Considerations on the Newton's binomial expansion

$$(x+y)^n = x^n + y^n + xy\sum_{j=0}^{n-2}\left(x^j + y^j\right)(x+y)^{n-2-j}$$

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Abstract

The binomial formula, set by Isaac Newton, is of utmost importance and has been extensively used in many different fields. This study aims at coming up with alternative expressions to the Newton's formula.

Chapter 1

Another way to write Newton's binomial expansion.

1.1 Purpose of this chapter.

Newton's binomial expansion can be expressed differently. This new formulation allows in turn to perform other calculations which will highlight certain properties that the original formula may not be able to provide.

1.2 Another formula.

Let $n \in \mathbb{N}^*$, and $x \in \mathbb{R}^*$ and $y \in \mathbb{R}^*$. In all that follows, we assume $n \geq 3$. We can write

$$\frac{(x+y)^n - x^n}{(x+y) - x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j}\, x^j = \frac{(x+y)^n - x^n}{y}$$

and likewise

$$\frac{(x+y)^n - y^n}{(x+y) - y} = \sum_{j=0}^{n-1} (x+y)^{n-1-j}\, y^j = \frac{(x+y)^n - y^n}{x}$$

Let us add these two quantities

$$\frac{(x+y)^n - x^n}{y} + \frac{(x+y)^n - y^n}{x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

and we end up with the formula

$$(x+y)^{n+1} - \left(x^{n+1} + y^{n+1}\right) = xy \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

which, for convenience's sake, we write

$$(x+y)^n - (x^n + y^n) = xy \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.1}$$

The Newton's binomial expansion formula, that we recall here

$$(x+y)^n = \sum_{j=0}^{n} C_n^j x^{n-j} y^j \tag{1.2}$$

wherein

$$C_n^j = \frac{n!}{(n-j)!j!} \tag{1.3}$$

allows to establish the equality

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=1}^{n-1} C_n^j x^{n-j-1} y^{j-1}$$

or lastly

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j$$

1.3 Study of the new formula.

Let us pose

$$A_n(x,y) = \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.4}$$

Let us remark first that

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{n-2-j} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right)$$

with $p \in \mathbb{N}^*$ and $p < n$, or likewise

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{(n-p)+(p-2-j)} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-(j-(p-1)+p-1)} \left(x^{j-(p-1)+p-1} + y^{j-(p-1)+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right) + \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-2-(j+p-1)} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right) + \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and finally

$$A_n(x,y) = (x+y)^{n-p} A_p(x,y) + \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

Let us now consider the case wherein $n = p+1$, then

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \sum_{j=0}^{0} (x+y)^{-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

or likewise

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \left(x^{p-1} + y^{p-1}\right)$$

but

$$x^{p-1} + y^{p-1} = (x+y)^{p-1} - xyA_{p-1}(x,y)$$

and so

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + (x+y)^{p-1} - xyA_{p-1}(x,y) \tag{1.5}$$

Let us concentrate now more specifically on $A_n(x,y)$ and let us develop this quantity from the formula 1.4 in page 2. Then

$$\begin{aligned}
A_n(x,y) &= 3(x+y)^{n-2} + \sum_{j=0}^{n-4} (x+y)^{n-4-j}\left(x^{j+2}+y^{j+2}\right) \\
&= 3(x+y)^{n-2} + \left(x^2+y^2\right)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right) \\
&= 3(x+y)^{n-2} + (x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right) \\
&= 4(x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)
\end{aligned}$$

As we continue our calculations in the same manner, we get

$$A_n(x,y) = 5(x+y)^{n-2} - 5xy(x+y)^{n-4} + \sum_{j=0}^{n-6} (x+y)^{n-6-j}\left(x^{j+4}+y^{j+4}\right)$$

$$A_n(x,y) = 6(x+y)^{n-2} - 9xy(x+y)^{n-4} + 2x^2y^2(x+y)^{n-6} + \sum_{j=0}^{n-7} (x+y)^{n-7-j}\left(x^{j+5}+y^{j+5}\right)$$

$$A_n(x,y) = 7(x+y)^{n-2} - 14xy(x+y)^{n-4} + 7x^2y^2(x+y)^{n-6} + \sum_{j=0}^{n-8} (x+y)^{n-8-j}\left(x^{j+6}+y^{j+6}\right)$$

$$A_n(x,y) = 8(x+y)^{n-2} - 20xy(x+y)^{n-4} + 16x^2y^2(x+y)^{n-6} - 2x^3y^3(x+y)^{n-8} + \sum_{j=0}^{n-9} (x+y)^{n-9-j}\left(x^{j+7}+y^{j+7}\right)$$

$$A_n(x,y) = 9(x+y)^{n-2} - 27xy(x+y)^{n-4} + 30x^2y^2(x+y)^{n-6} - 9x^3y^3(x+y)^{n-8} + \sum_{j=0}^{n-10} (x+y)^{n-10-j}\left(x^{j+8}+y^{j+8}\right)$$

$$A_n(x,y) = 10(x+y)^{n-2} - 35xy(x+y)^{n-4} + 50x^2y^2(x+y)^{n-6} - 25x^3y^3(x+y)^{n-8} + 2x^4y^4(x+y)^{n-10} + \sum_{j=0}^{n-11} (x+y)^{n-11-j}\left(x^{j+9}+y^{j+9}\right)$$

$$A_n(x,y) = 11(x+y)^{n-2} - 44xy(x+y)^{n-4} + 77x^2y^2(x+y)^{n-6} - 55x^3y^3(x+y)^{n-8} + 11x^4y^4(x+y)^{n-10} + \sum_{j=0}^{n-12}(x+y)^{n-12-j}\left(x^{j+10}+y^{j+10}\right)$$

