Incompatibility of the Dirac-like Field Operators with the Majorana Anzatzen

Valeriy V. Dvoeglazov

UAF, Universidad Autónoma de Zacatecas, México

E-mail: valeri@fisica.uaz.edu.mx

Abstract

In the present article we investigate the spin-1/2 and spin-1 cases in different bases. Next, we look for relations with the Majorana-like field operator. We show explicitly incompatibility of the Majorana anzatzen with the Dirac-like field operators in both the original Majorana theory and its generalizations. Several explicit examples are presented for higher spins too. It seems that the calculations in the helicity basis give mathematically and physically reasonable results only.

1 Introduction.

In Refs. [1]-[6] we considered the procedure of construction of the field operators *ab initio* (including for neutral particles). The Bogoliubov-Shirkov method has been used.

In the present article we investigate the spin-1/2 and spin-1 cases in different bases. Next, we look for relations with the Majorana-like field operator. It seems that the calculations in the helicity basis give mathematically and physically reasonable results only.

2 The Spin-1/2.

The Dirac equation is:

$$[i\gamma^\mu \partial_\mu - mc/\hbar]\Psi(x) = 0\,, \tag{1}$$

$\mu = 0, 1, 2, 3$. The most known methods of its derivation are [7, 8, 9]:

- the Dirac one (the Hamiltonian should be linear in $\partial/\partial x^i$, and be compatible with $E_p^2 - \mathbf{p}^2 c^2 = m^2 c^4$);

- the Sakurai one (based on the equation $(E_p - c\sigma \cdot \mathbf{p})(E_p + c\sigma \cdot \mathbf{p})\phi = m^2 c^4 \phi$, ϕ is the 2-component spinor);

- the Ryder one (the relation between 2-spinors at rest is $\phi_R(\mathbf{0}) = \pm\phi_L(\mathbf{0})$, and boosts).

The γ^μ are the Clifford algebra matrices

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\,, \tag{2}$$

$g^{\mu\nu}$ is the metric tensor. Usually, everybody uses the following definition of the field operator [10] in the pseudo-Euclidean metrics:

$$\Psi(x) = \frac{1}{(2\pi)^3} \sum_h \int \frac{d^3\mathbf{p}}{2E_p} [u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip_\mu \cdot x^\mu} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})e^{+ip_\mu \cdot x^\mu}]\,, \tag{3}$$

as given *ab initio*. After actions of the Dirac operator on $\exp(\mp ip_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p}-m)u_h(p) = 0$ and $(\hat{p}+m)v_h(p) = 0$, respectively; the h is the polarization index. It is easy to prove from the characteristic equations $Det(\hat{p}\mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relations $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$ for both $u-$ and $v-$ solutions.

The general scheme of construction of the field operator has been presented in [11]. In the case of the $(1/2, 0) \oplus (0, 1/2)$ representation we have:

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \int dp e^{ip\cdot x} \tilde{\Psi}(p)\,. \tag{4}$$

We know the condition of the mass shell: $(p^2 - m^2)\tilde{\Psi}(p) = 0$. Thus, $\tilde{\Psi}(p) = \delta(p^2 - m^2)\Psi(p)$. Next, we adjust the notation to the modern one:

$$\begin{aligned}
\Psi(x) &= \frac{1}{(2\pi)^3}\int dp\, e^{ip\cdot x}\delta(p^2 - m^2)(\theta(p_0) + \theta(-p_0))\Psi(p) = \quad (5)\\
&= \frac{1}{(2\pi)^3}\int dp\, \left[e^{ip\cdot x}\delta(p^2 - m^2)\Psi^+(p) + e^{-ip\cdot x}\delta(p^2 - m^2)\Psi^-(p)\right],
\end{aligned}$$

where

$$\Psi^+(p) = \theta(p_0)\Psi(p)\,,\text{ and } \Psi^-(p) = \theta(p_0)\Psi(-p)\,. \quad (6)$$

$$\Psi^+(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2E_p}e^{+ip\cdot x}\Psi^+(p)\,, \quad (7)$$

$$\Psi^-(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2E_p}e^{-ip\cdot x}\Psi^-(p)\,. \quad (8)$$

We continue:

$$\begin{aligned}
\Psi(x) &= \frac{1}{(2\pi)^3}\int d^4p\, \delta(p^2 - m^2)e^{-ip\cdot x}\Psi(p) =\\
&= \frac{1}{(2\pi)^3}\sum_h \int d^4p\, \delta(p_0^2 - E_p^2)e^{-ip\cdot x}u_h(p_0,\mathbf{p})a_h(p_0,\mathbf{p}) =\\
&= \frac{1}{(2\pi)^3}\int \frac{d^4p}{2E_p}[\delta(p_0 - E_p) + \delta(p_0 + E_p)][\theta(p_0) + \theta(-p_0)]e^{-ip\cdot x}\times\\
&\times \sum_h u_h(p)a_h(p) = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^4p}{2E_p}[\delta(p_0 - E_p) + \delta(p_0 + E_p)]\times \quad (9)\\
&\times \left[\theta(p_0)u_h(p)a_h(p)e^{-ip\cdot x} + \theta(p_0)u_h(-p)a_h(-p)e^{+ip\cdot x}\right] =\\
&= \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}\theta(p_0)\left[u_h(p)a_h(p)|_{p_0 = E_p}e^{-i(E_pt - \mathbf{p}\cdot\mathbf{x})} +\right.\\
&+ \left. u_h(p)a_h(-p)|_{p_0 = E_p}e^{+i(E_pt - \mathbf{p}\cdot\mathbf{x})}\right]
\end{aligned}$$

