Real Numbers in the Neighborhood of Infinity

Jonathan W. Tooker

October 27, 2019

Abstract

We demonstrate the existence of a broad class of real numbers which are not elements of any number field: those in the neighborhood of infinity. After considering the reals and the affinely extended reals, we prove that numbers in the neighborhood of infinity are ordinary real numbers of the type detailed in Euclid's Elements. We show that real numbers in the neighborhood of infinity obey the Archimedes property of real numbers. The main result is an application in complex analysis. We show that the Riemann zeta function has infinitely many non-trivial zeros off the critical line in the neighborhood of infinity.

§1 Introduction

It is the popular theme in modern mathematics to define $\mathbb{R}$ by an algebraic number field approach but here we return to the geometric approach given by the Euclid magnitude [1]. Following Euclid, a real number is the length of a line segment. This length exists completely separately from the axioms of a complete ordered field. We will examine at the end of this paper the Riemann hypothesis which predates Dedekind's work on $\mathbb{R}$ [2] by several years so we are, therefore, perfectly well motivated to eschew the Dedekind cut definition of $\mathbb{R}$ in favor of the Euclid definition. Euclid's Elements is something of a grand canon of mathematics and Euclid's definition of $\mathbb{R}$ dominated the mathematical landscape for thousands of years until the most recent chapter of the history of mathematics began around the turn of the 20th century. Most importantly, Riemann formulated his famous hypothesis during the era in which the Euclid definition of $\mathbb{R}$ was the one in common usage. The axioms of a complete ordered field had not modified the ancient definition of $\mathbb{R}$ at that time, and the Dedekind cut did not exist at that time.

Motivating the present approach, Pugh writes the following in Reference [3].

> "The current teaching trend treats the real number system $\mathbb{R}$ as a given—it is defined axiomatically. Ten or so of its properties are listed, called axioms of a complete ordered field, and the game becomes: deduce its other properties from the axioms. This is something of a fraud, considering that the entire structure of analysis is built on the real number system. For what if a system satisfying the axioms failed to exist? Then one would be studying the empty set!"

Although the Euclid magnitude was the definition of $\mathbb{R}$ throughout most of the past two millennia, modern mathematicians have chosen to adopt, for the time being at least, the currently trendy field axioms. Through such axioms, it is claimed that $\mathbb{R}$ is such that the set

$$\mathcal{R} = \{\mathbb{R}, +, \times\} \quad ,$$

satisfies all of the field axioms. Here we will define $\mathbb{R}$ as a cut in the real number line (a line with the label "real number line" attached) but we will not adopt the full structure of Dedekind cuts.[1] In the present convention, $\mathbb{R}$ will be such that $\mathcal{R}$ does not universally satisfy the field axioms. Nothing is lost in the present definition, however, because there shall exist $\mathbb{R}_0 \subset \mathbb{R}$ such that the set

$$\mathcal{R}' = \{\mathbb{R}_0, +, \times\} \quad ,$$

does satisfy all of the field axioms. Indeed, $\mathcal{R}'$ shall be **exactly identical** to what is usually called the real number field. By taking the more general approach through a geometric definition of $\mathbb{R}$, we do not lose any of the power of the field axioms because $\mathbb{R}$ contains a subset $\mathbb{R}_0$ (called real numbers in the neighborhood of the origin) which does 100% of the work done by the real number field when it is defined as in $\mathcal{R}$. Nothing at all is lost in this geometric definition of $\mathbb{R}$ but, as we will show, very much is gained when we define $\mathbb{R}$ geometrically as opposed to through its algebraic operations $+$ and $\times$. That which is gained shall be called real numbers in the neighborhood of infinity: $\mathbb{R} \setminus \mathbb{R}_0$. As we will show, the field axioms preclude the existence of numbers in the neighborhood of infinity but the existence of these numbers is very much implied by the historical a-real-number-is-a-cut-in-an-infinite-line approach to $\mathbb{R}$. As Pugh states in the above excerpt from Reference [3], our only burden is to show that there does exist a non-empty set which satisfies the definition. Since our definition is vastly simpler than the ordered field definition, this task is accomplished easily. The main hurdle will be to show that numbers in the neighborhood of infinity satisfy the Archimedes property of real numbers which we concede as an axiomatic requirement for any valid definition of $\mathbb{R}$.

In the next section of this paper, we will give the geometric definition of $x \in \mathbb{R}$ as a cut in a number line. In the third section, we will discuss the affinely extended real number line which is the real line together with its endpoints at infinity. This is the two-point Stone–Čech compactification of $\mathbb{R}$ [4–6]. In the fourth section, we give an axiomatic definition for infinity and we never require at any point that $\infty \in \mathbb{R}$. Infinity is not a real number! After developing infinity, we define real numbers in the neighborhood of infinity. We give their properties and show that such numbers, as presently defined, do not satisfy the axioms of a complete ordered field. It is precisely this property

[1] It is likely that Dedekind ended up with an eponymous variety of cut after he was compelled toward the study of cuts as the endpoints of Euclid's magnitudes during his primary mathematics training. Such magnitudes are cropped from an infinite line by "cutting."

which makes the geometric definition of $\mathbb{R}$ better than the algebraic definition; the geometric definition contains all the numbers admitted by the algebraic definition—numbers in the neighborhood of the origin—but also admits another class of numbers in the neighborhood of infinity whose properties are exciting, interesting, and non-trivial. In the sixth section, we examine the axiomatized arithmetic operations bestowed upon numbers in the neighborhood of infinity throughout the previous sections. In the seventh section, we make a direct extension from $\mathbb{R}$ to $\mathbb{C}$. In the final section, we will discuss the Archimedes property of real numbers and show that the Riemann ζ function has infinitely many non-trivial zeros off the critical line. All of these zeros are in the neighborhood of infinity.

The main purpose of this paper to develop axiomatically enough of the properties of the neighborhood of infinity for us to prove a few theorems regarding the behavior of the Riemann ζ function in that neighborhood. An extended analysis of the topological properties of $\mathbb{R}$ as given here appears in Reference [7], as does the formal construction by Cauchy sequences.

§2 Real Numbers

Definition 2.1 The real numbers are defined in interval notation as

$$\mathbb{R} \quad \equiv \quad (-\infty, \infty) \quad ,$$

where the interval $(-\infty, \infty)$ is an infinite line.

Definition 2.2 A cut in a line $x \in \mathbb{R}$ separates one line into two pieces as

$$\mathbb{R} \setminus x \quad \equiv \quad (-\infty, x) \cup (x, \infty) \quad .$$

Definition 2.3 A real number $x \in \mathbb{R}$ is a cut in the real number line.

Remark 2.4 A number is a cut in a line. A line is defined in Reference [1]. All lines can be cut so all lines are number lines. A given line is the real number line by definition. A real number separates the real number line into a set of "larger" real numbers and a set of "smaller" real numbers.

Definition 2.5 Call real numbers in the neighborhood of the origin $x \in \mathbb{R}_0$. Define them such that

$$\mathbb{R}_0 \quad \equiv \quad \{\, x \mid (\exists n \in \mathbb{N})[-n < x < n] \} \quad .$$

Here we define $\mathbb{R}_0$ as the set of all x such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$.

Definition 2.6 Call real numbers in the neighborhood of infinity $x \in \mathbb{R}_\infty$. Define them as all real numbers except for real numbers in the neighborhood of the origin:

$$\mathbb{R}_\infty \quad \equiv \quad \mathbb{R} \setminus \mathbb{R}_0 \quad .$$

Remark 2.7 A main result of this paper (Main Theorem 5.5) is to demonstrate that $\mathbb{R}_\infty$ is not the empty set.

Definition 2.8 For $x \in \mathbb{R}$, we have the property

$$\lim_{x\to 0^\pm} \frac{1}{x} = \text{diverges} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \text{diverges} \quad .$$

§3 Affinely Extended Real Numbers

Remark 3.1 In this section, we present a few standard properties of the affinely extended real numbers which can be found in References [8–11]. The affinely extended real numbers are a Stone–Čech compactification of $\mathbb{R}$ [4–6]. We will use this well known two-point compactification to introduce the neighborhood of infinity. Then we will prove that neighborhood exists in uncompactified, ordinary $\mathbb{R}$ in the Euclidean sense. Since the real number line has two distinct branches, positive and negative, we will introduce the neighborhood of the infinity though the two-point compactification purely for convenience.

Definition 3.2 Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$

$$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \infty \quad .$$

Definition 3.3 The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 3.4 The affinely extended real numbers are defined in interval notation as

$$\overline{\mathbb{R}} \quad \equiv \quad [-\infty, \infty] \quad .$$

Definition 3.5 An affinely extended real number $x \in \overline{\mathbb{R}}$ is $\pm\infty$ or it is a cut in the affinely extended real number line.

Axiom 3.6 In $\overline{\mathbb{R}}$, $\pm\infty$ are such that the limit of any monotonic sequence of real numbers which diverges in $\mathbb{R}$ is equal to ∞ or $-\infty$.

Theorem 3.7 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 3.3.

Axiom 3.8 Infinity is such that

$$\infty - \infty = \text{undefined} \quad , \qquad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Axiom 3.9 Infinity does not have the distributive property of multiplication over addition.

Remark 3.10 Axiom 3.9 is given precisely because we are using here the two-point compactification of $\mathbb{R}$ [4–6]. Axiom 3.9 may be discarded if one prefers instead to use the one-point Alexandroff compactification [12]. Peaking ahead to the treatment of complex variables of the form $x + iy$ in Section 7, in the two-point compactification we will take

$$\infty \cdot \big(x + iy\big) = \text{undefined} \quad ,$$

because ∞ and $i\infty$ are unified by a one-point compactification. In the one-point compactification, we would eschew Axiom 3.9 to take

$$\infty \cdot \big(x + iy\big) = \infty \quad .$$

§4 Infinity Hat

Definition 4.1 Additive absorption is a property of ∞ such that non-zero $\mathbb{R}_0$ numbers are additive identities of $\pm\infty$. The additive absorptive property is

$$\pm\infty + x = x \pm \infty = \pm\infty \quad , \qquad \text{for} \qquad x \in \mathbb{R}_0 \setminus 0 \quad .$$

Remark 4.2 In avoidance of certain contradictions, the arithmetic operations which we shall axiomatize for real numbers in the neighborhood of infinity will require the removal of the zero additive identity element from infinity. For this reason we have specified $x \in \mathbb{R}_0 \setminus 0$ in Definition 4.1.

