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Anomaly in sign function - probability function integration

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Abstract In the paper it is demonstrated that integration of products of sign functions and probability density functions such as in Bell's formula for ± 1 measurement functions, leads to inconsistencies.

Keywords Inconsistency, Bell's theorem.

1 Introduction

In 1964, John Bell wrote a paper [1] on the possibility of hidden variables [2] causing the entanglement correlation $E(\vec{a},\vec{b})$ between two particles. In his famous paper, Einstein [2] argued that the quantum description must be supplemented with extra variables to explain the entanglement phenomenon. von Neuman [4] presented a mathematical proof that any hidden variables theory is in conflict with quantum mechanics. However, one can doubt if von Neuman's view on the matter was completely related to the physics [5]. An interested reader can find more detail in a preprint of Dieks [6]. In the present paper, an inconsistency in the starting formula of Bell [1] will be demonstrated.

Bell based his hidden variable description on particle pairs with entangled spin, originally formulated by Bohm [3]. Bell used hidden variables λ that are elements of a universal set Λ and are distributed with a density $\rho(\lambda) \geq 0$. Suppose, $E(\vec{a},\vec{b})$ is the correlation between measurements with distant A and B that have unit-length, i.e. $||\vec{a}|| = ||\vec{b}|| = 1$, real 3 dim parameter vectors $\vec{a}$ and $\vec{b}$.

Then with the use of the λ we can write down the classical probability correlation between the two simultaneously measured spins of the particles. This is what we will call Bell's formula.

$$E(\vec{a},\vec{b}) = \int_{\lambda\in\Lambda} \rho(\lambda) A(\vec{a},\lambda) B(\vec{b},\lambda) d\lambda \tag{1.1}$$

The spin measurement functions are, $A(\vec{a},\lambda) \in \{-1,1\}$ and $B(\vec{b},\lambda) \in \{-1,1\}$. The probability density is normalized, $\int \rho(\lambda) d\lambda = 1$.

2 Example of inconsistency

Bell's formula (1.1) is general. That means that it has to be valid for all kinds sub-cases where ± 1 functions are used.

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In the present example we will concentrate the attention on two different expressions for e.g. $A = A(\vec{a}, \lambda) \in \{-1, 1\}$. Let us also make use of one single Gaussian density function and change the notation slightly. We have

$$\mathbb{P}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \tag{2.1}$$

Hence, $\rho_{Gauss}(x) = \frac{d}{dx}\mathbb{P}(x)$ in this example. To be more precise, we concentrate on a sub-case of Bell's formula in which e.g.

$$E = \int_{-\infty}^{+\infty} A(a, x) \rho_{Gauss}(x) dx \tag{2.2}$$

is a part of the computation of a more complete correlation. As an instructive example we may look at the case where $A(\vec{a}, \lambda) = A^{(1)}(a_1, \lambda_1) A^{(2)}(a_2, \lambda_2) A^{(3)}(a_3, \lambda_3) = \pm 1$. Each $A^{(m)} = \pm 1$, with, $m = 1, 2, 3$. Similarly for B with $B(\vec{b}, \lambda) = B^{(1)}(b_1, \lambda_4) B^{(2)}(b_2, \lambda_5) B^{(3)}(b_3, \lambda_6) = \pm 1$ and a six-fold normal density with $\{\lambda_k\}_{k=1}^{6}$ independent variables. The subsequently presented cases, already anticipated in (2.2), refer to e.g. $A^{(1)}(a_1, \lambda_1)$. For ease of notation a represents a_1 and x represents λ_1. Each $B^{(m)} = \pm 1$, with, $m = 1, 2, 3$.

2.1 Definition of to be used ± 1 functions

Let us define, for $a, x \in \mathbb{R}$,

$$A_1(a, x) = \frac{4H(x-a) - 1}{2H(x-a) + 1} = \begin{cases} +1, & x \geq a \\ -1, & x < a \end{cases} \tag{2.3}$$

and

$$A_2(a, x) = \frac{2H(x-a) + 1}{4H(x-a) - 1} = \begin{cases} +1, & x \geq a \\ -1, & x < a \end{cases} \tag{2.4}$$

To remain close to the physics of the problem, we can take $a \in [-1, 1]$. In both expressions, $H(x) = 1 \Leftrightarrow x \geq 0$ and $H(x) = 0 \Leftrightarrow x < 0$. The closed limit Heaviside form is here equal to:

$$H(x) = \lim_{n \to \infty} \exp\left(-\frac{e^{-nx}}{n}\right) \geq 0 \tag{2.5}$$

It must be noted that (2.3) and (2.4) are perfectly in order when in a Bell formula we are looking for $A \in \{-1, 1\}$. Equation (2.5) is a valid expression for the Heaviside function. No need to absolutely must have $H(0) = 1/2$.

