THE ARBELOS IN WASAN GEOMETRY, PROBLEMS OF IZUMIYA AND NAITŌ

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Abstract. We generalize two sangaku problems involving an arbelos proposed by Izumiya and Naitō, and show the existence of six non-Archimedean congruent circles.

1. Introduction

In this article we generalize two sangaku problems involving an arbelos proposed by Izumiya (泉屋徳太郎静政) and Naitō (内藤豊次郎). Let α, β and γ be the three semicircles with diameters AO, BO and AB, respectively for a point O on the segment AB constructed on the same side of AB. The area surrounded by the three semicircles is called arbelos (see Figure 1). The radical axis of α and β is called the axis. Let a and b be the radii of α and β, respectively, and let δ_α (resp. δ_β) be the incircle of the curvilinear triangle made by α (resp. β), γ and the axis. The two circles δ_α and δ_β have common radius $r_{\mathrm{A}} = ab/(a+b)$ and are called the twin circles of Archimedes.

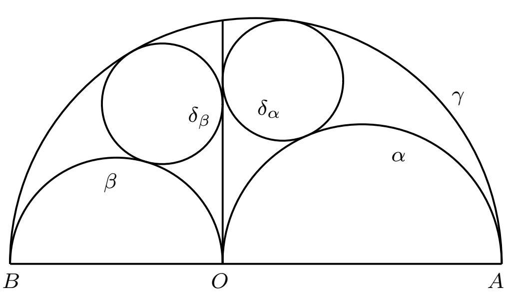

Figure 1.

Izumiya's problems appeared in a sangaku in Saitama hung in 1866, which is as follows [6] (see Figure 2).

Problem 1. If α and β are congruent and the tangent of α from B meets γ in a point C, show that the inradius of the curvilinear triangle formed by α, γ and the perpendicular from C to AB equals $a/9$.

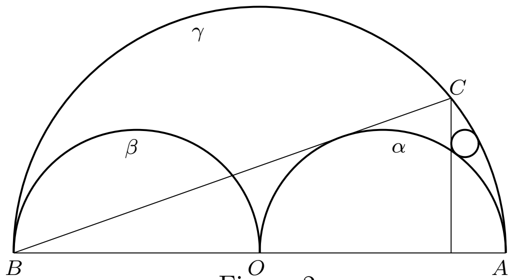

Figure 2.

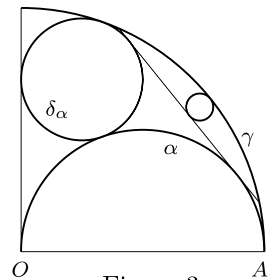

Figure 3.

Naitō's problem appeared in a sangaku in Fukushima hung in 1983 (the sangaku seems to be made in modern day times), which is as follows [3] (see Figure 3).

Problem 2. If α and β are congruent, show that the radius of the circle touching the remaining external common tangent of α and δ_α and the arc of γ cut by the tangent at the midpoint equals $a/9$.

2. Generalization

We now consider the case in which the semicircles α and β are not always congruent. We use the next proposition (see Figure 4).

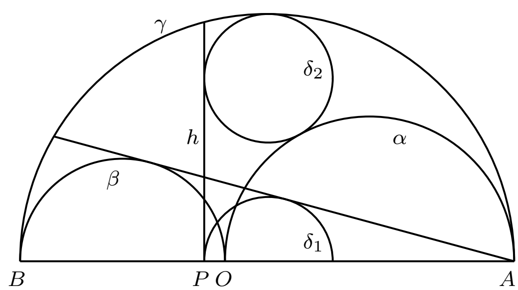

Figure 4.

Proposition 2.1. *For a point P on the segment AB, let h be the perpendicular to AB at P. If δ_1 is the circle touching h at P from the side opposite to B and the tangent of β from A and δ_2 is the circle touching α externally γ internally and h from the same side as δ_1, then δ_1 and δ_2 are congruent.*

Proof. The radius of δ_2 is proportional to the distance between its center and the radical axis of α and γ [1, p. 108], while δ_2 coincides with β if $P = B$. Also the radius of δ_1 is proportional to the distance between its center and the point A, and δ_1 coincides with β if $P = B$. □