Definition 4.3 Let the symbol $\widehat{\infty}$ be called "infinity hat" and endow it with every property of ∞ except additive absorption. $\pm\widehat{\infty}$ are explicitly two different infinities such that

$$-\big(\pm\widehat{\infty}\big) = \mp\widehat{\infty} \quad .$$

Axiom 4.4 Infinity and infinity hat both describe the same affinely extended real number, *i.e.*: $\|\widehat{\infty}\| = \infty$.

Remark 4.5 Although the operations of $\widehat{\infty}$ differ from those of ∞, we say they are the same number because the present treatment identifies unique numbers with unique points along the extended real number line. The operations neither contribute to nor appear in the definition of the number. Therefore, Axiom 4.4 is not contradictory in any way although such an axiom is impossible within the framework of the ordered field definition of $\mathbb{R}$.

Definition 4.6 If an expression using $\widehat{\infty}$ causes a contradiction via additive non-absorptivity, then the hat must be removed to alleviate the contradiction. This property requiring removal of the hat in certain instances is called the non-contradiction property of $\widehat{\infty}$.

Remark 4.7 Although it is not difficult to obtain statements requiring us to remove the hat from $\widehat{\infty}$ through the non-contradiction property, there is a very broad class of structures in which the hat does not imply any contradiction and **this class of structures should be studied**.

Axiom 4.8 The hat which differentiates infinity hat $\widehat{\infty}$ from canonical infinity ∞ is inserted and removed by choice except in the case where it invokes a contradiction and must be removed by definition.

Example 4.9 An example of a statement in which the hat does not invoke a contradiction and may be left in place is

$$x = \widehat{\infty} - b \quad .$$

Example 4.10 An example of a statement in which the hat invokes a contradiction and may not be left in place is given by two sequences

$$x_n = \sum_{k=1}^{n} k \quad , \qquad \text{and} \qquad y_n = c_0 + \sum_{k=1}^{n} k \quad ,$$

where c_0 is some non-zero real number. Since ∞ and $\widehat{\infty}$ are the same number (Axiom 4.4), we can use Definition 3.2 to write

$$\lim_{n\to\infty} x_n = \infty = \widehat{\infty} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty = \widehat{\infty} \quad .$$

We may also write, however,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} c_0 + \lim_{n\to\infty} x_n = c_0 + \widehat{\infty} \quad .$$

This delivers an equality

$$\widehat{\infty} = c_0 + \widehat{\infty} \quad ,$$

which contradicts the additive non-absorption of $\widehat{\infty}$. At this point, we must obey the non-contradiction property of $\widehat{\infty}$ (Definition 4.6) by removing the hat. Then

$$\infty = c_0 + \infty \quad ,$$

demonstrates the usual additive absorptive property of infinity and there is no contradiction.

Axiom 4.11 $\widehat{\infty}$ is such that for any non-zero $b \in \mathbb{R}_0$

$$\begin{aligned}
\pm\widehat{\infty} + b &= b \pm \widehat{\infty} \\
\pm\widehat{\infty} - b &= -b \pm \widehat{\infty} \\
\pm\widehat{\infty} + \big(-b\big) &= \pm\widehat{\infty} - b \\
\pm\widehat{\infty} + b &= \pm\widehat{\infty} - \big(-b\big) \\
-\big(\pm\widehat{\infty}\big) &= \mp\widehat{\infty} \\
\pm\widehat{\infty} \cdot b = b \cdot \big(\pm\widehat{\infty}\big) &= \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases} \\
\frac{\pm\widehat{\infty}}{b} &= \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases} \\
\frac{b}{\pm\widehat{\infty}} &= 0 \quad .
\end{aligned}$$

Axiom 4.12 Regarding $0 \in \mathbb{R}_0$, $\widehat{\infty}$ is such that

$$\begin{aligned}
\pm\widehat{\infty} + 0 = 0 \pm \widehat{\infty} = \pm\widehat{\infty} - 0 &= \text{undefined} \\
\pm\widehat{\infty} \cdot 0 = 0 \cdot \pm\widehat{\infty} &= \text{undefined} \\
\frac{\pm\widehat{\infty}}{0} &= \text{undefined} \\
\frac{0}{\pm\widehat{\infty}} &= 0 \quad .
\end{aligned}$$

Remark 4.13 We will revisit the lack of zero as an additive identity element for $\widehat{\infty}$ in Example 5.12. We will show that it is required to remove the zero additive identity from infinity if we want the freedom to add and subtract real numbers in the neighborhood of infinity.

Remark 4.14 When the ∞ symbol appears as $\widehat{\infty}$, we consider the hat to be an instruction to delay the additive absorption of ∞ indefinitely or until such a delay causes a contradiction. The instruction to "delay additive absorption" should be understood to mean that additive absorption is not a property of $\widehat{\infty}$

but that the additive absorptive property can be implemented trivially after an *ad hoc* decision to remove the hat by choice, or after its removal is required by the non-contradiction property.

Infinity without the hat is afforded with some freedom to do the algebraic operations of its expressions in different orders, and there is no requirement in mathematics that every expression must be simplified as much as possible. Since there is no *a priori* requirement for us to immediately execute the additive absorptive operation in all cases, we add absolutely nothing to infinity with the hat. Rather, the hat is simply a superior alternative to a declaration, "Remember not to simplify this expression via the additive absorptive operation which does not necessarily have to be completed immediately." Instead, we will put the hat there as a reminder not to do that operation while the hat is in place.

We have not added anything to infinity with the hat. The hat is merely an instruction about how to use the algebraic freedom which already exists in the order of operations. While we have not added anything to infinity, we have added something to mathematics. Peeking ahead to Axiom 5.10, an ideal example of that which is gained through the hat is

$$\big(\widehat{\infty}-b\big)-\big(\widehat{\infty}-a\big)=a-b \quad .$$

After delaying the absorptive operation on the left, there is no infinity remaining on the right into which we might absorb. Such statements are not possible without the axioms of $\widehat{\mathbb{R}}$ (given in the following section.)

§5 Real Numbers in the Neighborhood of Infinity

Definition 5.1 The set of large real numbers in the neighborhood of infinity is

$$\widehat{\mathbb{R}} \quad \equiv \quad \Big\{ \pm\big(\widehat{\infty}-b\big) \mid b\in\mathbb{R}_0,\ b>0 \Big\} \quad .$$

Remark 5.2 We call $\widehat{\mathbb{R}}$ large numbers in the neighborhood of infinity to distinguish them from all numbers in the neighborhood of infinity: $\mathbb{R}_\infty \equiv \mathbb{R}\setminus\mathbb{R}_0$. As we show in Reference [7], it is not the case that $\mathbb{R}_\infty\setminus\widehat{\mathbb{R}}\equiv\emptyset$.

Axiom 5.3 The ordering of $\widehat{\mathbb{R}}$ numbers is

$$\begin{array}{rcc}
\pm\big(\widehat{\infty}-b\big)=\pm\big(\widehat{\infty}-a\big) & \iff & a=b \\
\big(\widehat{\infty}-b\big)>\big(\widehat{\infty}-a\big) & \iff & a>b \\
-\big(\widehat{\infty}-b\big)>-\big(\widehat{\infty}-a\big) & \iff & a<b \\
\big(\widehat{\infty}-b\big)>-\big(\widehat{\infty}-a\big) & \forall & b,a\in\mathbb{R}_0 \\
\big(\widehat{\infty}-b\big)>x & \forall & b,x\in\mathbb{R}_0
\end{array}$$

$$\begin{aligned}
-\big(\widehat{\infty}-b\big) &< x &\qquad \forall &\qquad b,x\in\mathbb{R}_0\\
\big(\widehat{\infty}-b\big) &< \infty &\qquad \forall &\qquad b\in\mathbb{R}_0\\
-\big(\widehat{\infty}-b\big) &> -\infty &\qquad \forall &\qquad b\in\mathbb{R}_0 \quad .
\end{aligned}$$

Theorem 5.4 $\widehat{\mathbb{R}}$ *is a subset of* $\overline{\mathbb{R}}$.

Proof. Definition 3.3 gives $\overline{\mathbb{R}}\equiv[-\infty,\infty]$. By Definition 3.5, an affinely extended real number $x\in\overline{\mathbb{R}}$ is a cut in, or endpoint of, the affinely extended real number line. Definition 2.2 requires that a cut separates one line into two pieces. Observe that

$$\begin{aligned}
\overline{\mathbb{R}}\setminus(\widehat{\infty}-b) &\equiv [-\infty,\widehat{\infty}-b)\cup(\widehat{\infty}-b,\infty]\\
\overline{\mathbb{R}}\setminus\big(-\widehat{\infty}+b\big) &\equiv [-\infty,-\widehat{\infty}+b)\cup(-\widehat{\infty}+b,\infty] \quad .
\end{aligned}$$

All $x\in\widehat{\mathbb{R}}$ conform to the definition of affinely extended real numbers so $\widehat{\mathbb{R}}\subset\overline{\mathbb{R}}$.

Main Theorem 5.5 $\widehat{\mathbb{R}}$ *is a subset of* $\mathbb{R}$.

Proof. If a number is an affinely extended real number $x\in\overline{\mathbb{R}}$ and $x\neq\pm\infty$, then, by Theorem 3.7, we have $x\in\mathbb{R}$. Theorem 5.4 proves

$$x\in\widehat{\mathbb{R}} \quad\implies\quad x\in\overline{\mathbb{R}} \quad ,$$

so, in the absence of additive absorption,

$$\pm\big(\widehat{\infty}-b\big)\neq\pm\widehat{\infty}=\pm\infty \quad .$$

(Definition 5.1 requires $b\neq 0$.) This proves the theorem.

