Theorem on the Convergence Speed of Tetration

Marco Ripà

sPIqr Society, World Intelligence Network
Rome, Italy
e-mail: marcokrt1984@yahoo.it

Published Online: 21 Nov. 2018 **Revised:** 24 Nov. 2018

Abstract: We provide a preliminary proof of Ripà's Conjecture 3 about the convergence speed of tetration, published in October 2018, which states that $\forall v \in \mathbb{N} \setminus \{0\}$, $\exists a$, not a multiple of 10, such that $V(a) = v$, where $V(a)$ represents the convergence speed of the tetration ${}^{b}a$.

Keywords: Number theory, Power tower, Tetration, Chinese reminder theorem, Charmichael function, Euler's totient funtion, Exponentiation, Convergence speed, Modular arithmetic, Stable digits, Rightmost digits.

2010 Mathematics Subject Classification: 11A07, 11F33.

1. Introduction

We provide a preliminary proof of Ripà's Conjecture 3 [1], which states that it is possible to find (at least) one base a for any arbitrarily large natural number $v \coloneqq V(a)$, where $V(a)$ represents the "convergence speed" of the tetration ${}^{b}a = a^{a^{a^{\cdots}}}$ (b-times).
Let we now introduce the definition of convergence speed of ${}^{b}a$ [1] .

Defintion 1: Let $a \in \mathbb{N} \setminus \{1\}$ be an arbitrary base which is not a multiple of 10 and let $b \in \mathbb{N} \setminus \{0, 1\}$ be such that ${}^{(b-1)}a \equiv {}^{b}a \ (mod\ 10^{d}) \wedge {}^{(b-1)}a \not\equiv {}^{b}a \left(mod\ 10^{(d+1)}\right)$, where $d \in \mathbb{N}$, we consider $V(a) \ni' \ {}^{b}a \equiv {}^{(b+1)}a \left(mod\ 10^{((d+V(a)))}\right) \wedge {}^{b}a \not\equiv {}^{(b+1)}a \left(mod\ 10^{(d+V(a)+1)}\right)$.

2. Proof of Conjecture 3

In this section we present Theorem 1, the old Conjecture 3, thanks to the the preliminary version of the first proof of Ripà's third conjecture. This brief proof is based on some results published in [3-4]. Further improvements will follow in the near future, in order to complete the original paper [1].

Theorem 1: $\forall v \in \mathbb{N}\backslash\{0\}$, $\exists a$, not a multiple of 10, such that $V(a) = v$.

Remark: In order to prove Theorem 1, it is sufficient to verify that, for any n-digits long base $a \coloneqq a_n \dots a_2 a_1$, where $a_1 = a_2 = \cdots = a_n = 9$, $V(a = 9 \dots 9) = n$ $(\forall b)$ (see [2], pp. 25-26).