It is of course possible to extend our calculations as far as we desire. As n is taking on the values 3, 4, 5, 6,..., we can deduct the respective new developments of $A_3(x,y)$, $A_4(x,y)$, $A_5(x,y)$, $A_6(x,y)$, etc...

Let us assume now that the following formulas hold for all natural integers less than or equal to $2k$ and $2k+1$, wherein $k \in \mathbb{N}^*$

$$A_{2k}(x,y) = \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} \tag{1.6}$$

$$A_{2k+1}(x,y) = (x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} \tag{1.7}$$

The coefficients D_{2k}^{j} and D_{2k+1}^{j} are to be made explicit if possible (and will be indeed further down in this study).

Let us go back to the equation 1.5 page 3 and rewrite in the form

$$A_{2k+2}(x,y) = (x+y)A_{2k+1}(x,y) + (x+y)^{2k} - xyA_{2k}(x,y)$$

Let us develop now this relation

$$A_{2k+2}(x,y) = (x+y)^2\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} + (x+y)^{2k} - xy\sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)}$$

$$\Longleftrightarrow$$

$$A_{2k+2}(x,y) = (x+y)^2\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} + (x+y)^{2k} + \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^{j+1}(xy)^{j+1}(x+y)^{2(k-1-j)}$$

Let us carry on with our calculations. We obtain in an equivalent manner

$$
\begin{aligned}
A_{2k+2}(x,y) = \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} \\
+ (x+y)^{2k} \\
+ \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)}
\end{aligned}
$$

$$\Longleftrightarrow$$

$$
\begin{aligned}
A_{2k+2}(x,y) = \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} \\
+ (x+y)^{2k} \\
+ \sum_{j=1}^{k} D_{2k}^{j-1} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}
\end{aligned}
$$

$$\Longleftrightarrow$$

$$
\begin{aligned}
A_{2k+2}(x,y) = \sum_{j=1}^{k-1} \left(D_{2k+1}^{j} + D_{2k}^{j-1} \right) (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} \\
+ D_{2k+1}^{0} (x+y)^{2k} + (x+y)^{2k} + D_{2k}^{k-1} (xy)^{k}
\end{aligned}
$$

and we can write

$$A_{2k+2}(x.y) = \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$
$$D_{2k+2}^{k} = D_{2k}^{k-1}$$

and

$$(\forall j \in \mathbb{N}) (1 \leq j \leq k-1) \left(D_{2k+2}^{j} = D_{2k+1}^{j} + D_{2k}^{j-1} \right)$$

Similarly, we have

$$A_{2k+3}(x,y) = (x+y) A_{2k+2}(x,y) + (x+y)^{2k} - xyA_{2k+1}(x,y)$$

Let us make it more explicit

$$
\begin{aligned}
A_{2k+3}(x,y) = (x+y) \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} \\
+ (x+y)^{2k+1} \\
- xy(x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)}
\end{aligned}
$$

hence

$$A_{2k+3}(x,y) = \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1} + (x+y)^{2k+1} + \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)+1}$$

which is equivalent to

$$A_{2k+3}(x,y) = (x+y)^{2k+1} + \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1} + \sum_{j=1}^{k-1} D_{2k+1}^{j-1} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1}$$

and also

$$A_{2k+3}(x,y) = \left(D_{2k+2}^{0}+1\right)(x+y)^{2k+1} + \sum_{j=1}^{k} \left(D_{2k+2}^{j} + D_{2k+1}^{j-1}\right) (-1)^{j} (xy)^{j} (x+y)^{2(k-j)+1}$$

and we can finally write

$$A_{2k+3}(x,y) = (x+y) \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$

and

$$(\forall j \in \mathbb{N})\,(1 \le j \le k)\left(D_{2k+3}^{j} = D_{2k+2}^{j} + D_{2k+1}^{j-1}\right)$$

This concludes our mathematical induction and we can write at last as a conclusion

$$(\forall k \in \mathbb{N}^{*}) \left(A_{2k} = \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)}\right) \tag{1.8}$$

with

$$D_{2k}^{0} = D_{2k-1}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k \tag{1.9}$$

and

$$D_{2k}^{k-1} = D_{2k-2}^{k-2} = \cdots = D_{4}^{1} = 2 \tag{1.10}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k-1)\left(D^{j}_{2k} = D^{j}_{2k-1} + D^{j-1}_{2k-2}\right) \tag{1.11}$$

and as well

$$(\forall k \in \mathbb{N}^{*})\left(A_{2k+1} = (x+y)\sum_{j=0}^{k-1} D^{j}_{2k+1}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}\right) \tag{1.12}$$

with

$$D^{0}_{2k+1} = D^{0}_{2k} + 1 \Longleftrightarrow D^{0}_{2k} = 2k+1 \tag{1.13}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k)\left(D^{j}_{2k+1} = D^{j}_{2k} + D^{j-1}_{2k-1}\right) \tag{1.14}$$