Alternatively, by the ordering of Axiom 5.3, we have

$$\mathbb{R}\setminus(\widehat{\infty}-b) \equiv (-\infty,\widehat{\infty}-b)\cup(\widehat{\infty}-b,\infty) \quad .$$

All numbers $x\in\widehat{\mathbb{R}}$ satisfy the definition of $x\in\mathbb{R}$ through Definitions 2.2 and 2.3. This also proves the theorem.

Theorem 5.6 $\widehat{\mathbb{R}}$ *is a subset of* $\mathbb{R}_\infty$.

Proof. We have shown in Main Theorem 5.5 that

$$\widehat{\mathbb{R}}\subset\mathbb{R} \quad ,$$

so we will satisfy the definition of $\mathbb{R}_\infty$ (Definition 2.6) if we show that

$$\widehat{\mathbb{R}} \cap \mathbb{R}_0 \quad \equiv \quad \emptyset \ \ .$$

Definition 2.5 requires that elements of $\mathbb{R}_0$ satisfy

$$-n < x < n \ \ ,$$

For proof by contradiction, assume $\pm\big(\widehat{\infty} - b\big) \in \mathbb{R}_0$. This requires

$$-n < \pm\big(\widehat{\infty} - b\big) < n \ \ .$$

Since $b \in \mathbb{R}_0$, we know it has an additive inverse. Add or subtract b to obtain

$$-n + b < \widehat{\infty} < n + b \ \ , \qquad \text{and} \qquad -n - b < -\widehat{\infty} < n - b \ \ .$$

We obtain a contradiction as $\widehat{\infty}$ cannot be less than the sum of two $\mathbb{R}_0$ numbers and $-\widehat{\infty}$ cannot be greater than the difference of two $\mathbb{R}_0$ numbers. All $\widehat{\mathbb{R}}$ numbers satisfy the definition of $\mathbb{R}_\infty$ (Definition 2.6.)

Remark 5.7 The remainder of this section defines and makes remarks on the arithmetic operations for $\widehat{\mathbb{R}}$ numbers, and we will also treat these operations in the following section. The purpose in defining operations for $\widehat{\mathbb{R}}$ is to supplement the canonical operations for $\mathbb{R}_0$ and those of $\infty \sim \widehat{\infty}$. (The canonical operations of $\mathbb{R}_0$ are those given when $\mathcal{R}' = \{\mathbb{R}_0, +, \times\}$ satisfies the field axioms.) Every $\widehat{\mathbb{R}}$ number can be decomposed and its pieces manipulated separately but the main purpose in defining special operations for $\widehat{\mathbb{R}}$ is to define new operations for expressions which are undefined under the arithmetic operations of $\mathbb{R}_0$ and $\widehat{\infty}$ alone, or whose structure vanishes under additive absorption.

Axiom 5.8 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\mathbb{R}_0$ numbers are

$$\begin{aligned}
-\big(\widehat{\infty} - b\big) &= -\widehat{\infty} + b \\
-\big(- \widehat{\infty} + b\big) &= \widehat{\infty} - b \\
\pm\big(\widehat{\infty} - b\big) + x = x \pm \big(\widehat{\infty} - b\big) &= \begin{cases} \pm\widehat{\infty} \mp \big(b \mp x\big) & \text{if} \quad b \neq x \\ \pm\widehat{\infty} & \text{if} \quad b = x \end{cases} \\
\pm\big(\widehat{\infty} - b\big) \cdot x = x \cdot \pm\big(\widehat{\infty} - b\big) &= \begin{cases} \pm\big(\widehat{\infty} - xb\big) & \text{if} \quad x \neq 0 \\ \text{undefined} & \text{if} \quad x = 0 \end{cases} \\
\frac{\pm\big(\widehat{\infty} - b\big)}{x} &= \begin{cases} \pm\widehat{\infty} \mp \dfrac{b}{x} & \text{if} \quad x \neq 0 \\ \text{undefined} & \text{if} \quad x = 0 \end{cases} \\
\frac{x}{\pm\big(\widehat{\infty} - b\big)} &= 0 \ \ .
\end{aligned}$$

Remark 5.9 Although the axiom of closure is not always explicitly included in the axioms of a complete ordered field, it is usually taken for granted that number fields are closed under their operations. By Axiom 5.8, however,

$$x \in \mathbb{R}_0 \quad , \quad x > b \qquad \Longrightarrow \qquad \big[\big(\widehat{\infty} - b \big) + x \big] \notin \widehat{\mathbb{R}} \quad .$$

Although we have not given the ordering for numbers of the form $\big(\widehat{\infty} + a\big)$ with $a > 0$, it reasonably follows from Axiom 5.3 that such numbers are greater than infinity. Therefore, with $\mathbb{R}$ as presently defined, the set $\mathcal{R} = \{\mathbb{R}, +, \times\}$ is not a number field because it is not closed under its operations.

Axiom 5.10 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\widehat{\mathbb{R}}$ numbers are

$$\begin{aligned} \pm\big(\widehat{\infty} - b\big) \pm \big(\widehat{\infty} - a\big) &= \pm\widehat{\infty} \mp \big(a + b\big) \\ \pm\big(\widehat{\infty} - b\big) \mp \big(\widehat{\infty} - a\big) &= \pm\big(a - b\big) \\ \pm\big(\widehat{\infty} - b\big)\big(\widehat{\infty} - a\big) &= \pm\widehat{\infty} \\ \frac{\pm\big(\widehat{\infty} - b\big)}{\widehat{\infty} - a} &= \pm 1 \quad . \end{aligned}$$

Remark 5.11 Axiom 5.10 states that

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad .$$

Although this implies the existence of an additive inverse for every $\widehat{\mathbb{R}}$ number, it does not imply an additive inverse for $\widehat{\infty}$ because the case of $a = b = 0$ is ruled out by the definition of $\widehat{\mathbb{R}}$ (Definition 5.1.)

Example 5.12 Axiom 4.12 states that infinity does not have a zero additive identity element. We were not able to demonstrate this requirement in Section 4 because we needed first to define the axioms of $\widehat{\mathbb{R}}$ which make it impossible for ∞ or $\widehat{\infty}$ to have zero as an additive identity.[2] This example gives an illustration of the type of contradictions which are avoided by removing the zero additive identity element of infinity. Consider the limit

$$\lim_{x\to\infty} \big(x^2 - x\big) = \infty \quad ,$$

which can also be computed as

$$\lim_{x\to\infty} \big(x^2 - x\big) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty \quad .$$

This is a typical example used to demonstrate the lack of an additive inverse for ∞. If infinity is bestowed with an additive inverse, then we obtain from

[2] The additive identity element of $\pm\widehat{\infty}$ is $\pm\widehat{\infty}$ through the multiplicative absorptive property and the definition of multiplication that $2x = x + x$. Every $x \in \mathbb{R}_0 \backslash 0$ is an additive identity element of ∞.

the above a contradiction $\infty = 0$. The expression $\infty - \infty$, thus, is undefined. If we added the hats to infinity then we could insert the additive identity on the right side of $\infty = \infty - \infty$ to write

$$\begin{aligned}\widehat{\infty} &= \widehat{\infty} - \widehat{\infty} \\ &= \widehat{\infty} - \widehat{\infty} + 0 \\ &= \widehat{\infty} - \widehat{\infty} + 1 - 1 \\ &= \big(\widehat{\infty} - 1\big) - \big(\widehat{\infty} - 1\big) = 0 \quad .\end{aligned}$$

We see that unhatted infinity likewise cannot have zero as an additive identity because we could write

$$\begin{aligned}\infty &= \infty - \infty \\ &= \infty - \infty + \big(1 - 1\big) \\ &= \widehat{\infty} - \widehat{\infty} + 1 - 1 \\ &= \big(\widehat{\infty} - 1\big) - \big(\widehat{\infty} - 1\big) = 0 \quad ,\end{aligned}$$

where we have simply chosen within the freedom afforded to the order of algebraic operations not to do the additive absorptive operation at the second step. By allowing infinity to have zero as an additive identity element, we induce the same contradiction which forbids an additive inverse for infinity.

Remark 5.13 The expressions ∞ and $\widehat{\infty}$ are perfectly well defined but $\infty + 0$ and $\widehat{\infty} + 0$ are examples of an undefined composition. Since ∞ is not an $\widehat{\mathbb{R}}$ number, this property cannot create problems for the algebra of $\widehat{\mathbb{R}}$ numbers. Essentially, we have traded the zero additive identity element of infinity for the freedom to add and subtract $\widehat{\mathbb{R}}$ numbers.

Axiom 5.14 The additive operation is not associative for $\widehat{\mathbb{R}} + \widehat{\mathbb{R}}$.

Example 5.15 The example demonstrates why $\widehat{\mathbb{R}} + \widehat{\mathbb{R}}$ cannot have the associative property. Through associativity we may easily derive a contradiction from Axiom 5.10 which gives

$$\big(\widehat{\infty} - b\big) + \big(\widehat{\infty} - a\big) = \widehat{\infty} - \big(b + a\big) \quad .$$

To obtain that contradiction, subtract an $\widehat{\mathbb{R}}$ number from both sides as

$$\Big[\big(\widehat{\infty} - b\big) + \big(\widehat{\infty} - a\big)\Big] - \big(\widehat{\infty} - c\big) = \widehat{\infty} - \big(b + a\big) - \big(\widehat{\infty} - c\big) \quad .$$

Assuming the associative property of addition, we may arrange brackets to write

$$\big(\widehat{\infty} - b\big) + \Big[\big(\widehat{\infty} - a\big) - \big(\widehat{\infty} - c\big)\Big] = \widehat{\infty} - \big(b + a\big) - \big(\widehat{\infty} - c\big)$$

$$\widehat{\infty} + \big[c - \big(b+a\big)\big] = c - \big(b+a\big) \quad .$$

Subtracting the $\mathbb{R}_0$ part from both sides yields

$$\widehat{\infty} = 0 \quad ,$$

which is not allowed. Since additive associativity is required among the elements of a number field, this example further demonstrates that the present definition of $\mathbb{R}$ is not such that $\mathcal{R} = \{\mathbb{R}, +, \times\}$ satisfies the field axioms in all cases.

