0$$

for $\nu, \lambda = 0, 1, \dots$ [1, 2]. If $\nu = 0$, the associated Laguerre polynomials reduce to the Laguerre polynomials $L_\lambda(x)$ [1, 2].

The polynomials $L'_\lambda(x)$ are given by the closed-form expression [1, 3]

$$L'_\lambda(x) = \sum_{n=0}^{\lambda} \frac{(-1)^n}{n!} \binom{\lambda + \nu}{\lambda - n} x^n,$$

where $\binom{\lambda + \nu}{\lambda - n}$ is the binomial coefficient, i.e.

$$\binom{\lambda + \nu}{\lambda - n} = \frac{(\lambda + \nu)!}{(\lambda - n)!(\nu + n)!}$$

The polynomials $L'_\lambda(x)$ satisfy the orthogonality relation [1-3]

$$\int_0^{\infty} dx x^\nu \exp(-x) L'_\lambda(x) L'_{\lambda'}(x) = 0,$$

for $\lambda \neq \lambda'$.

2. The factorial identity

Using the closed-form expression of the associated Laguerre polynomials, the orthogonality integral takes the form

$$\begin{aligned} & \int_0^{\infty} dx x^\nu \exp(-x) L'_\lambda(x) L'_{\lambda'}(x) = \\ & = \int_0^{\infty} dx x^\nu \exp(-x) \left(\sum_{m=0}^{\lambda} \frac{(-1)^m}{m!} \binom{\lambda + \nu}{\lambda - m} x^m \right) \left(\sum_{n=0}^{\lambda'} \frac{(-1)^n}{n!} \binom{\lambda' + \nu}{\lambda' - n} x^n \right) = \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\infty} dx x^{\nu} \exp(-x) \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} x^{m+n} = \\
&= \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} \int_0^{\infty} dx x^{m+n+\nu} \exp(-x)
\end{aligned}$$

That is

$$\int_0^{\infty} dx x^{\nu} \exp(-x) L_{\lambda}^{\nu}(x) L_{\lambda'}^{\nu}(x) = \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} \int_0^{\infty} dx x^{m+n+\nu} \exp(-x)$$

The integral on the right-hand side is easily calculated using the Gamma function, since

$$\int_0^{\infty} dx x^{m+n+\nu} \exp(-x) = \Gamma(m+n+\nu+1) = (m+n+\nu)!,$$

and then

$$\begin{aligned}
&\int_0^{\infty} dx x^{\nu} \exp(-x) L_{\lambda}^{\nu}(x) L_{\lambda'}^{\nu}(x) = \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} (m+n+\nu)! = \\
&= \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \frac{(\lambda+\nu)!}{(\lambda-m)!(m+\nu)!} \frac{(\lambda'+\nu)!}{(\lambda'-n)!(n+\nu)!} (m+n+\nu)! = \\
&= (\lambda+\nu)! (\lambda'+\nu)! \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n} (m+n+\nu)!}{m!n! (\lambda-m)! (\lambda'-n)! (m+\nu)! (n+\nu)!}
\end{aligned}$$

That is

$$\int_0^{\infty} dx x^{\nu} \exp(-x) L_{\lambda}^{\nu}(x) L_{\lambda'}^{\nu}(x) = (\lambda+\nu)! (\lambda'+\nu)! \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n} (m+n+\nu)!}{m!n! (\lambda-m)! (\lambda'-n)! (m+\nu)! (n+\nu)!}$$

Then, since $(\lambda+\nu)! (\lambda'+\nu)! \neq 0$, the orthogonality relation of the associated Laguerre polynomials reduces to the following factorial identity

$$\sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n} (m+n+\nu)!}{m!n! (\lambda-m)! (\lambda'-n)! (m+\nu)! (n+\nu)!} = 0 \tag{1}$$

where $\lambda, \lambda', \nu = 0, 1, \dots$ and $\lambda \neq \lambda'$.

If $\lambda' = 0$, then $n = 0$ too, and (1) reads

$$\sum_{m=0}^{\lambda \neq 0} \frac{(-1)^m (m+\nu)!}{m!0!(\lambda-m)!(0-0)!(m+\nu)!(0+\nu)!} = 0 \stackrel{0!=1}{\Rightarrow} \frac{1}{\nu!} \sum_{m=0}^{\lambda \neq 0} \frac{(-1)^m}{m!(\lambda-m)!} = 0,$$

and since $1/\nu!$ is non-zero,

$$\sum_{m=0}^{\lambda \neq 0} \frac{(-1)^m}{m!(\lambda-m)!} = 0 \quad (2)$$

The series in (2) has $\lambda+1$ terms. If λ is odd, the series has an even number of terms, while m and $\lambda-m$ have different parity, i.e. if m is even/odd then $\lambda-m$ is odd/even, and thus

$$\frac{(-1)^{\lambda-m}}{(\lambda-m)!(\lambda-(\lambda-m))!} = \frac{(-1)^{\lambda-m}}{(\lambda-m)!m!} = -\frac{(-1)^m}{m!(\lambda-m)!}$$