1.4 Values taken by the coefficients D^{j}_{h} wherein $(h \in \mathbb{N})$ and $(h \geq 3)$.

We have, as we just established it

$$(\forall h \in \mathbb{N})\,(h \geq 3)\left(D^{0}_{h} = h\right)$$

Let us now take $j = 1$. We have

$$D^{1}_{h} = D^{1}_{h-1} + D^{0}_{h-2}$$

We can then write

$$\left.\begin{array}{r} D^{1}_{h} = D^{1}_{h-1} + D^{0}_{h-2} \\ D^{1}_{h-1} = D^{1}_{h-2} + D^{0}_{h-3} \\ \cdots \\ \cdots \\ \cdots \\ D^{1}_{5} = D^{1}_{4} + D^{0}_{3} \end{array}\right\} \Longrightarrow D^{1}_{h} = \sum_{j=0}^{h-5} D^{0}_{h-2-j} + D^{1}_{4}$$

but

$$D^{0}_{h-2-j} = h-2-j$$

and, according to the relation 1.10 established in page 7

$$D^{1}_{4} = 2$$

hence we get

$$D^{1}_{h} = \sum_{j=0}^{h-5} (h-2-j) + 2 = ((h-2) + (h-3) + (h-4) + \cdots + 3) + 2$$

and therefore

$$2D^{1}_{h} = h\,(h+3)$$

and finally

$$(\forall h \in \mathbb{N}^{*})\,(h \geq 3)\left(D^{1}_{h} = \frac{h\,(h-3)}{2}\right) \tag{1.15}$$

Clearly

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^1 \in \mathbb{N}\right)$$

Making similar calculations, we find for every natural integer $h \geq 3$

$$D_h^2 = \frac{h\,(h-4)\,(h-5)}{6} \tag{1.16}$$

$$D_h^3 = \frac{h\,(h-5)\,(h-6)\,(h-7)}{24} \tag{1.17}$$

There as well

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^2 \in \mathbb{N}\right)$$

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^3 \in \mathbb{N}\right)$$

We then remark that the relations 1.11 and 1.14 established in page 8, as well as those ((see relations 1.15, 1.16 and 1.17) established in pages 8 and 9 allow us to affirm

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(\forall j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}\right)\left(D_h^j \in \mathbb{N}\right)$$

Let us assume now, h being chosen as even, and for all $j \in \left\{0, 1, \cdots, \frac{h-2}{2}\right\}$ the formula

$$D_h^j = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} \tag{1.18}$$

true until rank h, for all even natural integer lower than or equal to h.

Let us assume as well that, for all $j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}$, until rank $h-1$, the formula

$$D_{h-1}^j = \frac{(h-1)\,((h-1)-(j+2))!}{(j+1)!\,((h-1)-2\,(j+1))!} \tag{1.19}$$

is true. Then

$$D_{h-1}^{j-1} = \frac{(h-1)\,(h-1-(j+1)!)}{j!\,(h-1-2j)!} = \frac{(h-1)\,(h-(j+2)!)}{j!\,(h-1-2j)!}$$

The relation 1.11 established in page 8 allows us to write

$$D_{h+1}^{j} = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} + \frac{(h-1)\,(h-(j+2))!}{j!\,(h-1-2j)!}$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{j!}\left(\frac{h}{(j+1)\,(h-2\,(j+1))!} + \frac{(h-1)}{(h-1-2j)!}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{j!}\left(\frac{h\,(h-1-2j)+(h-1)\,(j+1)}{(h-1-2j)!\,(j+1)}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\,(h\,(h-1)-2jh+(h-1)\,j+(h-1))$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\left(h^2-1-(h+1)\,j\right)$$

and finally

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\,(h+1)\,(h-1-j)$$

We can therefore write

$$D_{h+1}^{j} = \frac{(h+1)\,(h-(j+1))!}{(j+1)!\,(h+1-2\,(j+1))!} \tag{1.20}$$

We could make similar calculations if we take h as odd

We verify that

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(D_h^0 = h\right)$$

and, as we denote the ensemble of even natural integers as $2\mathbb{N}$

$$(\forall h = 2k \in 2\mathbb{N}^*)\,(h \geq 4)\left(D_{2k}^{k-1} = 2\right)$$

At the end of this mathematical induction, we have therefore established

$$\begin{gathered}(\forall k \in \mathbb{N}^*)\,(\forall j \in \{0,1,2,\cdots,k-1\}) \\ \left(D_{2k+1}^{j} = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}\right) \\ \left(D_{2(k+1)}^{j} = \frac{2\,(k+1)\,(2\,(k+1)-1-(j+1))!}{(j+1)!\,(2\,(k+1)-2\,(j+1))!}\right)\end{gathered} \tag{1.21}$$