Theorem 5.16 *It is possible to make cuts in the real number line which are numbers greater than any $n \in \mathbb{N}$, i.e.: certain cuts in the real number line are real numbers in the neighborhood of infinity.*

Proof. Suppose there exists a line segment AB. Every line segment can be bisected by a cut at its midpoint C. We say C is a midpoint of AB if and only if

$$\operatorname{len} AC = \operatorname{len} CB \quad , \qquad \text{and} \qquad \operatorname{len} AC + \operatorname{len} CB = \operatorname{len} AB \quad .$$

Suppose x is a chart on AB such that $x \in [0, \pi/2]$. Then define a conformal chart x' such that

$$x' = \tan(x) \quad , \qquad \text{and} \qquad x' \in [0, \infty] \quad .$$

The greater bound on the x' chart is derived through Definition 3.2 as

$$\lim_{\theta \to \pi/2} \tan(\theta) = \lim_{\theta \to \pi/2} \frac{\sin(\theta)}{\cos(\theta)} = \lim_{\substack{x \to 0 \\ y \to 1}} \frac{y}{x} = \lim_{x \to 0} \frac{1}{x} = \infty \quad .$$

We know it is possible to bisect any line segment AB at a midpoint C. If we add a chart to AB, it cannot affect this fundamental geometric property that every line segment can be bisected. Adding a chart such that $x'(B) = \infty$ does not disrupt the fundamental geometric properties of line segment. To prove the present theorem by contradiction, assume $x'(C)$ is less than some $n \in \mathbb{N}$ (where $x'(C)$ refers to the value of x' at the geometric midpoint of AB.) Then

$$\operatorname{len} AC < n \quad , \qquad \text{and} \qquad \operatorname{len} CB < n \quad ,$$

so it follows that

$$\operatorname{len} AC + \operatorname{len} CB < 2n \quad .$$

Here we obtain a contradiction because, by the definition of a midpoint, we have

$$\operatorname{len} AC + \operatorname{len} CB = \operatorname{len} AB = \infty \quad , \qquad \text{but} \qquad \infty \not< 2n \quad .$$

Therefore, when AB is charted in x', the magnitude of the cut at the midpoint C is greater than any natural number. Because $\operatorname{len}[0, \infty] = \operatorname{len}(0, \infty)$, it follows that we can make a cut in the positive branch of the real line at a magnitude greater than any natural number.

Remark 5.17 In Theorem 5.16, we have derived a requirement for two lengths which are equal, less than infinity, and whose sum is equal to infinity. This requirement is given a dedicated treatment in Reference [7].

Theorem 5.18 *An $\widehat{\mathbb{R}}$ number does not have a multiplicative inverse.*

Proof. Axiom 5.10 gives

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = 1 \quad ,$$

which allows us to give a simple proof by contradiction. Assume that the numerator has a multiplicative inverse e_b. Multiplying both sides by e_b gives

$$\frac{e_b\big(\widehat{\infty} - b\big)}{\widehat{\infty} - a} = e_b \qquad \Longrightarrow \qquad e_b = \frac{1}{\widehat{\infty} - a} \quad .$$

By Axiom 5.8, the expression on the left identically zero. If $e_b = 0$, however, then the product $e_b\big(\widehat{\infty} - b\big)$ is undefined (Axiom 5.8.) Therefore, $x \in \widehat{\mathbb{R}}$ cannot have a multiplicative inverse.

Remark 5.19 In Example 6.3, we will discuss the notion that division cannot be defined as multiplication by the inverse for numbers which do not have a multiplicative inverse.

§6 Limit Considerations for Axiomatized Operations

Example 6.1 This example shows we that cannot always substitute the limit definition of infinity to directly compute all expressions involving ∞. If we use Definition 3.2 to write

$$\infty - \infty = \left(\lim_{x\to 0} \frac{1}{x}\right) - \left(\lim_{x\to 0} \frac{1}{x}\right) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0 \quad ,$$

then we contradict Axiom 3.8 which gives

$$\infty - \infty = \text{undefined} \quad .$$

This example motivates the axiomatization of certain operations on expressions using $\widehat{\infty}$.

Theorem 6.2 *The quotient of a number $x \in \mathbb{R}_0$ divided by a number $y \in \widehat{\mathbb{R}}$ is identically zero (Axiom 5.8.)*

Proof. For proof by contradiction, et z be any non-zero real number such that

$$\frac{x}{y} = z \quad .$$

By Axiom 5.3, $||x|| < ||y||$ so we have $||z|| < 1$ which implies $z \in \mathbb{R}_0$. (The case of $||z|| = 1$ would imply $x \in \widehat{\mathbb{R}}$, a contradiction.) All $\mathbb{R}_0$ numbers have a multiplicative inverse so we find, therefore, that

$$\frac{x}{zy} = 1 \qquad \Longleftrightarrow \qquad x = zy \quad .$$

The product zy is given by Axiom 5.8 as

$$zy = z \cdot \big(\pm \widehat{\infty} \mp b \big) = \pm \big(\widehat{\infty} - zb \big) \quad .$$

This delivers a contradiction because it requires that $x = zy$ is a real number in the neighborhood of infinity while we have already defined x to be a real number in the neighborhood of the origin. Therefore, the only possible numerical value for x/y is 0.

Alternatively, the limit definition of infinity gives the same result. Observe that for some $n, b \in \mathbb{R}_0$ we have

$$\frac{n}{\widehat{\infty} - b} = \lim_{x \to 0} \frac{n}{\frac{1}{x} - b} = \lim_{x \to 0} \frac{nx}{1 - bx} = 0 \quad .$$

Example 6.3 This example treats the $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ operation. In Axiom 5.8, we have given

$$0 \cdot \big(\widehat{\infty} - b \big) = \text{undefined} \quad .$$

This follows from

$$0 \cdot \big(\widehat{\infty} - b \big) = 0 \cdot \widehat{\infty} - 0 \cdot b \quad ,$$

with $0 \cdot \widehat{\infty}$ being undefined, as per usual. In Axiom 5.10, however, we have given

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = 1 \quad ,$$

which can be written as

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \big(\widehat{\infty} - b \big) \left(\frac{1}{\widehat{\infty} - a} \right) = \big(\widehat{\infty} - b \big) \cdot 0 \quad .$$

Apparently, $\widehat{\mathbb{R}}/\widehat{\mathbb{R}} = \pm 1$ contradicts Axiom 5.8 which says such expressions are undefined. For this, we need to give special attention to the $\div$ operator which is usually defined as multiplication by the inverse. Theorem 5.18 shows that $x \in \widehat{\mathbb{R}}$ does not have a multiplicative inverse so

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} \neq \big(\widehat{\infty} - b \big) \big(\widehat{\infty} - a \big)^{-1} \quad .$$

We need a separate definition for the $\div$ operation. In this case, we resort to the limit

$$\frac{\widehat{\infty}-b}{\widehat{\infty}-a}=\lim_{x\to0}\frac{\frac{1}{x}-b}{\frac{1}{x}-a}=\lim_{x\to0}\frac{1-bx}{1-ax}=1 \quad .$$

This is the result given in Axiom 5.10.

Example 6.4 This example treats the $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ operation. If we axiomatize this operation with the limit definition of infinity, then we obtain

$$\begin{aligned}\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big)&=\lim_{x\to0}\left(\frac{1}{x}-b\right)\left(\frac{1}{x}-a\right)\\&=\lim_{x\to0}\left(\frac{1}{x^2}-\frac{b+a}{x}+ba\right)\\&=\lim_{x\to0}\left(\frac{1-x\big(b+a\big)+x^2ba}{x^2}\right)=\text{diverges}\\&=\infty \quad .\end{aligned}$$

This is the value that appears in Axiom 5.10. If we gave any value other than that supported by the limit definition of infinity, then that would be contrived and arbitrary because we have already used the limit definition for $\widehat{\mathbb{R}}\div\widehat{\mathbb{R}}$ in that same axiom. However, we could equally well choose some definition for $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ and then require for consistency that $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ be computed in the same way. Therefore, in this example we will demonstrate the invalidity of a few other possible definitions for $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ to support the limit as the correct computation to determine the relevant quotient and product.

Firstly, Axiom 3.9 states that ∞ does not have the distributive property of multiplication over addition so $\widehat{\infty}$, likewise, does not have this property. Therefore, we cannot compute $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ with the FOIL method. If we could do that, then by the assumption $\widehat{\infty}^2=\widehat{\infty}$ we would obtain

$$\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big)=\widehat{\infty}-\widehat{\infty}-\widehat{\infty}+ba \quad .$$

As written, this expression is undefined through $\widehat{\infty}-\widehat{\infty}$. If we rearranged the $\mathbb{R}_0$ term as

$$\widehat{\infty}-\widehat{\infty}-\widehat{\infty}+2ba-ba=\big(\widehat{\infty}-ba\big)-\big(\widehat{\infty}-ba\big)-\big(\widehat{\infty}-ba\big) \quad ,$$

then the non-associativity of addition in $\widehat{\mathbb{R}}$ gives a contradiction

$$\begin{aligned}\Big[\big(\widehat{\infty}-ba\big)-\big(\widehat{\infty}-ba\big)\Big]-\big(\widehat{\infty}-ba\big)&=\big(\widehat{\infty}-ba\big)-\Big[\big(\widehat{\infty}-ba\big)+\big(\widehat{\infty}-ba\big)\Big]\\0-\big(\widehat{\infty}-ba\big)&=\big(\widehat{\infty}-ba\big)-\big(\widehat{\infty}-2ba\big)\\-\big(\widehat{\infty}-ba\big)&=ba \quad .\end{aligned}$$

The expression on the left is not equal to the expression on the right. Indeed, if we subtract away the $\mathbb{R}_0$ part, then we obtain the familiar contradiction $-\widehat{\infty} = 0$.