Proof: Let $a := 9 \dots 9$, where a is a n-digits long string of trail 9s, Charmichael's lambda function assures that $a^{(\lambda(10^n)+1)} \equiv a(mod\ 10^n)$, because 9...9 and 10 are relatively prime.
For any $n \geq 4$, $\lambda(10^n) = 5 \cdot 10^{(n-2)}$, so ${}^{2}9 \dots 9 \equiv 9 \dots 9^{(200 \cdot \lambda(10^n)-1)} \equiv 1 \cdot \dots \cdot 1 \cdot 9 \dots 9^{(\lambda(10^n)-1)}$.
Since $9 \dots 9^2 \equiv 1(mod\ 10^n)$, it follows that $9 \dots 9^{9 \dots 9} \cdot 1 \equiv 9 \dots 9^{9 \dots 9} \cdot 9 \dots 9^2 \equiv 9 \dots 9^{(9 \dots 9+2)} \equiv 9 \dots 9^{(\lambda(10^n)+1)} \equiv 9 \dots 9(mod\ 10^n)$. Hence $9 \dots 9^{9 \dots 9} \equiv 9 \dots 9(mod\ 10^n)$.
Therefore, for any $n(a) \geq 4$, ${}^{(b \geq 2)}99 \dots 9 \equiv {}^{(b-1)}99 \dots 9 \equiv \cdots \equiv 99 \dots 9\ (mod\ 10^n)$.
This proves that $V(a = 9 \dots 9, b = 1) = n$, for any $n \geq 4$.
Let $n < 4$, there are only 3 cases:
- $999^{999} \equiv 999^{10 \cdot \lambda(1000)-1}(mod\ 1000)$, thus ${}^{2}999 \equiv {}^{2}999 \cdot 999^2 \equiv 999^1(mod\ 1000)$.
Therefore, $V(a = 999, b = 1) = 3 = n$.
- $99^{99} \equiv 99^{5 \cdot \lambda(100)-1}(mod\ 100)$, thus ${}^{2}99 \equiv {}^{2}99 \cdot 99^2 \equiv 99^1(mod\ 100)$.
Therefore, $V(a = 99, b = 1) = 2 = n$.
- $9^9 \equiv 9^{2 \cdot \lambda(10)+1}(mod\ 10)$, thus ${}^{2}9 \equiv 9^1\ (mod\ 10)$. Therefore, $V(a = 9, b = 1) = 1 = n$.
In order to complete the proof of the general conjecture, we need to prove that $V(a = 9 \dots 9) = n$ also for any $b \geq 2$, which means that, for any $b \in \mathbb{N}$, there is an exactly $(b \cdot n)$-digits long sequence of stable figures $x(a, b) \coloneqq x_{b \cdot n} x_{(b \cdot n-1)} \dots x_{(n+1)} 9 \dots 9$ at the end of the result of the tetration ${}^{b}9$, so ${}^{b}9 \dots 9 \equiv {}^{(b+1)}9 \dots 9\big(mod\ 10^{(b \cdot n)}\big)$.
Moreover, $\forall k \in \mathbb{N}_0$, we have that ${}^{b}9 \dots 9 \equiv {}^{(b+k)}9 \dots 9\big(mod\ 10^{(b \cdot n)}\big)$.
This result follows from [4] (*ibid.,* see Theorem 3, case 1), considering that $\gcd(a \coloneqq 9 \dots 9, 10) = 1$ and $\varphi(10^n) = 4 \cdot 10^{(n-1)}$, $\forall n \in \mathbb{N}\backslash\{0\}$.
J. Germain, completed and extended the aforementioned outcome via the Chinese Reminder Theorem (see [4], Sections 4 and 5).
Referring to [3], Lemma 2 assures the same result (we can invoke it since $\gcd(9 \dots 9, 10^n) = 1$).
In fact, $n < x \coloneqq a$ and the n-digits long base $a = 99 \dots 9$ satisfies the congruence relation $9 \dots 9^x :\equiv x(mod\ \varphi(10^n)$. □

$\forall n \in \mathbb{N}\backslash\{0\}$, $\varphi\big(10^{(n+1)}\big) = 4 \cdot 10^n$ and the Fermat-Euler Theorem assures us that, $\forall k \in \mathrm{N}_0$, $99 \dots 9^{(2 \cdot k)} \equiv 1(mod\ 4 \cdot 10^n)$. We know that, $x = 99 \dots 9 = a$, since $99 \dots 9^{99 \dots 9} :\equiv 99 \dots 9(mod\ \varphi(10^{(n+1)}))$. In fact, $1 \equiv 99 \dots 9^{(2 \cdot k)} \equiv 99 \dots 9^{(49 \dots 9 \cdot 2+1)}(mod\ 4 \cdot 10^n)$ and it follows that $99 \dots 9^{99 \dots 9} \equiv 99 \dots 9 \cdot 99 \dots 9^{99 \dots 8} \equiv 99 \dots 9 \cdot 1 \equiv 99 \dots 9(mod\ \varphi(10^{(n+1)}))$.
We can now use Lemma 2 from [3] and say that $99 \dots 9^{99 \dots 9^x} \equiv 99 \dots 9^x\big(mod\ 10^{(n+1)}\big) \Rightarrow {}^{2}99 \dots 9 \equiv {}^{3}99 \dots 9\big(mod\ 10^{(n+1)}\big)$.
Thus, our recurrence relation holds $(10^{(n+1)} = 2^{(n+1)} \cdot 5^{(n+1)}$ and $s(b) = b$ [3]) and we can also prove this by induction (see [3], Proposition 11), $V(a = 99 \dots 9, b \geq 1) = n + 1$, for any $n \in \mathbb{N}$ (which implies that $V(a = 99 \dots 9)$ is constant for any b).

3. Conclusion

The proof provided in Section 2 is not enough in order to prove the general hypothesis [1], but it shows how it is possible to find one (or more) base characterized by any convergence speed $V(a) = n$, where n goes from 1 to any arbitrarily large natural number.
Thanks in advance to everybody who will contribute to help us to improve this work and to prove the other important conjectures introduced in [1].

References

[1] Ripà, M., On the Convergence Speed of Tetration [v2], *viXra*, 1810.0223, 23 Oct. 2018.

[2] Ripà, M., La strana coda della serie n^n^...^n. *UNI Service*, Trento, 2011.

[3] Germain, J., On the Equation $a^x \equiv x \pmod{b}$. *Integers: Learning, Memory, and Cognition*, 9(6), 2009, 629–638.

[4] Urroz, J. J., Yebra J. L. A., On the Equation $a^x \equiv x \pmod{b^n}$, *Journal of Integer Sequences*, 8(8), 2009.