Let us remark that for all natural integer h

$$h-2\,(j+1)+(j+1) = h-(j+1)$$

We can then write

$$D_h^{j} = \frac{h\,(h-(j+1))!}{(h-(j+1))\,(j+1)!\,(h-2\,(j+1))!}$$

and also

$$D_h^{j} = \frac{h}{h-(j+1)}C_{h-(j+1)}^{j+1}$$

1.5 Study on the coefficients D_h^jEtude sur les coefficients D_h^j.

For the following odd natural integers $h = 2k + 1$, we verify the relations

$$k = 1 \Longleftrightarrow h = 2k + 1 = 3$$
$$D_3^0 = 3C_0^0$$

$$k = 2 \Longleftrightarrow h = 2k + 1 = 5$$
$$D_5^0 = 5C_1^0$$
$$D_5^1 = 5C_1^1$$

$$k = 3 \Longleftrightarrow h = 2k + 1 = 7$$
$$D_7^0 = 7C_2^0$$
$$D_7^1 = 7C_2^1$$
$$D_7^2 = 7C_2^2$$

$$k = 4 \Longleftrightarrow h = 2k + 1 = 9$$
$$D_9^0 = 9C_3^0$$
$$D_9^1 = 9C_3^1$$
$$D_9^2 = 9C_3^2 + 3C_0^0$$
$$D_9^1 = 9C_3^3$$

$$k = 5 \Longleftrightarrow h = 2k + 1 = 11$$
$$D_{11}^0 = 11C_4^0$$
$$D_{11}^1 = 11C_4^1$$
$$D_{11}^2 = 11\left(C_4^2 + C_1^0\right)$$
$$D_{11}^3 = 11\left(C_4^3 + C_1^1\right)$$
$$D_{11}^4 = 11C_4^4$$

$$k = 6 \Longleftrightarrow h = 2k + 1 = 13$$
$$D_{13}^0 = 13C_5^0$$
$$D_{13}^1 = 13C_5^1$$
$$D_{13}^2 = 13\left(C_5^2 + 2C_2^0\right)$$
$$D_{13}^3 = 13\left(C_5^3 + 2C_2^1\right)$$
$$D_{13}^4 = 13\left(C_5^4 + 2C_2^2\right)$$
$$D_{13}^5 = 13C_5^5$$

$k = 7 \Longleftrightarrow h = 2k + 1 = 15$

$$D^0_{15} = 15C^0_6$$
$$D^1_{15} = 15C^1_6$$
$$D^2_{15} = 15\left(C^2_6 + 3C^0_3\right)$$
$$D^3_{15} = 15\left(C^3_6 + 3C^1_3\right)$$
$$D^4_{15} = 15\left(C^4_6 + 3C^2_3 + 3C^0_0\right)$$
$$D^5_{15} = 15\left(C^5_6 + 3C^3_3\right)$$
$$D^6_{15} = 15C^6_6$$

$k = 8 \Longleftrightarrow h = 2k + 1 = 17$

$$D^0_{17} = 17C^0_7$$
$$D^1_{17} = 17C^1_7$$
$$D^2_{17} = 17\left(C^2_7 + 5C^0_4\right)$$
$$D^3_{17} = 17\left(C^3_7 + 5C^1_4\right)$$
$$D^4_{17} = 17\left(C^4_7 + 5C^2_4 + C^0_0\right)$$
$$D^5_{17} = 17\left(C^5_7 + 5C^3_4 + C^1_1\right)$$
$$D^6_{17} = 17\left(C^6_7 + 5C^4_4\right)$$
$$D^7_{17} = 17C^7_7$$

$k = 9 \Longleftrightarrow h = 2k + 1 = 19$

$$D^0_{19} = 19C^0_8$$
$$D^1_{19} = 19C^1_8$$
$$D^2_{19} = 19\left(C^2_8 + 7C^0_5\right)$$
$$D^3_{19} = 19\left(C^3_8 + 7C^1_5\right)$$
$$D^4_{19} = 19\left(C^4_8 + 7C^2_5 + 3C^0_2\right)$$
$$D^5_{19} = 19\left(C^5_8 + 7C^3_5 + 3C^1_2\right)$$
$$D^5_{19} = 19\left(C^6_8 + 7C^4_5 + 2C^2_2\right)$$
$$D^6_{19} = 19\left(C^7_8 + 7C^5_5\right)$$
$$D^8_{19} = 19C^8_8$$

$k = 10 \Longleftrightarrow h = 2k + 1 = 21$

$$
\begin{aligned}
D_{21}^{0} &= 21 C_{9}^{0} \\
D_{21}^{1} &= 21 C_{9}^{1} \\
D_{21}^{2} &= 21 \left(C_{9}^{2} + 19 C_{6}^{0} \right) \\
D_{21}^{3} &= 21 \left(C_{9}^{3} + 19 C_{6}^{1} \right) \\
D_{21}^{4} &= 21 \left(C_{9}^{4} + 19 C_{6}^{2} + 14 C_{3}^{0} \right) \\
D_{21}^{5} &= 21 \left(C_{9}^{5} + 19 C_{6}^{3} + 14 C_{3}^{1} \right) \\
D_{21}^{6} &= 21 \left(C_{9}^{6} + 19 C_{6}^{4} + 14 C_{3}^{2} + 3 C_{0}^{0} \right) \\
D_{21}^{7} &= 21 \left(C_{9}^{7} + 19 C_{6}^{5} + 14 C_{3}^{3} \right) \\
D_{21}^{8} &= 21 \left(C_{9}^{7} + 19 C_{6}^{6} \right) \\
D_{21}^{9} &= 21 C_{9}^{9}
\end{aligned}
$$