As another non-limit method for computing $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$, we should examine the multiplicative absorptive property of $\widehat{\infty}$. Due to the lack of multiplicative distributivity over addition, it is impossible get either of $\big(\widehat{\infty} - b\big)$ or $\big(\widehat{\infty} - a\big)$ on its own multiplied by $\widehat{\infty}$. Therefore, the multiplicative absorptive property of $\widehat{\infty}$ cannot uniquely determine the product of $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$.

Remark 6.5 Consider $\mathbb{R}_0 \times \widehat{\mathbb{R}}$ as in

$$x\big(\widehat{\infty} - b\big) = \widehat{\infty} - xb \quad .$$

If x is a positive number, then by Axiom 5.3 the magnitude of the product decreases as the magnitude of x increases. Therefore, there is some radical change in behavior as x increases from $\mathbb{R}_0$ to $\widehat{\mathbb{R}}$ because the product $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ is greater than any $x \in \widehat{\mathbb{R}}$. This exotic behavior is studied more closely in Reference [7].

§7 Complex Numbers

Definition 7.1 The set of all complex numbers is

$$\mathbb{C} \quad \equiv \quad \{x + iy \mid x, y \in \mathbb{R},\ i = \sqrt{-1}\,\}$$

Definition 7.2 The set of all complex numbers in the neighborhood of the origin is

$$\mathbb{C}_0 \quad \equiv \quad \{x + iy \mid x, y \in \mathbb{R}_0,\ i = \sqrt{-1}\,\}$$

Axiom 7.3 As ∞ does not absorb -1 in 1D, in 2D (meaning in $\mathbb{C}$) we have the condition that infinity absorbs neither -1 nor $\pm i$. In other words, we carry the Stone–Čech compactification over into $\mathbb{C}$ and do not adopt a one-point compactification.

Definition 7.4 The affinely extended complex plane is

$$\overline{\mathbb{C}} \quad \equiv \quad \mathbb{C} \cup \{\pm\infty\} \cup \{\pm i\infty\} \quad .$$

Remark 7.5 As the extended real line $\overline{\mathbb{R}}$ has two distinct infinities, the extended complex plane $\overline{\mathbb{C}}$ has four: $\{+\infty, +i\infty, -\infty, -i\infty\}$.

Axiom 7.6 The multiplicative operations for $\pm\widehat{\infty}$ and $\pm i\,\widehat{\infty}$ with i are

$$\pm\widehat{\infty} \cdot i = i \cdot \big(\pm\widehat{\infty}\big) = \pm i\,\widehat{\infty}$$

$$\pm i\,\widehat{\infty}\cdot i = i\cdot\big(\pm i\,\widehat{\infty}\big) = \mp\widehat{\infty} \quad .$$

Remark 7.7 The non-distributive property of $\pm\infty$ by multiplication over addition (Axiom 3.9) was practically redundant in 1D but for $z\in\mathbb{C}$ this feature gains significance. For the 1D case, if $a,b\in\mathbb{R}_0$, then there exists a $c\in\mathbb{R}_0$ such that $a+b=c$. This allows us to mimic the distributive property through multiplicative absorption as

$$\widehat{\infty}\cdot\big(a+b\big) = \widehat{\infty}\cdot c = \widehat{\infty} \quad .$$

To the contrary, if $z\in\mathbb{C}_0$, then

$$\widehat{\infty}\cdot\big(x+iy\big) = \text{undefined} \neq \operatorname{sign}(x)\widehat{\infty} + i\operatorname{sign}(y)\widehat{\infty}.$$

Axiom 7.8 The multiplicative operations for $\pm\widehat{\infty}$ with complex numbers $z\in\mathbb{C}_0$ are

$$\pm\widehat{\infty}\cdot z = z\cdot\pm\widehat{\infty} = \begin{cases} \pm\widehat{\infty} & \text{if} \quad \operatorname{Re}(z)>0 \quad \text{and} \quad \operatorname{Im}(z)=0 \\ \mp\widehat{\infty} & \text{if} \quad \operatorname{Re}(z)<0 \quad \text{and} \quad \operatorname{Im}(z)=0 \\ \pm i\,\widehat{\infty} & \text{if} \quad \operatorname{Im}(z)>0 \quad \text{and} \quad \operatorname{Re}(z)=0 \\ \mp i\,\widehat{\infty} & \text{if} \quad \operatorname{Im}(z)<0 \quad \text{and} \quad \operatorname{Re}(z)=0 \\ \text{undefined} & \text{if} \quad \operatorname{Im}(z)\neq 0 \quad \text{and} \quad \operatorname{Re}(z)\neq 0 \\ \text{undefined} & \text{if} \quad z=0 \end{cases} \quad .$$

Remark 7.9 The multiplicative operations for $\pm i\,\widehat{\infty}$ with complex numbers $z\in\mathbb{C}_0$ follow from Axiom 7.8. The arithmetic operations for complex numbers $z\in\mathbb{C}$ whose real and/or imaginary parts are $\widehat{\mathbb{R}}$ numbers follow directly from the other axioms.

§8 The Riemann Hypothesis

Remark 8.1 The Riemann hypothesis [13–26] dates to Riemann's 1859 paper [27]. Since the axioms of a complete ordered field date to Dedekind's 1872 paper [2], it would be **patently absurd** to claim that the Riemann hypothesis is formulated in terms of the ordered field definition of $\mathbb{R}$. While we cannot directly show what definition of $\mathbb{R}$ Riemann had in mind when formulating his hypothesis, we can point out that his program of Riemannian geometry is a direct extension of Euclidean geometry. This qualitatively supports the notion that Riemann had in mind the cut-in-a-number-line definition of $\mathbb{R}$ given by Euclid in the Elements. When one examines the Elements [1], the very many diagrams, definitions, and postulates make it exceedingly obvious that Euclid's definition of a real number $x\in\mathbb{R}$ is exactly the one given here in Definition

2.2
$$\mathbb{R} \setminus x = (-\infty, x) \cup (x, \infty) \quad ,$$
formulated as the alternative identical statement
$$x \in \mathbb{R} \quad , \quad x > 0 \qquad \Longrightarrow \qquad (0, \infty) = (0, x] \cup (x, \infty) \quad .$$

Riemann—the man himself being the premier mathematical analyst of the 19th century—went to no lengths whatsoever to formalize with rigor the definition of $\mathbb{R}$ used by him to formulate his hypothesis. What does this tell us that a man of the utmost standards of mathematical rigor did not even deem it worthwhile to mention his definition of $\mathbb{R}$? It tells us, in the opinion of this writer, that Riemann assumed it would be implicitly obvious to his intended audience that he was relying on the Euclidean definition of $\mathbb{R}$ which is totally equivalent to the definition given here in Section 2. Furthermore, even if one does not accept that Riemann meant to implicitly use Euclid's definition, upon seeing that Riemann did not give a definition for $\mathbb{R}$, one may reasonably conclude that the specifics of the aspect of Riemann's hypothesis relating to the definition of the domain of $\zeta(z)$ were not highly relevant. Rather, the object of relevance would be the behavior of $\zeta(z)$ at various z. It is reasonable to assume that when Riemann formulated his hypothesis he had in mind that any definition of $\mathbb{R}$ consistent with the Euclid magnitude and explicitly displaying the Archimedes property of real numbers would be sufficient. The domain of $\zeta(z)$, namely $\mathbb{C}$, would be constructed from two orthogonal copies of $\mathbb{R}$, one of them having the requisite phase factor i. As per Pugh [3] quoted in Section 1, if we prove that all $x \in \widehat{\mathbb{R}}$ satisfy the Archimedes property, then that should be sufficient reason to accept the present definition of $\mathbb{R}$ into applications regarding Riemann's hypothesis.

For some reason, the modern statement of the Archimedes property has evolved to include natural numbers in its predicate but this is not at all supported by the statement of the property as it is given in Euclid's Elements [1]. The modern statement depending on natural numbers is
$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$
Numbers in the neighborhood of infinity do not conform to the natural number statement of the Archimedes property but they do absolutely conform to the statement that appears in Euclid's elements (given below.) Regarding the natural number statement of the property, consider Remark 6.5. Those remarks point out that for any $n \in \mathbb{N}$ and any positive $x \in \widehat{\mathbb{R}}$, we have $nx \leq x$. It follows that if $x < y$, then $nx \not> y$. Therefore, $x \in \widehat{\mathbb{R}}$ do not exhibit the Archimedes property of real numbers when it is formulated without precedent in terms of natural numbers. However, we will show in this section that the statement depending on natural numbers has no precedent in the ancient history of mathematics and that it is only a modern (over-)simplification of the

genuine Archimedes property of antiquity. Indeed, without reference to any one technical statement or another, the main gist of the Archimedes property is that there is no greatest real number. It is obvious that the present definition of $\mathbb{R}$ satisfies the main gist of the Archimedes property. The round bracket notation

$$\mathbb{R} \quad \equiv \quad (-\infty, \infty) \ ,$$

directly requires "no greatest element."

In this section, we will examine the Archimedes property to show that all $x \in \widehat{\mathbb{R}}$ do satisfy the property given in Euclid's Elements and that, therefore, they are fully qualified for applications to the Riemann hypothesis. Then we will show that the Riemann ζ function has infinitely many non-trivial zeros off the critical line in the neighborhood of infinity.

Definition 8.2 The statement of the Archimedes property which appears in Euclid's Elements, and which was attributed by Archimedes to his predecessor Eudoxus, and which is very often taken to be *the* definitive statement of the Archimedes property of real numbers, appears as Definition 4 in Book 5 of Euclid's Elements [1]. The original Greek is translated as follows.