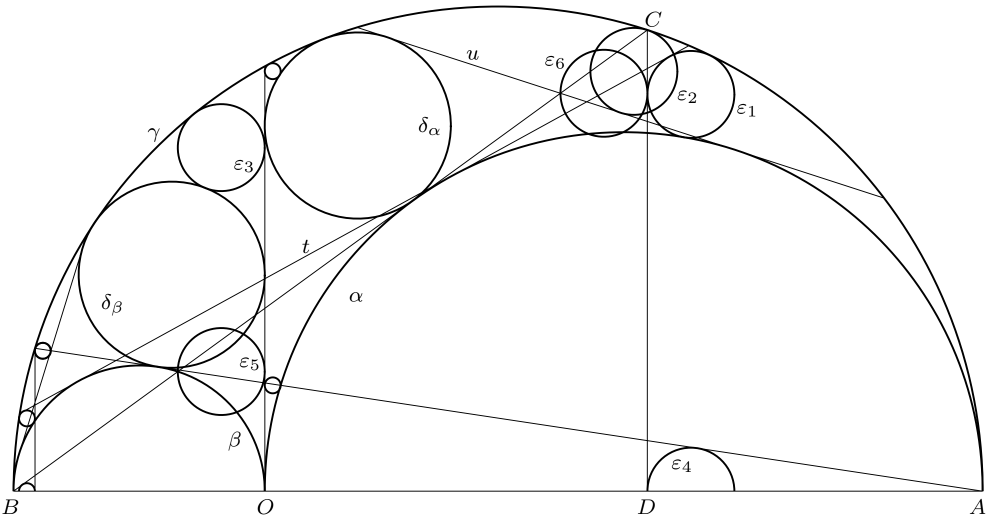

Figure 5.

Theorem 2.2. *Let C be the point of intersection of γ and the tangent of α from B and let D be the foot of perpendicular from C to AB. The incircle of the curvilinear triangle made by α, γ and CD is denoted by ε_1. Let u be the remaining external common tangent of α and δ_α. The circle touching u and the arc of γ cut by u at the midpoint is denoted by ε_2. The incircle of the curvilinear triangle made by γ, δ_β and the axis is denoted by ε_3. The circle touching the tangent of β from A and CD at D from the side opposite to B is denoted by ε_4. The smallest circle passing through the point of intersection of β and BC and touching the axis is denoted by ε_5. The smallest circle passing through the point of intersection of BC and u and touching the line CD is denoted by ε_6. Then the following statements hold.*

(i) *The six circles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_6$ are congruent and have common radius*

$$\frac{a^2b}{(a+2b)^2}.$$

(ii) *The circle ε_1 touches the line t, and the circle ε_2 touches γ at C.*

Proof. We assume that r_i is the radius of ε_i, $d = a + 2b$, E is the point of intersection of BC and β, F is the foot of perpendicular from E to the axis, G is the point of tangency of α and BC, H is the center of α, and BC meets the axis and u in points J and K, respectively (see Figure 6).

Since the three segments CA, GH and EO are parallel and H is the midpoint of AO, G is the midpoint of CE. While the line BC is the internal common tangent of α and δ_α [2, p. 212]. Therefore G is also the midpoint of JK. Hence $|EJ| = |CK|$, i.e., the circles ε_5 and ε_6 are congruent. Since the triangles BGH, BEO and OFE are similar, $a/d = |OE|/(2b) = |EF|/|OE|$. Therefore $|OE| = 2ab/d$ and $|EF| = 2a^2b/d^2$. Hence $r_5 = a^2b/d^2 = r_6$, and $|OF| = 4ab\sqrt{(a+b)b}/d^2$ from the right triangle OFE.

The last equation implies $|EF| = a|OF|/(2\sqrt{(a+b)b})$. Let $x = |BD|$. Then $|CD| = ax/(2\sqrt{(a+b)b})$ from the similar triangles OFE and BDC. Therefore we have $x(2(a+b)-x) = |CD|^2 = a^2x^2/(4(a+b)b)$. Solving the equation for x, we get $x = 8b(a+b)^2/d^2$. Therefore $|AD| = 2(a+b)-x = 2a^2(a+b)/d^2$. Therefore $r_4 = b|AD|/|AB| = a^2b/d^2 = r_1$

by Proposition 2.1. Meanwhile ε_3 and the incircle of the curvilinear triangle made by α, γ and t have radius a^2b/d^2 [5, Theorem 9]. Therefore the last circle coincides with ε_1, i.e., ε_1 touches t. While we have also shown that ε_1 and ε_2 are congruent in [4]. This proves (i) and the first half part of (ii).