We are led to assume that for all odd natural integer $2k + 1$, greater than or equal to 3, each coefficient D_{2k+1}^{j} can be expressed as follows

$$
D_{2k+1}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} \tag{1.22}
$$

with

$$
0 \leq k - 1 - 3l \leq k - 1 \tag{1.23}
$$

and we will write by convention

$$
(\forall j)\,(j - 2l < 0) \left(C_{k-1-3l}^{j-2l} = 0 \right) \tag{1.24}
$$

In order to demonstrate the validity of this formula for all natural integer k, we are going to develop, to the extent possible, the coefficient F_{2k+1}^{l} against k and l

For any natural integer k, we verify the relations

$$
D_{2k+1}^{0} = (2k + 1)\, C_{k-1}^{0}
$$

$$
D_{2k+1}^{1} = (2k + 1)\, C_{k-1}^{1}
$$

We can always write, with $k \geq 4$

$$
D_{2k+1}^{2} = (2k + 1)\, C_{k-1}^{2} + \left(D_{2k+1}^{2} - (2k + 1)\, C_{k-1}^{2} \right) C_{k-4}^{0}
$$

But, in accordance with the relations 1.3 and 1.21 established in pages 2 and 10

$$
D_{2k+1}^{2} - (2k + 1)\, C_{k-1}^{2} = (2k + 1) \left(\frac{(2k - 3)!}{3!\,(2k + 1 - 6)!} - \frac{(k - 1)!}{2!\,(k - 3)!} \right)
$$

and similarly

$$
D_{2k+1}^{2} - (2k + 1)\, C_{k-1}^{2} = (2k + 1) \left(\frac{(2k - 3)!}{3!\,(2k - 5)!} - \frac{(k - 1)!}{2!\,(k - 3)!} \right)
$$

and

$$D^2_{2k+1} - (2k+1)\,C^2_{k-1} = (2k+1)\left(\frac{(2k-3)\,(2k-4)}{3!} - \frac{(k-1)\,(k-2)}{2!}\right)$$

and

$$D^2_{2k+1} - (2k+1)\,C^2_{k-1} = (2k+1)\left(\frac{(2k-3)\,(k-2)}{3} - \frac{(k-1)\,(k-2)}{2}\right)$$

and also

$$D^2_{2k+1} - (2k+1)\,C^2_{k-1} = (2k+1)\left(\frac{2\,(2k-3)\,(k-2) - 3\,(k-1)\,(k-2)}{6}\right)$$

and finally

$$D^2_{2k+1} - (2k+1)\,C^2_{k-1} = \frac{(2k+1)\,(k-2)\,(k-3)}{6}$$

Let us pose

$$F^1_{2k+1} = D^2_{2k+1} - (2k+1)\,C^2_{k-1} = \frac{(2k+1)\,(k-2)\,(k-3)}{3!}$$

In a similar way, we could find

$$D^3_{2k+1} = (2k+1)\left(C^3_{k-1} + F^1_{2k+1} C^1_{k-4}\right)$$

and

$$D^4_{2k+1} = (2k+1)\left(C^4_{k-1} + F^1_{2k+1} C^2_{k-4} + F^2_{2k+1} C^0_{k-7}\right)$$

which gives us

$$F^2_{2k+1} = \left(\left(D^4_{2k+1} - (2k+1)\,C^4_{k-1}\right) - \left(D^2_{2k+1} - (2k+1)\,C^2_{k-1}\right) C^2_{k-4}\right)$$

Making similar calculations as the previous ones, and with coefficients D^j_{2k+1} and C^l_{k-j} being made explicit, we find

$$F^2_{2k+1} = \frac{(2k+1)\,(k-3)\,(k-4)\,(k-5)\,(k-6)}{5!}$$

We are therefore led to assume that, for all natural integer $k \geq 1$, the equality

$$F^l_{2k+1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} \tag{1.25}$$

is true, with the natural integer l such that

$$0 \leq l \leq \lfloor \frac{k}{3} \rfloor$$

Let us calculate the difference F^l_{2k+1} and F^l_{2k-1}

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} - \frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Then

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(k-2-l)!}{(2l+1)!\,(k-2-3l)!}\left(\frac{(2k+1)\,(k-1-l)-(2k-1)\,(k-1-3l)}{(k-1-3l)}\right)$$

and

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left((2k+1)\,(k-1-l)-(2k-1)\,(k-1-3l)\right)$$

and also

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left(2\,(2l+1)\,(k-1)\right)$$

and finally

$$F^l_{2k+1}-F^l_{2k-1}=\frac{2\,(k-1)\,(k-2-l)!}{(2l)!\,(k-1-3l)!} \tag{1.26}$$

Our hypothesis 1.22 stated in pages 13 leads us to use a mathematical induction to show the existence of the relation

$$(\forall k\in\mathbb{N}^*)\,(k\geq 1)\,(\forall j\in\mathbb{N})\,(0\leq j\leq k-1)\left(D^j_{2k+1}=\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}F^l_{2k+1}C^{j-2l}_{k-1-3l}\right)$$

wherein each coefficient F^l_{2k+1} is expressed by the formula 1.25 established in page 14.