2.2 A_1 form integration

Let us compute with partial integration

$$E_1 = \int_{-\infty}^{+\infty} \left(\frac{d}{dx}\mathbb{P}(x)\right) \frac{4H - 1}{2H + 1} dx \tag{2.6}$$

Here and below, H, is an abbreviation of $H(x - a)$. As far as we know partial integration is a valid step in the concrete mathematics behind Bell's correlation formula (1.1). It can then be checked that

$$E_1 = \mathbb{P}(x) \frac{4H - 1}{2H + 1} |_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \mathbb{P}(x) \left(\frac{d}{dx} \frac{4H - 1}{2H + 1}\right) dx$$

$$= 1 + \int_{-\infty}^{+\infty} \frac{\mathbb{P}(x)}{(2H+1)^2} \{2(4H-1) - 4(2H+1)\} \frac{dH}{dx}$$

$$= 1 - 6 \int_{-\infty}^{+\infty} \mathbb{P}(x) \frac{\frac{dH}{dx}}{(2H+1)^2} dx$$

$$= 1 + 3 \int_{-\infty}^{+\infty} \mathbb{P}(x) \left(\frac{d}{dx} (2H+1)^{-1} \right) dx \tag{2.7}$$

$$1 + 3 \frac{\mathbb{P}(x)}{2H+1} |_{-\infty}^{+\infty} - 3 \int_{-\infty}^{+\infty} \left(\frac{d}{dx} \mathbb{P}(x) \right) \frac{dx}{2H+1}$$

$$= 1 + 1 - 3 \int_{-\infty}^{a} \left(\frac{d}{dx} \mathbb{P}(x) \right) dx - 3 \int_{a}^{+\infty} \left(\frac{d}{dx} \mathbb{P}(x) \right) \frac{dx}{3}$$

$$2 - 3\mathbb{P}(a) - \mathbb{P}(+\infty) + \mathbb{P}(a) = 1 - 2\mathbb{P}(a)$$

The Gaussian in (2.1) gives $\mathbb{P}(+\infty) = 1$.

2.3 A_2 form integration

Let us also compute with partial integration

$$E_2 = \int_{-\infty}^{+\infty} \left(\frac{d}{dx} \mathbb{P}(x) \right) \frac{2H+1}{4H-1} dx \tag{2.8}$$

It can also be checked that

$$E_2 = \mathbb{P}(x) \frac{2H+1}{4H-1} |_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \mathbb{P}(x) \left(\frac{d}{dx} \frac{2H+1}{4H-1} \right) dx$$

$$= 1 - \int_{-\infty}^{+\infty} \frac{\mathbb{P}(x)}{(4H-1)^2} \{2(4H-1) - 4(2H+1)\} \frac{dH}{dx} dx$$

$$= 1 + 6 \int_{-\infty}^{+\infty} \mathbb{P}(x) \frac{\frac{dH}{dx}}{(4H-1)^2} dx$$

$$= 1 - \frac{3}{2} \int_{-\infty}^{+\infty} \mathbb{P}(x) \left(\frac{d}{dx} (4H-1)^{-1} \right) dx \tag{2.9}$$

$$= 1 - \frac{3}{2} \frac{\mathbb{P}(x)}{4H-1} |_{-\infty}^{+\infty} + \frac{3}{2} \int_{-\infty}^{+\infty} \left(\frac{d}{dx} \mathbb{P}(x) \right) \frac{dx}{4H-1}$$

$$= 1 - \frac{3}{2} \frac{1}{4-1} - \frac{3}{2} \int_{-\infty}^{a} \left(\frac{d}{dx} \mathbb{P}(x) \right) dx + \frac{1}{2} \int_{a}^{+\infty} \left(\frac{d}{dx} \mathbb{P}(x) \right) dx$$

$$= 1 - \frac{1}{2} - \frac{3}{2} \mathbb{P}(a) + \frac{1}{2} - \frac{1}{2} \mathbb{P}(a) = 1 - 2\mathbb{P}(a)$$

Of course, $\mathbb{P}(-\infty) = 0$. Despite the fact that $A_2 = \frac{1}{A_1}$, the outcome of integration is the same for A_1 as well as for A_2. We might have expected that, based on the derivation:

$$\frac{d}{dx} A_2(a,x) = \frac{d}{dx} (A_1(a,x))^{-1} = -\frac{1}{\{A_1(a,x)\}^2} \frac{d}{dx} A_1(a,x) = -\frac{d}{dx} A_1(a,x)$$

and $\{A_1(a,x)\}^2 = 1$, the integrals, E_1 in (2.7) and E_2 in (2.9) would differ. So, where is the anomaly?