Then, the terms with $m=0$ and $m=\lambda$ are opposite, as are the terms with $m=1$ and $m=\lambda-1$, as are the terms with $m=2$ and $m=\lambda-2$, etc. Thus, in this case, the series consists of $(\lambda+1)/2$ pairs of opposite terms, and the identity (2) is rather trivial. However, if λ is even, m and $\lambda-m$ have the same parity, and also the series has an odd number of terms, thus it does not consist of pairs of opposite terms. Therefore, in the case where λ is even, the identity (2) is not trivial. Moreover, setting $\lambda \rightarrow 2\lambda$, with $\lambda = 1, 2, \dots$, the series in (2) is written as

$$\begin{aligned} \sum_{m=0}^{2\lambda} \frac{(-1)^m}{m!(2\lambda-m)!} &= \sum_{m=0}^{\lambda} \frac{(-1)^{2m}}{(2m)!(2\lambda-2m)!} + \sum_{m=1}^{\lambda} \frac{(-1)^{2m-1}}{(2m-1)!(2\lambda-(2m-1))!} = \\ &= \sum_{m=0}^{\lambda} \frac{1}{(2m)!(2(\lambda-m))!} - \sum_{m=1}^{\lambda} \frac{1}{(2m-1)!(2(\lambda-m)+1)!}, \end{aligned}$$

and (2) takes the form

$$\sum_{m=0}^{\lambda} \frac{1}{(2m)!(2(\lambda-m))!} = \sum_{m=1}^{\lambda} \frac{1}{(2m-1)!(2(\lambda-m)+1)!} \quad (3)$$

with $\lambda = 1, 2, \dots$

Let us verify (3) for $\lambda = 1, 2, 3$.

For $\lambda = 1$, we have

$$\sum_{m=0}^1 \frac{1}{(2m)!(2(1-m))!} = \frac{1}{0!2!} + \frac{1}{2!0!} = \frac{1}{2} + \frac{1}{2} = 1$$

and

$$\sum_{m=1}^1 \frac{1}{(2m-1)!(2(1-m)+1)!} = \frac{1}{1!(2(1-1)+1)!} = \frac{1}{1!1!} = 1$$

For $\lambda = 2$, we have

$$\sum_{m=0}^2 \frac{1}{(2m)!(2(2-m))!} = \frac{1}{0!4!} + \frac{1}{2!2!} + \frac{1}{4!0!} = \frac{2}{4!} + \frac{1}{2!2!} = \frac{1}{12} + \frac{1}{4} = \frac{1}{12} + \frac{3}{12} = \frac{4}{12} = \frac{1}{3}$$

and

$$\sum_{m=1}^2 \frac{1}{(2m-1)!(2(2-m)+1)!} = \frac{1}{1!3!} + \frac{1}{3!1!} = \frac{2}{3!} = \frac{1}{3}$$

For $\lambda = 3$, we have

$$\begin{aligned} \sum_{m=0}^3 \frac{1}{(2m)!(2(3-m))!} &= \frac{1}{0!6!} + \frac{1}{2!4!} + \frac{1}{4!2!} + \frac{1}{6!0!} = \frac{2}{6!} + \frac{2}{2!4!} = \frac{1}{3*4*5*6} + \frac{1}{2*3*4} = \\ &= \frac{1}{3*4*5*6} + \frac{15}{3*4*5*6} = \frac{16}{3*4*5*6} = \frac{4}{3*5*6} = \frac{2}{3*5*3} = \frac{2}{45} \end{aligned}$$

and

$$\begin{aligned} \sum_{m=1}^3 \frac{1}{(2m-1)!(2(3-m)+1)!} &= \frac{1}{1!5!} + \frac{1}{3!3!} + \frac{1}{5!1!} = \frac{2}{5!} + \frac{1}{3!3!} = \frac{1}{3*4*5} + \frac{1}{2*3*2*3} = \\ &= \frac{1}{3*4*5} + \frac{1}{3*4*3} = \frac{3}{3*4*5*3} + \frac{5}{3*4*3*5} = \frac{8}{3*4*5*3} = \frac{2}{3*5*3} = \frac{2}{45} \end{aligned}$$

3. References

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- [3] <http://mathworld.wolfram.com/AssociatedLaguerrePolynomial.html>.