Let us assume that, for all natural integer $j\leq k-2$, the relation

$$D^j_{2k-1}=\sum_{l=0}^{\lfloor\frac{k-1}{3}\rfloor}F^l_{2k-1}C^{j-2l}_{k-2-3l} \tag{1.27}$$

is true until the rank $2k-1$, with

$$F^l_{2k-1}=\frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Let us calculate now the difference

$$D^j_{2k-1}-D^{j-1}_{2k-3}=D^j_{2k-2}$$

and also

$$D^j_{2k-2}=\sum_{l=0}^{\lfloor\frac{k-1}{3}\rfloor}F^l_{2k-1}C^{j-2l}_{k-2-3l}-\sum_{l=0}^{\lfloor\frac{k-2}{3}\rfloor}F^l_{2k-3}C^{j-2l-1}_{k-3-3l} \tag{1.28}$$

Then, we are faced with two cases.

1.5.1 Case 1: $\lfloor\frac{k-1}{3}\rfloor=\lfloor\frac{k-2}{3}\rfloor=m$

We have

$$\lfloor\frac{k-1}{3}\rfloor=m\Longleftrightarrow k-1=3m+\rho>3m$$

The only values that ρ can take a priori are 0, 1 and 2

1.5.1.1 $\rho = 0$

$$\begin{aligned} \rho = 0 &\Longrightarrow k-1 = 3m \\ &\Longleftrightarrow k-2 = 3m-1 < 3m \end{aligned}$$

1.5.1.2 $\rho = 1$

$$\begin{aligned} \rho = 1 &\Longrightarrow k-1 = 3m+1 \\ &\Longleftrightarrow k-2 = 3m \end{aligned}$$

1.5.1.3 $\rho = 2$

$$\begin{aligned} \rho = 2 &\Longrightarrow k-1 = 3m+2 \\ &\Longrightarrow k-2 = 3m+1 \end{aligned}$$

Clearly, ρ cannot be equal to 0. We also notice that in this Case 1

$$2k+1 \not\equiv 0 \quad (3) \tag{1.29}$$

Let us recall that

$$\left(C_{k-2-3l}^{j-2l} = C_{k-3-3l}^{j-2l-1} + C_{k-3-3l}^{j-2l}\right) \Longleftrightarrow \left(C_{k-3-3l}^{j-2l-1} = C_{k-2-3l}^{j-2l} - C_{k-3-3l}^{j-2l}\right) \tag{1.30}$$

We then have (see the relation relation 1.28 established page 15)

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(F_{2k-1}^{l} C_{k-2-3l}^{j-2l} - F_{2k-3}^{l} C_{k-3-3l}^{j-2l-1}\right) \tag{1.31}$$

which is equivalent to

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(F_{2k-1}^{l} C_{k-2-3l}^{j-2l} - F_{2k-3}^{l} \left(C_{k-2-3l}^{j-2l} - C_{k-3-2l}^{j-2l}\right)\right)$$

and

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(\left(F_{2k-1}^{l} - F_{2k-3}^{l}\right) C_{k-2-3l}^{j-2l} + F_{2k-3}^{l} C_{k-3-2l}^{j-2l}\right)$$

with $m = \lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k-1}{3} \rfloor$. In particular, among the natural integers $2k+1$ wherein k satisfies this property, we find all the prime integers strictly greater to 3.

1.5.2 Case 2: $\lfloor \frac{k-1}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor + 1 = m$

We have

$$\lfloor \frac{k-1}{3} \rfloor = m \Longleftrightarrow k-1 = 3m+\rho$$

As previously,

1.5.2.1 $\rho = 0$

$$\begin{aligned}\rho = 0 &\Longrightarrow k-1 = 3m \\ &\Longleftrightarrow k-2 = 3m-1\end{aligned}$$

1.5.2.2 $\rho = 1$

$$\begin{aligned}\rho = 1 &\Longrightarrow k-1 = 3m+1 \\ &\Longleftrightarrow k-2 = 3m\end{aligned}$$

1.5.2.3 $\rho = 2$

$$\begin{aligned}\rho = 2 &\Longrightarrow k-1 = 3m+2 \\ &\Longleftrightarrow k-2 = 3m+1\end{aligned}$$

And in this case, ρ can only be equal to 0. We also notice

$$\begin{aligned}2k+1 \equiv 0 &\Longleftrightarrow k \equiv 1 \quad (3) \\ &\Longleftrightarrow k-1 \equiv 0 \quad (3)\end{aligned}$$