> "Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another."

Remark 8.3 As it appears in Euclid's Elements, the straightforward mathematical statement of the property should be

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \ .$$

There is no mention of multiplication by a positive integer $n \in \mathbb{N}$. It is obvious that this property—the statement of the Archimedes property in which the multiplier is $z \in \mathbb{R}$ rather than $n \in \mathbb{N}$—holds for all $x, y \in \mathbb{R}$ as presently defined. If we have

$$x = 1 \ , \quad y = \widehat{\infty} - 1 \qquad \Longrightarrow \qquad x < y \ ,$$

then choosing $z = \widehat{\infty} - 0.9$ gives $zx > y$. If we have

$$x = \widehat{\infty} - 2 \ , \quad y = \widehat{\infty} - 1 \qquad \Longrightarrow \qquad x < y \ ,$$

then choosing $z = 1/3$ gives $zx > y$. It is obvious that the present definition of $\mathbb{R}$ defines a set which exhibits the Archimedes property. For some reason, however, many modern mathematicians often choose to express the property mathematically as

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \ .$$

This is the statement of a property that $\mathbb{R}$ does not have in its current incarnation (Definition 2.3) because we needed to choose $z \notin \mathbb{N}$ for the two cases given in this remark. However, the statement depending on $\mathbb{N}$ is **not** the Archimedes property of real numbers! Nowhere did Euclid mention integers but it is claimed, apparently, that Euclid's definition of multiplication should be taken only to mean multiplication by $n \in \mathbb{N}$ or that, perhaps, it should be obvious from the context that Euclid was writing about multiplication by natural numbers alone. Indeed, careful (or even cursory) examination of the context in Reference [1] shows no such thing.

The Archimedes property given here in Definition 8.2 appears as Definition 4 of Book 5 of Euclid's Elements [1]. We may directly extract from Definitions 1 and 2 of Book 5 that Euclid did not use a definition of multiplication restricted to $n \in \mathbb{N}$. Definition 1 of Book 5 is

> "A magnitude is a part of another magnitude, the lesser of the greater, when it measures the greater."

Fitzpatrick, the English translator of Euclid's Elements cited here as Reference [1], adds the following footnote to this definition.

> "In other words, α is said to be a part of β if $\beta = m\alpha$."

This makes it perfectly obvious that Euclid's multiplication was never restricted to $n \in \mathbb{N}$. Euclid was certainly aware that is possible to measure one magnitude of, say, ten Archimedean length units, and another length having 25 such units. This proves that the multiplier in Euclid's definitions was never intended to be restricted to $\mathbb{N}$.

However, so that we need not cite the translator's footnote in the determination that Euclid had no intention whatsoever to restrict his multiplier implicitly as $n \in \mathbb{N}$, we should also consider Book 5, Definition 2 [1].

> "And the greater is a multiple of the lesser whenever it is measured by the lesser."

Is a length of 25 units an integer multiple of a length of ten units? Obviously not. Are we to believe that Euclid meant to forbid the existence of the number 25 once one has discovered the number ten? Obviously not! A magnitude of 25 units is the greater of the lesser magnitude of ten units with multiplier 2.5. Surely this was known to Euclid!

We finish this remark with Fitzpatrick's footnote to Book 5, Definition 4 which is the Archimedes property proper. Fitzpatrick's footnote to,

> "Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another,"

is,

> "In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n."

Although it is common to use the variables m and n to refer to natural numbers, such is not the case in this context. If the multiplier was restricted as $m, n \in \mathbb{N}$, then it could not be said that four is a part of five, or two a part of three, and it would follow that among four and five or two and three, neither is the greater and neither is the lesser! Clearly this would be an affront to reason! There is absolutely no historical precedent for any statements of the Archimedes property dependent on natural numbers. Such statements should be called, "Archimedes properties of the second kind," or some such thing like, "the Archimedes property of natural numbers," to distinguish them from the Archimedes property of real numbers which famously appears in Euclid's Elements [1].

In closing, if we could add a second footnote to Definition 4 of Book 5, it would be the following.

> "The Archimedes property of real numbers states that there is no largest real number."

The $\widehat{\mathbb{R}}$ proven to be $\widehat{\mathbb{R}} \subset \mathbb{R}$ in Main Theorem 5.5 satisfies the requirement that real numbers have the Archimedes property. Now that we have properly motivated the application of $\widehat{\mathbb{R}}$ to the Riemann hypothesis, we will present the application in the remainder of this section.

Theorem 8.4 *If $b, y_0 \in \mathbb{R}_0$, if $z_0 = (\widehat{\infty} - b) + iy_0$, and if $\zeta(z)$ is the Riemann ζ function, then $\zeta(z_0) = 1$.*

Proof. Observe that the Dirichlet sum form of ζ [27] takes z_0 as

$$\begin{aligned}
\zeta(z_0) &= \sum_{n=1} \frac{1}{n^{(\widehat{\infty}-b)+iy_0}} \\
&= \sum_{n=1} \frac{n^b}{n^{\widehat{\infty}}}\bigg(\cos(y_0 \ln n) - i\sin(y_0 \ln n)\bigg) \\
&= 1 + \sum_{n=2} 0 \bigg(\cos(y_0 \ln n) - i\sin(y_0 \ln n)\bigg) = 1 \quad .
\end{aligned}$$

Main Theorem 8.5 *The Riemann ζ function has non-trivial zeros at certain $z \in \mathbb{C}$ outside of the critical strip.*

Proof. Riemann's functional form of ζ [27] is

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

Theorem 8.4 gives $\zeta(\widehat{\infty}-b)=1$ when we set $y_0=0$ so we will use Riemann's equation to prove this theorem by computing $\zeta(z)$ at $z_0=-(\widehat{\infty}-b)+1$. (This value for z_0 follows from $1-z_0=\widehat{\infty}-b$.) We have

$$\begin{aligned}\zeta\big[-(\widehat{\infty}-b)+1\big] &= \lim_{z\to-(\widehat{\infty}-b)+1}\left(\frac{(2\pi)^z}{\pi}\sin\left(\frac{\pi z}{2}\right)\right)\lim_{z\to(\widehat{\infty}-b)}\Big(\Gamma(z)\zeta(z)\Big)\\ &= \lim_{z\to-(\widehat{\infty}-b)+1}\left(2\sin\left(\frac{\pi z}{2}\right)\right)\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\zeta(z)\Big)\quad .\end{aligned}$$

For the limit involving Γ, we will compute the limit as a product of two limits. We separate terms as

$$\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\zeta(z)\Big)=\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\Big)\lim_{z\to(\widehat{\infty}-b)}\zeta(z)\quad .$$

From Theorem 8.4, we know the limit involving ζ is equal to one. For the remaining limit, we will insert the identity and again compute it as the product of two limits. If z approaches $(\widehat{\infty}-b)$ along the real axis, then it follows from Axiom 5.10 that

$$1=\frac{z-(\widehat{\infty}-b)}{z-(\widehat{\infty}-b)}\quad .$$

Inserting the identity yields

$$\lim_{z\to(\widehat{\infty}-b)}\Big((2\pi)^{-z}\Gamma(z)\Big)=\lim_{z\to(\widehat{\infty}-b)}\left((2\pi)^{-z}\Gamma(z)\frac{z-(\widehat{\infty}-b)}{z-(\widehat{\infty}-b)}\right)\quad .$$

Let

$$A=\Gamma(z)\Big(z-(\widehat{\infty}-b)\Big)\quad,\qquad\text{and}\qquad B=\frac{(2\pi)^{-z}}{z-(\widehat{\infty}-b)}\quad .$$

To get the limit of A into workable form, we will use the property $\Gamma(z)=z^{-1}\Gamma(z+1)$ to derive an expression for $\Gamma[z-(\widehat{\infty}-b)+1]$. If we can write $\Gamma(z)$ in terms of $\Gamma[z-(\widehat{\infty}-b)+1]$, then the limit as z approaches $(\widehat{\infty}-b)$ will be very easy to compute. Observe that

$$\Gamma\big[z-(\widehat{\infty}-b)+1\big]=\Gamma\big[z-(\widehat{\infty}-b)+2\big]\Big(z-(\widehat{\infty}-b)+1\Big)^{-1}\quad .$$

On the RHS, we see that Γ's argument is increased by one with respect to the Γ function that appears on the LHS. The purpose of inserting the identity $z-(\widehat{\infty}-b)[z-(\widehat{\infty}-b)]^{-1}=1$ was precisely to exploit this self-referential identity of the Γ function which is most generally expressed as

$$\Gamma\big(z\big)=\Gamma\big(z+1\big)z^{-1}\quad .$$

By taking a limit of recursion, we will let z approach a number in the neighborhood of infinity. Then through the axiomatized addition of such numbers

(Axiom 5.10), we will cast the argument of Γ into the neighborhood of the origin where its properties are well known. The limit is

$$\Gamma\big[z-(\widehat{\infty}-b)+1\big]=\Gamma(z)\lim_{n\to(\widehat{\infty}-b)}\prod_{k=1}^{n}\Big(z-(\widehat{\infty}-b)+k\Big)^{-1} \quad .$$

Moving the infinite product to the other side yields

$$\Gamma(z)=\Gamma\big[z-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=1}^{n}\Big(z-(\widehat{\infty}-b)+k\Big) \quad .$$

We have let $A=\Gamma(z)(z-(\widehat{\infty}-b))$ where the coefficient $z-(\widehat{\infty}-b)$ can be expressed as the $k=0$ term in the infinite product. It follows that

$$A=\Gamma\big[z-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=0}^{n}\Big(z-(\widehat{\infty}-b)+k\Big) \quad .$$

To evaluate the limit of AB, we will take the limits of A and B separately. The limit of A is

$$\lim_{z\to(\widehat{\infty}-b)}A=\Gamma\big[(\widehat{\infty}-b)-(\widehat{\infty}-b)+1\big]\lim_{n\to(\widehat{\infty}-b)}\prod_{k=0}^{n}\Big((\widehat{\infty}-b)-(\widehat{\infty}-b)+k\Big) \quad .$$

