A general definition of division in a field

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Abstract. The historical definition of division by zero given by Brahmagupta is correct.

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Let F be a field. The definition of division / in F may be stated as $a/b = ab^{-1}$ in the case $b \neq 0$, and a/b is not defined if $b = 0$. But we can easily define / including the case $b = 0$ as follows:

$$a/b = ab^{-1} \text{ if } b \neq 0, \text{ and } a/b = ab \text{ if } b = 0.$$

Notice that this is well-defined. It is also made in a natural way. For if we try to choose an element $x \in F$ such that $a/b = ax$ in the case $b = 0$, there is only one choice $x = 0$ remaining, since $a \longmapsto a^{-1}$ is a one-to-one correspondence on $F\backslash\{0\}$.

Essentially the same definition was made by the Indian mathematician and astronomer Brahmagupta (598-665+α). But his definition has been considered wrong until today. However if his definition is wrong, the above simple definition must be wrong, or the concept of the field must be wrong. Our definition is also justified by Takahasi's theorem [2].

Theorem 1. *Let F be a filed with characteristic different from 2. Let f be a mapping $f : F \times F \to F$ satisfying*

(i) $f(a, b) = ab^{-1}$ *if* $b \neq 0$,

(ii) $f(a, b)f(c, d) = f(ac, bd)$.

Then we have $f(a, 0) = 0$ for any $a \in F$.

Our general definition of division is essentially the same to defining the inverse of 0 being 0 itself, i.e., $0^{-1} = 0$. Hence the definition of the inverse should be renewed under our definition as follows:

a and b are inverse to each other if and only if $a = 1/b$ holds.

Since our general definition of division has not been considered until today, it works as something like a new axiom when we consider mathematics over a field. Therefore we have to check whether the definition gives us fruitful and meaningful results. It should get canceled, if it do not give any meaningful results. Experimenting works with this aspect have been made for more than four years under the leadership of Saburou Saitoh, and a listing with more than eight hundreds interesting results is made, some of which can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. We now consider that too much time has passed to admit this definition.

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