We then have (see the relation 1.28 established page 15)

$$\begin{aligned}D^{j}_{2k-2} &= \sum_{l=0}^{m} F^{l}_{2k-1} C^{j-2l}_{k-2-3l} - \sum_{l=0}^{m-1} F^{l}_{2k-3} C^{j-2l-1}_{k-3-3l} \\ &= F^{m}_{2k-1} C^{j-2(m+1)}_{k-2-3(m+1)} + \sum_{l=0}^{m-1} F^{l}_{2k-1} C^{j-2l}_{k-2-3l} - \sum_{l=0}^{m-1} F^{l}_{2k-3} C^{j-2l-1}_{k-3-3l} \\ &= F^{m}_{2k-1} C^{j-2(m+1)}_{k-2-3(m+1)} + \sum_{l=0}^{m-1} \left(\left(F^{l}_{2k-1} - F^{l}_{2k-3} \right) C^{j-2l}_{k-2-3l} + F^{l}_{2k-3} C^{j-2l}_{k-3-3l} \right)\end{aligned}$$

with $m = \lfloor \frac{k-1}{3} \rfloor$ and $m-1 = \lfloor \frac{k-2}{3} \rfloor$

Let us return to Case 1 and let us take our hypothesis 1.27 stated page 15

$$\sum_{l=0}^{m} F^{l}_{2k-3} C^{j-2l}_{k-3-3l} = D^{j}_{2k-3}$$

then, in accordance with the relation relation 1.31 set out page 16

$$D^{j}_{2k-2} - D^{j}_{2k-3} = D^{j-1}_{2k-4}$$

and finally, we get the equality

$$D^{j-1}_{2k-4} = \sum_{l=0}^{\lfloor \frac{k-2}{3} \rfloor} F^{l}_{2k-4} C^{j-2l}_{k-2-3l} \tag{1.32}$$

with, in accordance with the relation 1.26 established page 15

$$F_{2k-4}^{l} = F_{2k-1}^{l} - F_{2k-3}^{l}$$

We still have to establish that the equality 1.32 in page 17 is true when $k \geq 4$ describes $\mathbb{N}$. We make sure first, by a simple calculation, that this equality indeed holds when k takes successively the values 4, 5 and $6 \cdots$, when j takes its values in its domain.

We then assume that this equality holds for any given natural integer less or equal to $2k$, for all $j \leq (k-1)$, that is

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

We can now remark that the calculations made to get the formula of $D_{h=2k}^{j}$ depending on the coefficients F_{2k}^{l} and the binomial coefficients C_{k-3l}^{j+1-2l} are generalizable to any value of h in $\mathbb{N}$. We just have to verify by mathematical induction the correctness of the formulation of the odd index coefficients $D_{h=2k+1}^{j}$ to obtain a result that is valid, irrespective of the parity of this index h.

Let us go back to the initial hypothesis on the odd index coefficients (see our hypothesis 1.27 stated page 15)and let us utilize what we just established. We verify

$$D_{2k+1}^{j} = D_{2k}^{j} + D_{2k-1}^{j-1}$$

with

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

and

$$D_{2k-1}^{j-1} = \sum_{l=0}^{\lfloor \frac{k-1}{3} \rfloor} F_{2k-1}^{l} C_{k-2-3l}^{j-2l}$$

Further to the calculations we just made in pages 16 and 17, we have

$$\begin{aligned}
D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k-2}^{l} C_{k-1-3l}^{j-2l} + F_{2k-1}^{l} C_{k-2-3l}^{j-2l} \\
\Longleftrightarrow D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} \left(F_{2k+1}^{l} - F_{2k-1}^{l}\right) C_{k-1-3l}^{j-2l} + F_{2k-1}^{l} C_{k-2-3l}^{j-2l} \\
\Longleftrightarrow D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} - \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k-1}^{l} C_{k-1-3l}^{j-2l-1} \\
\Longleftrightarrow D_{2k}^{j} &= D_{2k+1}^{j} - D_{2k-1}^{j-1}
\end{aligned}$$

This result is in agreement with the equality 1.14 established in page 8.

As we know how to express the coefficients F_{2k-2}^{l} and F_{2k-1}^{l} against l and k, we can now calculate F_{2k+1}^{l}. We thus find

$$D_{2k+1}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l}$$

with

$$F_{2k+1}^{l} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

Our mathematical induction is therefore complete for every coefficient D_{h}^{j}, with odd or even indices h.