Let ζ be the circle with center C passing through G. We invert the figure in ζ. Then the circles α and δ_α are orthogonal to ζ, i.e, they are fixed by the inversion. The line u, which intersects ζ, is inverted to a circle intersecting ζ touching α and δ_α passing through C. Therefore γ is the inverse of u. This implies that the points of intersection of γ and u lie on ζ. Hence C is the midpoint of the arc of γ cut by u. Therefore ε_2 touches γ at C. This proves the second half part of (ii). □

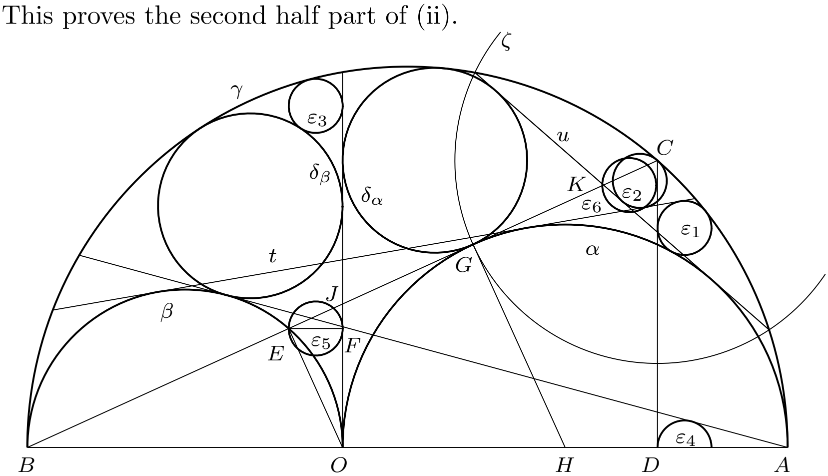

Figure 6.

Circles of radius r_A are called Archimedean circles [2]. Therefore we now have six non-Archimedean congruent circles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_6$. Exchanging the roles of α and β, we get another six non-Archimedean congruent circles of radius $ab^2/(2a+b)^2$, which are denoted in Figure 5.

3. The circle associated with a point on γ

For a circle δ touching α externally and γ internally, if P is the point of intersection of γ and the internal common tangent of δ and α closer to B, we say that δ is associated with P. As mentioned in the proof of Theorem 2.2, the circle δ_α is associated with the point B (see Figure 6). We can also consider that the point circle A is associated with the point A itself, because the perpendicular to AB at A can be considered as the internal common tangent of the point circle A and α. Let I be the point of intersection of γ and the axis. The next theorem gives the circle associated with the point I.

Theorem 3.1. *The internal common tangent of α and ε_1 passes through I.*

Proof. Let ρ be the circle with center I passing through O. We invert the figure in ρ (see Figure 7). Then α and β are fixed. While t, which intersects ρ, is inverted into the circle with center I touching α and β intersecting ρ. Therefore γ is the inverse of t. Hence the figure consisting of α, γ and t is fixed by the inversion. This implies that ε_1 is also fixed. Since α and ε_1 are orthogonal to ρ, their point of tangency lies on ρ, and their common internal tangent passes through I. □

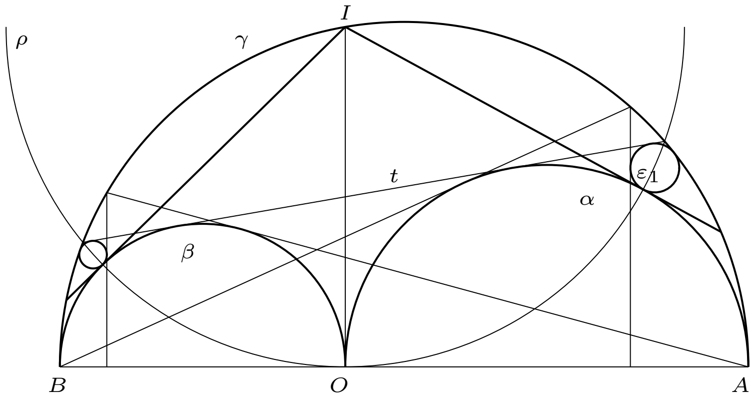

Figure 7.

The proof also shows that the points of intersection of γ and t lies on ρ. Therefore I is the midpoint of the arc of γ cut by t.

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