Question 481 : On Integrals

EDGAR VALDEBENITO

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Abstract. In this note we give some integrals.
Resumen. En esta nota mostramos algunas fórmulas integrales.

En la tabla de integrales GRADSHTEYN and RYZHIK (seventh edition), pag.534 (Logarithmic Functions), aparecen las fórmulas (4228.5,4228.6):

$$\int_0^{\pi/4} \sqrt{\ln \cot x}\, dx = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{(2n+1)^3}} \tag{1}$$

$$\int_0^{\pi/4} \frac{1}{\sqrt{\ln \cot x}}\, dx = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}} \tag{2}$$

Combinando (1) y (2) se obtienen las siguientes fórmulas:

$$\frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{\sqrt{2n+1}} + \frac{1}{\sqrt{(2n+1)^3}} \right) = \frac{\pi}{2} + \int_2^{\infty} \left(\frac{\pi}{4} - \tan^{-1}\left(\exp\left(-\left(\frac{x-\sqrt{x^2-4}}{2} \right)^2 \right) \right) \right) dx +$$
$$+ \int_2^{\infty} \tan^{-1}\left(\exp\left(-\left(\frac{x+\sqrt{x^2-4}}{2} \right)^2 \right) \right) dx \tag{3}$$

$$\frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{\sqrt{2n+1}} - \frac{1}{\sqrt{(2n+1)^3}} \right) =$$
$$= \int_0^{\infty} \left(\frac{\pi}{4} - \tan^{-1}\left(\exp\left(-\left(\frac{x-\sqrt{4+x^2}}{2} \right)^2 \right) \right) \right) dx - \int_0^{\infty} \left(\tan^{-1}\left(\exp\left(-\left(\frac{x+\sqrt{4+x^2}}{2} \right)^2 \right) \right) \right) dx \tag{4}$$

Observación:

$$\frac{\sqrt{\pi}}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{\sqrt{2n+1}}\pm\frac{1}{\sqrt{(2n+1)^3}}\right)=\frac{\sqrt{\pi}}{2}\left(2\beta\left(\frac{1}{2}\right)\pm\beta\left(\frac{3}{2}\right)\right) \tag{5}$$

donde

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}\quad ,s>0 \tag{6}$$

Es la función beta de Dirichlet.
Algunas integrales equivalentes son:

$$\begin{aligned}&\frac{\sqrt{\pi}}{2}\left(2\beta\left(\frac{1}{2}\right)+\beta\left(\frac{3}{2}\right)\right)=\\&=\frac{\pi}{2}+2\int_0^{\infty}\left(\frac{\pi}{4}-\tan^{-1}\left(e^{-e^{-2x}}\right)\right)\sinh x\,dx+2\int_0^{\infty}\tan^{-1}\left(e^{-e^{2x}}\right)\sinh x\,dx\end{aligned} \tag{7}$$

$$\begin{aligned}&\frac{\sqrt{\pi}}{2}\left(2\beta\left(\frac{1}{2}\right)-\beta\left(\frac{3}{2}\right)\right)=\\&=2\int_0^{\infty}\left(\frac{\pi}{4}-\tan^{-1}\left(e^{-e^{-2x}}\right)\right)\cosh x\,dx-2\int_0^{\infty}\tan^{-1}\left(e^{-e^{2x}}\right)\cosh x\,dx\end{aligned} \tag{8}$$

$$\frac{\sqrt{\pi}}{2}\left(2\beta\left(\frac{1}{2}\right)+\beta\left(\frac{3}{2}\right)\right)=\frac{\pi}{2}+2\int_0^{\infty}\tan^{-1}\left(\frac{1+e^{-2\cosh(2x)}-e^{-e^{-2x}}+e^{-e^{2x}}}{1+e^{-2\cosh(2x)}+e^{-e^{-2x}}-e^{-e^{2x}}}\right)\sinh x\,dx \tag{9}$$

$$\frac{\sqrt{\pi}}{2}\left(2\beta\left(\frac{1}{2}\right)-\beta\left(\frac{3}{2}\right)\right)=2\int_0^{\infty}\tan^{-1}\left(\frac{1-e^{-2\cosh(2x)}-e^{-e^{-2x}}-e^{-e^{2x}}}{1-e^{-2\cosh(2x)}+e^{-e^{-2x}}+e^{-e^{2x}}}\right)\cosh x\,dx \tag{10}$$

Referencias

1. GRADSHTEYN, I.S., and RYZHIK, I.M.: TABLE of INTEGRALS, SERIES and PRODUCTS. seventh edition. Edited by Alan Jeffrey and Daniel Zwillinger. Academic Press, 2007.