Let us now summarize all the results we have obtained over the previous pages (see equations 1.8 and 1.12 in pages 7 and 8)

$$(\forall n \in \mathbb{N})\,(n \geq 3) \left((x^n + y^n) = x^n + y^n + xy \sum_{j=1}^{n-2} A_n\,(x, y) \right)$$

with, for $n = 2k$ (see equation 1.8 in page 7)

$$A_{2k}\,(x, y) = \sum_{j=0}^{k-1} D_{2k}^{j}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}$$

and

$$(\forall k \in \mathbb{N}^{*})\,(k > 1)\,(\forall j \in \mathbb{N})\,(j \leq k-1) \left(D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l} \right)$$

and

$$F_{2k}^{l} = \frac{2k\,(k-1-l)!}{(2l)!\,(k-3l)!}$$

and for $n = 2k+1$ (see equation 1.12 in page 8)

$$A_{2k+1}\,(x, y) = (x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}$$

and

$$(\forall k \in \mathbb{N}^{*})\,(\forall j \in \mathbb{N})\,(j \leq k-1) \left(D_{2k+1}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} \right)$$

and

$$F_{2k+1}^{l} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

1.6 Study of $A_{2k+1}(x,y)$ wherein $k \in \mathbb{N}^*$.

We will show in this paragraph how we can further factorize the quantity $A_{2k+1}(x,y)$. Using the previous results, we can write

$$A_{2k+1}(x,y) = (x+y)\sum_{j=0}^{k-1}\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-j)}$$

We then have for each k, and for all j and all l

$$F_{2k+1}^{l} C_{k-1-3l}^{j-2l}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-j)}$$
$$= F_{2k+1}^{l} C_{k-1-3l}^{j-2l}(-1)^{j-2l+2l}(xy)^{j-2l+2l}(x+y)^{2(k-1-3l+3l-(j-2l)-2l)}$$
$$= \left(F_{2k+1}^{l} C_{k-1-3l}^{j-2l}(-1)^{j-2l}(xy)^{j-2l}(x+y)^{2(k-1-3l-(j-2l))}\right)(-1)^{2l}(xy)^{2l}(x+y)^{2l}$$

We can therefore write $A_{2k+1}(x.y)$ in the following manner

$$A_{2k+1}(x,y)$$
$$= (x+y)$$
$$\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2l}$$
$$\sum_{j=0}^{k-1} C_{k-1-3l}^{j-2l}(-1)^{j-2l}(xy)^{j-2l}(x+y)^{2(k-1-3l-(j-2l))}$$

If j varies from 0 to $k-1$, then $j-2l$ varies from 0 to $k-1-2l$, and as we necessarily have

$$j-2l \leq k-1-3l$$

we get

$$A_{2k+1}(x,y)$$
$$= (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2l}\sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-3l-j)}$$

but

$$\sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-3l-j)}$$
$$= \left((x+y)^{2}-xy\right)^{k-1-3l}$$
$$= \left(x^{2}+xy+y^{2}\right)^{k-1-3l}$$

and lastly

$$A_{2k+1}(x,y) = (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2l}\left(x^{2}+xy+y^{2}\right)^{k-1-3l}$$

If, in addition, we assume that $2k+1$ is an odd natural integer strictly greater than 3, and not a multiple of 3 then (see the equality 1.29 page 16)

$$k - 1 \not\equiv 0 \quad (3)$$

and therefore $k-1-3l$ does not vanish for any value taken by l. As a result, $A_{2k+1}(x,y)$ is always divisible by $\left(x^2+xy+y^2\right)$ and we can write for every natural integer $n = 2k+1 > 3$.

$$A_{2k+1}(x,y) =$$

$$(x+y)\left(x^2+xy+y^2\right)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2l}\left(x^2+xy+y^2\right)^{k-2-3l} \quad (1.33)$$

1.7 Various ways to express the Binomial expansion.

We are getting now close to the end of this study, the purpose of which was to express the Newton binomial expansion in other manners. As enounced (see relation 1.2 in page 2) and later established (see relation 1.1 in page 1), we have

$$\begin{aligned}
(x+y)^n &= \sum_{j=0}^{n} C_n^j x^{n-j} y^j \\
&= x^n + y^n + \sum_{j=1}^{n-1} C_n^j x^{n-j} y^j \\
&= x^n + y^n + xy\sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j \\
&= x^n + y^n + xy\sum_{j=0}^{n-2} (x+y)^{n-2-j}\left(x^j+y^j\right)
\end{aligned}$$

Moreover, depending on whether the natural integer n is even or odd, the binomial expansion can be equally expressed as follows

$n = 2k$ **pair**

$$(x+y)^{2k} = x^{2k} + y^{2k} + xy\sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

with

$$D_{2k}^{j} = \frac{2k\,(2k-1-(j+1))!}{(j+1)!\,(2k-2\,(j+1))!}$$

as established in page 7 (see equation 1.8).

$n = 2k+1$ **impair**

$$(x+y)^{2k+1} = x^{2k+1} + y^{2k+1} + xy\,(x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}$$

avec

$$D_{2k+1}^{j} = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}$$

as established in page 8 (see equation 1.12).

$n = 2k+1 > 3$ **and** $n \not\equiv 0 \ (3)$

$$(x+y)^{n} = x^{n} + y^{n} + \\ xy\,(x+y)\left(x^{2}+xy+y^{2}\right) \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}\,(-1)^{2l}\,(x+y)^{2l}\left(x^{2}+xy+y^{2}\right)^{k-2-3l} \tag{1.34}$$

with

$$F_{2k+1}^{l} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} \tag{1.35}$$

$$\equiv 0 \quad (n = 2k+1)$$

as established in page 21 (see equation 1.33).

Let us notice that the set of the prime integer greater than 3 is a subset of these natural integers n.

Outlining these results concludes this study.