Note however, we may employ (2.4) in the evaluation of E_2. Let us, again, look at the step

$$E_2' = 1 + 6 \int_{-\infty}^{+\infty} \mathbb{P}(x) \frac{\frac{dH}{dx}}{(4H-1)^2} dx \tag{2.10}$$

in the derivation (2.9). We may rewrite, noting $2H+1>0$,

$$\begin{aligned} E_2' &= 1+6\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(4H-1)^2}\frac{(2H+1)^2}{(2H+1)^2}dx \\ &= 1+6\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(2H+1)^2}\frac{(2H+1)^2}{(4H-1)^2}dx \\ &= 1+6\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(2H+1)^2}\left(\frac{2H+1}{4H-1}\right)^2 dx \end{aligned} \tag{2.11}$$

In this form we can recognize (2.4) squared. So we also may write

$$\begin{aligned} E_2' &= 1+6\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(2H+1)^2}dx \\ &= 1-3\int_{-\infty}^{+\infty}\mathbb{P}(x)\left(\frac{d}{dx}(2H+1)^{-1}\right)dx \\ &= 1-3\frac{\mathbb{P}(x)}{2H+1}\Big|_{-\infty}^{+\infty}+3\int_{-\infty}^{+\infty}\frac{1}{2H+1}\left(\frac{d}{dx}\mathbb{P}(x)\right)dx \\ &= 1-\frac{3}{2+1}+3\int_{-\infty}^{a}\left(\frac{d}{dx}\mathbb{P}(x)\right)\frac{dx}{1}+3\int_{a}^{+\infty}\left(\frac{d}{dx}\mathbb{P}(x)\right)\frac{dx}{3} \\ &= 1-1+3\mathbb{P}(a)+\mathbb{P}(+\infty)-\mathbb{P}(a) \\ &= 1+2\mathbb{P}(a) \end{aligned} \tag{2.12}$$

Hence, the anomaly with sign functions and probability densities, expected from

$$\frac{d}{dx}A(a,x)\neq\frac{d}{dx}\left(A(a,x)\right)^{-1}$$

surfaces when use is made of $A_2^2=\left(\frac{2H(x-a)+1}{4H(x-a)-1}\right)^2=1$, in equation (2.11). Obviously, it is allowed to rewrite the integral by multiplying the integrand with $\frac{(2H+1)^2}{(2H+1)^2}=1$. Note also that, by definition, either $2H+1=1$ or $2H+1=3$. Moreover, reshuffling of terms

$$\frac{1}{(4H-1)^2}=\frac{1}{(4H-1)^2}\frac{(2H+1)^2}{(2H+1)^2}=\frac{1}{(2H+1)^2}\frac{(2H+1)^2}{(4H-1)^2}=\frac{1}{(2H+1)^2}\left(\frac{2H+1}{4H-1}\right)^2$$

is allowed. It enables the use of (2.4) squared, which is unity. So, it makes perfect sense to have

$$\frac{1}{(4H-1)^2}=\frac{1}{(2H+1)^2} \tag{2.13}$$

when $H=0$ or $H=1$, such as in (2.5). The step from (2.11) to (2.12) is therefore justified. The use of, e.g. partial integration is allowed as a part of the rulebook of concrete mathematics. If partial integration must be excluded from the list of valid operations, then we may ask if Bell's theorem is as generally valid as is claimed. In particular, looking at the step from (2.11) to (2.12), a sceptical reader has to <u>*demonstrate*</u>, observing (2.13), what is wrong with the step

$$\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(4H-1)^2}dx=\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(2H+1)^2}\left(\frac{2H+1}{4H-1}\right)^2dx=\int_{-\infty}^{+\infty}\mathbb{P}(x)\frac{\frac{dH}{dx}}{(2H+1)^2}dx$$

and then the subsequent steps in (2.12). Moreover, if looking at the above equation it is found wanting, then how can we be sure that the activities leading to Bell inequalities (e.g. the CHSH inequality) are correct? A similar type of reasoning, i.e. $A^2=1$ in the integrand, is followed in the derivation of the CHSH. For a detailed CHSH derivation see [7].

3 Conclusion

In the present paper support was found for the fact that Bell's formula is the origin of inconsistent mathematics. In this case, noting $A_2 = \frac{1}{A_1}$, the anomaly surfaced without the $A = \frac{1}{A}$ partial integration breakdown in [7]. Namely

$$\frac{d}{dx}A(a,x) \neq \frac{d}{dx}\left(A(a,x)\right)^{-1}$$

Nevertheless the anomaly, already argued for in [7], resurfaced as a case of inconsistent outcomes where valid mathematical operations on the integrand are performed. We found, $E_1 \neq E_2$, under a certain group of valid operations. The operations on the integrand coincide with those employed in the CHSH derivation.

It appears safe to conclude that the found anomaly in Bell's reasoning [7], is not something accidental. It is noted that the central role this theorem plays in foundation discussions, doesn't put this (physics) theorem beyond any possible criticism. We showed, that the form in which it shows the anomaly is different in different cases. The anomaly also explains why it is possible to come with a valid countermodel [7] and deduce concrete mathematical incompleteness. The latter refers to the Gödel phenomenon in concrete mathematics [8].

It must also be noted that the work of Norden [9] is relevant here because the Bell theorem might not necessarily be central in the foundation discussion. Obviously, the present mathematical and foundational discussion is relevant to a broad field of physics and chemistry, e.g. spin chemistry [10]. If one, armed with the Bell inequalities, claims that there are no Einsteinian hidden variables to explain entanglement, it is necessary that the mathematics upon which that claim is based is a correct representation of the entanglement physics. We showed, with a simple example with a Gaussian normal density, that the mathematics behind the inequalities is anomalous.

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