Axiom 5.10 gives $(\widehat{\infty}-b)-(\widehat{\infty}-b)=0$ so

$$\lim_{z\to(\widehat{\infty}-b)}A=\Gamma(1)\lim_{n\to(\widehat{\infty}-b)}\prod_{k=0}^{n}k=0 \quad .$$

Direct evaluation of the $z\to(\widehat{\infty}-b)$ limit of $B=(2\pi)^{-z}(z-(\widehat{\infty}-b))^{-1}$ gives $0/0$ so we need to use L'Hôpital's rule. Evaluation yields

$$\begin{aligned}\lim_{z\to(\widehat{\infty}-b)}B &\overset{*}{=} \lim_{z\to(\widehat{\infty}-b)}\left(\frac{\dfrac{d}{dz}(2\pi)^{-z}}{\dfrac{d}{dz}\Big(z-(\widehat{\infty}-b)\Big)}\right)\\ &=\lim_{z\to(\widehat{\infty}-b)}\frac{d}{dz}e^{-z\ln(2\pi)}\\ &=-\ln(2\pi)\;e^{-(\widehat{\infty}-b)\ln(2\pi)}\\ &=\frac{-1}{e^{\widehat{\infty}}}\ln(2\pi)\;e^{b\ln(2\pi)}=0 \quad .\end{aligned}$$

Therefore, we find that the limit of AB is 0. It follows that

$$\zeta\big[-(\widehat{\infty}-b)+1\big]=\lim_{z\to-(\widehat{\infty}-b)+1}2\sin\left(\frac{\pi z}{2}\right)\cdot 0=0 \quad .$$

Remark 8.6 In Main Theorem 8.5, we have considered the case of $y_0 = 0$ purely for convenience. In Main Theorem 8.7, we prove the case of $y_0 \neq 0$.

Main Theorem 8.7 *The Riemann ζ function has non-trivial zeros at certain $z \in \mathbb{C}$ outside of the critical strip with non-zero imaginary parts.*

Proof. Following the form of the proof Main Theorem 8.5, we will prove that

$$\zeta\big[-(\widehat{\infty} - b) + 1 - iy_0\big] = 0 \quad ,$$

if we can prove that

$$\lim_{z \to (\widehat{\infty}-b)+iy_0} \bigg((2\pi)^{-z}\Gamma(z) \bigg) = 0 \quad .$$

In proving Main Theorem 8.5, we have introduced the identity

$$1 = \frac{z - (\widehat{\infty} - b)}{z - (\widehat{\infty} - b)} \quad ,$$

by way of Axiom 5.10. If we approach z along the real axis, then this identity follows directly from the axiom. If we add an imaginary part as is required for the present theorem, then to conjure an identity we must note that if $\beta \in \mathbb{R}_0$, then

$$\frac{x}{y} = 1 \qquad \Longrightarrow \qquad \frac{x + i\beta}{y + i\beta} = 1 \quad ,$$

and that, therefore,

$$1 = \frac{z - (\widehat{\infty} - b) - iy_0}{z - (\widehat{\infty} - b) - iy_0} \quad .$$

This allows us to write

$$\lim_{z \to (\widehat{\infty}-b)+iy_0} \bigg((2\pi)^{-z}\Gamma(z) \bigg) = \lim_{z \to (\widehat{\infty}-b)+iy_0} \bigg((2\pi)^{-z}\Gamma(z) \frac{z - (\widehat{\infty} - b) - iy_0}{z - (\widehat{\infty} - b) - iy_0} \bigg) \quad .$$

Let

$$A = \Gamma(z)\bigg(z - (\widehat{\infty} - b) - iy_0 \bigg) \quad , \qquad \text{and} \qquad B = \frac{(2\pi)^{-z}}{z - (\widehat{\infty} - b) - iy_0} \quad ,$$

so that we may compute the limit of the product as the product of limits. To get the limit of A into workable form, we will use the property $\Gamma(z) = z^{-1}\Gamma(z+1)$ to derive an expression for $\Gamma[z - (\widehat{\infty} - b) + 1 - iy_0]$. If we can write $\Gamma(z)$ in terms of $\Gamma[z - (\widehat{\infty} - b) + 1 - iy_0]$, then the limit as z approaches $(\widehat{\infty} - b) + iy_0$ will be very easy to compute. Let $z' = z - iy_0$ so that

$$\Gamma\big[z' - (\widehat{\infty} - b) + 1\big] = \Gamma\big[z' - (\widehat{\infty} - b) + 2\big]\bigg(z' - \widehat{\infty} - b) + 1 \bigg)^{-1} \quad .$$

As in the proof of Main Theorem 8.5, we let the argument of Γ approach a number in the neighborhood of infinity by taking a limit of recursion as

$$\Gamma\big[z' - (\widehat{\infty} - b) + 1\big] = \Gamma(z') \lim_{n\to(\widehat{\infty}-b)} \prod_{k=1}^{n} \Big(z' - (\widehat{\infty} - b) + k \Big)^{-1} \quad .$$

It follows that

$$A = \Gamma\big[z' - (\widehat{\infty} - b) + 1\big] \lim_{n\to(\widehat{\infty}-b)} \prod_{k=0}^{n} \Big(z' - (\widehat{\infty} - b) + k \Big) \quad ,$$

and

$$\lim_{z'\to(\widehat{\infty}-b)} A = \Gamma\big[(\widehat{\infty} - b) - (\widehat{\infty} - b) + 1\big] \lim_{n\to(\widehat{\infty}-b)} \prod_{k=0}^{n} \Big((\widehat{\infty} - b) - (\widehat{\infty} - b) + k \Big).$$

Again, the $k = 0$ term of the infinite product is zero so

$$\lim_{z\to(\widehat{\infty}-b)+iy_0} A = 0 \quad .$$

The limit of B is

$$\begin{aligned}
\lim_{z\to(\widehat{\infty}-b)+iy_0} B &= \frac{\big(2\pi\big)^{-(\widehat{\infty}-b)-iy_0}}{(\widehat{\infty} - b) + iy_0 - (\widehat{\infty} - b) - iy_0} \\
&= \frac{\Big(\big(2\pi\big)^{-(\widehat{\infty}-b)} \left[\cos(y_0 \ln 2\pi) - i \sin(y_0 \ln 2\pi)\right] \Big)}{0} = \frac{0}{0} \quad .
\end{aligned}$$

This requires L'Hôpital's rule again:

$$\begin{aligned}
\lim_{z\to(\widehat{\infty}-b)+iy_0} B &\overset{*}{=} \lim_{z\to(\widehat{\infty}-b)+iy_0} \frac{d}{dz} e^{-z\ln(2\pi)} \\
&= -\ln(2\pi)\; e^{-(\widehat{\infty}-b)\ln(2\pi) - iy_0 \ln(2\pi)} \\
&= \frac{-\ln(2\pi)}{e^{\widehat{\infty}}} \big(e^{b\ln(2\pi)} e^{-iy_0\ln(2\pi)} \big) = 0 \quad .
\end{aligned}$$

Therefore, we find that the limit of AB is 0. It follows that

$$\zeta\big[-(\widehat{\infty} - b) + 1 - iy_0 \big] = \lim_{z\to-(\widehat{\infty}-b)+1-iy_0} 2\sin\left(\frac{\pi z}{2}\right) \cdot 0 = 0 \quad .$$

Remark 8.8 Now we have proven that ζ is equal to zero almost everywhere in the neighborhood of negative infinity for any y_0. We have proven it "almost everywhere" because the requirement $b > 0$ (Definition 5.1) gives $(b + 1) > 1$ and the strip $-\widehat{\infty} < \mathrm{Re}(z) \leq -(\widehat{\infty} - 1)$ is not covered by Main Theorem

8.7. Note how this strip of unit width at the extreme left of the left complex half-plane is quite similar in structure to the famous critical strip of unit width at the extreme left of the right complex half-plane.

Definition 8.9 The Riemann hypothesis as defined by the Clay Mathematics Institute [28] is the following.

> "The non-trivial zeros of the Riemann ζ function have real parts equal to one half."

Definition 8.10 According to the Clay Mathematics Institute [28], the trivial zeros of ζ are the even negative integers.

Remark 8.11 The zeros demonstrated in Main Theorems 8.5 and 8.7 are neither on the critical line $\text{Re}(z) = 1/2$ nor are they the negative even integers. Main Theorems 8.5 and 8.7, therefore, are the negation of the Riemann hypothesis as it is posed by the Clay Mathematics Institute.

Remark 8.12 What is called the theorem of Hadamard and de la Vallée-Poussin [29, 30] supposedly proves that all non-trivial zeros of ζ must lie in the region $0 < \text{Re}(z) < 1$ which is called the critical strip [13]. However, the theorem of Hadamard and de la Vallée-Poussin regards the prime number theorem [13] and the requirement that all non-trivial zeros lie inside the critical strip is only a corollary result of their theorem demonstrated through an exterior proposition: Proposition 8.13. Proposition 8.13 supposes that there can be no non-trivial zeros in the left complex half-plane due to the symmetry of Riemann's functional equation and the fact that there are no non-trivial zeros outside of the critical strip in the right complex half-plane. Below, we will show that this proposition is false.

Theorem 8.4 shows that ζ is equal to one everywhere in the neighborhood of positive real infinity but we have demonstrated in Main Theorem 8.5 non-trivial zeros in the neighborhood of negative real infinity. Now we will demonstrate that the corollary of the theorem of Hadamard and de la Vallée-Poussin requiring all non-trivial zeros to lie inside the critical strip is false because the symmetry about the critical line is not preserved by Riemann's functional equation in the neighborhood of infinity.

Proposition 8.13 *If* $\text{Re}(z) > 1$, then $\zeta(z) \neq 0$ because

$$\zeta(z) \prod_p (1 - p^{-z}) = 1 \quad .$$

Refutation. The argument in favor of this proposition goes as follows [31]. In the region $\mathrm{Re}(z) > 1$, $\zeta(z)$ absolutely converges to the Euler product so

$$\zeta(z) = \prod_p \frac{1}{1 - p^{-z}} \qquad \Longrightarrow \qquad \zeta(z) \prod_p (1 - p^{-z}) = 1 \quad .$$

Every prime number is a natural number so every term in $\Pi_p\,(1 - p^{-z})$ is contained in $\Pi_n\,(1 - n^{-z})$. It follows that the former product will converge if the latter does. It is a property of infinite products that $\Pi_n\,(1 + a_n)$ converges if and only if $\Sigma_n\, a_n$ converges. Since $\Sigma_n\, n^{-z}$ does converge for $\mathrm{Re}(z) > 1$, we know that $\Pi_n\,(1-n^{-z})$ absolutely converges. Therefore, $\Pi_p\,(1-p^{-z})$ absolutely converges and the condition that

$$\zeta(z) \prod_p (1 - p^{-z}) = 1 \quad ,$$

guarantees that $\zeta(z) \neq 0$ for any z such that $\mathrm{Re}(z) > 1$. If there was some z_0 such that $\zeta(z_0)$ was equal to zero, then the expression could not be equal to one. To show the failure of this argument, consider $z_0 \in \widehat{\mathbb{R}}$ such that

$$z_0 = \widehat{\infty} - b \qquad \Longrightarrow \qquad \mathrm{Re}(z_0) > 1 \quad .$$

Then,

$$\prod_p (1 - p^{-\widehat{\infty}+b}) = \prod_p \left(1 - \frac{1}{p^{-b}\,\widehat{\infty}} \right) = \lim_{x \to \widehat{\infty}} \left(1 - \frac{1}{\widehat{\infty}} \right)^x \quad .$$

This is an indeterminate form because $\widehat{\infty}$ is defined as a limit (Definition 3.2). Therefore, the expression

$$\zeta(z) \prod_p (1 - p^{-z}) = 1 \quad ,$$

cannot be used to rule out zeros of ζ for all z with $\mathrm{Re}(z) > 1$ because the expression on the LHS is not defined everywhere in that region.

Remark 8.14 When one closely examines the reasoning by which the theorem of Hadamard and de la Vallée-Poussin [29, 30] is said to disallow non-trivial zeros in the left complex half-plane, one finds in a certain region an undefined operation on an indeterminate form, as demonstrated in the treatment of Proposition 8.13. That the neighborhood of infinity has been neglected in history is demonstrated directly by the pseudo-trivial failure of the Hadamard–de la Vallée-Poussin result to carry beyond the neighborhood of the origin and into the neighborhood of negative real infinity in the far left complex half-plane.

Remark 8.15 In this section, we have proven that the Riemann ζ function has non-trivial zeros off of the critical line, and that, therefore, the famous Millennium Prize is solved. We have identified new non-trivial zeros in the left complex half-plane. Then we have discussed a result [29, 30] which is said to prove in a corollary fashion that no such zeros can exist. This corollary result, Proposition 8.13, fails pseudo-trivially in the neighborhood of infinity. The zeros demonstrated in Main Theorem 8.5 are in the neighborhood of negative real infinity, and the theorem of Hadamard and de la Vallée-Poussin [29, 30] does not disallow non-trivial zeros in that region.

Remark 8.16 Patterson writes the following in reference [13].

> "There is a second representation of ζ due to Euler in 1749 which is perhaps more fundamental and which is the reason for the significance of the zeta-function. This is
>
> $$\zeta(s) = \prod_{p\in\text{primes}} \left(1 - p^{-s}\right)^{-1} \quad .$$
>
> where the product is taken over all prime numbers p. This is called the Euler Product representation of the zeta-function and gives analytic expression to the fundamental theorem of arithmetic."

The fundamental theorem of arithmetic is given in Euclid's Elements [1] Book 7, Propositions 30, 31, and 32. A modern statement of the fundamental theorem of arithmetic is that every natural number greater than one is a prime number or it is a product of prime numbers. The ultimate goal of all of number theory being concerned with the distribution of the prime numbers, now we will demonstrate as a corollary result that the Euler product form of ζ [13, 32] shares at least some zeros with the the Riemann ζ function in the left complex half-plane where the convergence of the Euler product to the Riemann ζ function cannot be proven.

Corollary 8.17 *The Euler product has zeros in the neighborhood of negative real infinity.*

Proof. Let

$$z_0 = -\big(\widehat{\infty} - b\big) + iy_0 \quad , \qquad \text{and} \qquad \zeta(z) = \prod_{p\in\text{primes}} \frac{1}{1 - p^{-z}} \quad .$$

Then

$$\zeta(z_0) = \prod_p \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}}$$

$$
\begin{aligned}
&= \frac{1}{1 - P^{(\widehat{\infty}-b)-iy_0}} \prod_{p \neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}}-b)-iy_0}} \\
&= \frac{1}{1 - \dfrac{1}{P^b}(P^{\widehat{\infty}})\left[\cos(y_0 \ln P) - i \sin(y_0 \ln P)\right]} \prod_{p \neq P} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \quad .
\end{aligned}
$$

Let $y_0 \ln P = 2n\pi$ for some prime P and $n \in \mathbb{N} \cup \{0\}$. Then

$$
\zeta(z_0) = \left(\frac{1}{1 - \widehat{\infty}} \right) \prod_{p \neq P} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \quad .
$$

By Axiom 5.8, we have $1/(1 - \widehat{\infty}) = 0$ so

$$
\zeta(z_0) = 0 \quad .
$$

References

[1] Richard Fitzpatrick. *Euclid's Elements of Geometry.* Lulu, (2007).

[2] J.R.W. Dedekind. Essays on the Theory of Numbers. (1872).

[3] Charles Chapman Pugh. *Real Mathematical Analysis.* Springer, (2003).

[4] Marshall H. Stone. Applications of the Theory of Bolean Rings to General Topology. *Transactions of the American Mathematical Society, 41 (3): 375–481*, (1937).

[5] Eduard Cech. On Bicompact Spaces. *Annals of Mathematics, 38, 823–844*, (1937).

[6] James Munkres. *Topology, 2nd ed.* Pearson, (2014).

[7] Jonathan W. Tooker. Fractional Distance: The Topology of the Real Number Line. *(In preparation)*, (2019).

[8] Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis, 3rd ed.* Academic Press, (1998).

[9] N. A. Kolmogorov. *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia," 2nd ed., Vol. 3.* M. Hazewinkel, editor, Reidel, (1995).

[10] G. William Wlaster, E.R Hansen, and J.D. Pryce. Extended Real Intervals and the Topological Closure of Extended Real Relations. *Sun Microsystems, Inc.*, (2002).

[11] David W. Cantrell. Affinely Extended Real Numbers. *Mathworld*.

[12] Pavel S. Alexandroff. Uber die Metrisation der im Kleinen Kompakten Topologischen Raume. *Mathematische Annalen, 92, 294–301*, (1924).

[13] S.J. Patterson. *An Introduction to the Theory of the Riemann Zeta Function*. Cambridge University Press, (1988).

[14] G.H. Hardy and J.E. Littlewood. The Zeros of Reimann's Zeta Function on the Critical Line. *Math. Zeitschrift, 10, 283-317*, (1921).

[15] P. Borwein, S. Choi, B. Rooney, and A. Weirathmueller. *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*. Springer, (2008).

[16] E.C. Titchmarsh. *The Zeta Function of Riemann*. Cambridge University Press, (1930).

[17] E.C. Titchmarsh. *The Theory of the Riemann Zeta Function*. Oxford University Press, (1986).

[18] K. Chandesekharan. *Lectures on the Riemann Zeta Function*. Tata Institute of Fundamental Research, Bombay, India, (1958).

[19] H.M. Edwards. *Riemann's Zeta Function*. Academic Press, New York, (1974).

[20] D. Bump. On the Riemann Zeta Function. *Math. Zeitschrift, 192, 195-204*, (1986).

[21] H. Iwaniec. Lectures on the Riemann Zeta Function. *University Lecture Series, vol. 62, American Mathematical Society, Providence, RI*, (2014).

[22] D.B. Zagier. Zeta Functions in Number Theory. *Annual Meeting of the American Mathematical Society, Phoenix, AZ*, (1989).

[23] M. Kaneko T. Arakawa, T. Ibukiyama. *Bernoulli Numbers and Zeta Functions*. Springer Monographs in Mathematics, (2001).

[24] D. Bump, K.K. Choi, P. Kurlberg, and J.Vaaler. A Local Riemann ypothesis. *Math. Zeitschrift, 223, 1-18*, (2000).

[25] S. Feng. Zeros of the Riemann Zeta Function on the Critical Line. *Journal of Number Theory, 132, 511-542*, (2012).

[26] M. Julita. Zeros of the Zeta Function Near the Critical Line. *Studies in Pure Mathematics, Birkhauser, 385-394*, (1982).

[27] Bernhard Riemann. On the Number of Primes Less than a Given Quantity. *Monatsberichte der Berliner Akademie*, (1859).

[28] Enrico Bombieri. Problems of the Millennium: The Riemann Hypothesis. *Clay Mathematics Institute*, (2000).

[29] Jacques Hadamard. Sur la Distribution des Zeros de la Fonction Zeta(s) et ses Consequences Arithmetiques. *Bulletin de la Societe Mathematique de France, **14**, 199-220*, (1896).

[30] Ch. J. de la Vallee-Poussin. Recherches Analytiques sur la Theorie des Nombers Premiers. *Ann. Soc. Sci. Bruxelles, **20**, 183–256*, (1896).

[31] Jin Gyu Lee. Zeros of the Riemann Zeta Function. *arXiv:1305:0323*, (2013).

[32] Leonhard Euler. Various Observations about Infinite Series. *St. Petersburg Academy*, (1737).