During the calculations we had to represent $1 = \theta(p_0) + \theta(-p_0)$ above in order to get positive- and negative-frequency parts.[1] Moreover, we did not yet assumed, which equation this field operator (namely, the $u-$ spinor) satisfies, with negative- or positive- mass and/or $p^0 = \pm E_p$.

In general we should transform $u_h(-p)$ to the $v_h(p)$. The procedure is the following one [1, 2].

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[1]See Ref. [5b] for discussion.

2.1 The Standard Basis.

The explicit forms of the 4-spinors are:

$$u_\sigma(\mathbf{p}) = \frac{N_\sigma^+}{2\sqrt{m(E_p + m)}} \begin{pmatrix} [E_p + m + \boldsymbol{\sigma} \cdot \mathbf{p}]\phi_\sigma(\mathbf{0}) \\ [E_p + m - \boldsymbol{\sigma} \cdot \mathbf{p}]\chi_\sigma(\mathbf{0}) \end{pmatrix}, \quad v_\sigma(\mathbf{p}) = \gamma^5 u_\sigma(\mathbf{p}), (10)$$

in the spinorial basis $(\phi_\uparrow(\mathbf{0}) = \chi_\uparrow(\mathbf{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi_\downarrow(\mathbf{0}) = \chi_\downarrow(\mathbf{0}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix})$.
The transformation to the standard basis is produced with the $(\gamma^5 + \gamma^0)/\sqrt{2}$ matrix. The normalizations, projection operators, propagators, dynamical invariants *etc* have been given in [9], for example.

In the Dirac case we should assume the following relation in the field operator (9):

$$\sum_{h=\pm 1/2} v_h(p) b_h^\dagger(p) = \sum_{h=\pm 1/2} u_h(-p) a_h(-p). \tag{11}$$

We know that [9]

$$\bar{u}_\mu(p) u_\lambda(p) = +m\delta_{\mu\lambda}, \tag{12}$$
$$\bar{u}_\mu(p) u_\lambda(-p) = 0, \tag{13}$$
$$\bar{v}_\mu(p) v_\lambda(p) = -m\delta_{\mu\lambda}, \tag{14}$$
$$\bar{v}_\mu(p) u_\lambda(p) = 0, \tag{15}$$

μ, λ are now the polarization indices. However, we need $\Lambda_{\mu\lambda}(p) = \bar{v}_\mu(p) u_\lambda(-p)$. By direct calculations, we find

$$-m b_\mu^\dagger(p) = \sum_\lambda \Lambda_{\mu\lambda}(p) a_\lambda(-p). \tag{16}$$

Hence, $\Lambda_{\mu\lambda} = -im(\boldsymbol{\sigma} \cdot \mathbf{n})_{\mu\lambda}$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and

$$b_\mu^\dagger(p) = +i \sum_\lambda (\boldsymbol{\sigma} \cdot \mathbf{n})_{\mu\lambda} a_\lambda(-p). \tag{17}$$

Multiplying (11) by $\bar{u}_\mu(-p)$ we obtain

$$a_\mu(-p) = -i \sum_\lambda (\boldsymbol{\sigma} \cdot \mathbf{n})_{\mu\lambda} b_\lambda^\dagger(p). \tag{18}$$

The equations are self-consistent.

2.2 The Helicity Basis.

The 2-eigenspinors of the helicity operator

$$\frac{1}{2}\boldsymbol{\sigma} \cdot \widehat{\mathbf{p}} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{+i\phi} & -\cos\theta \end{pmatrix} \tag{19}$$

can be defined as follows [12, 13, 14]:

$$\phi_{\frac{1}{2}\uparrow} = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix}, \quad \phi_{\frac{1}{2}\downarrow} = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi/2} \\ -\cos\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix}, \tag{20}$$

for $\pm 1/2$ eigenvalues, respectively.

We start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i. e., two degrees of freedom); $c = \hbar = 1$:

$$(E + \boldsymbol{\sigma} \cdot \mathbf{p})(E - \boldsymbol{\sigma} \cdot \mathbf{p})\phi = m^2\phi. \tag{21}$$

It can be re-written in the form of the set of two first-order equations for 2-spinors. We observe at the same time that they may be chosen as eigenstates of the helicity operator which present in (21):[2]

$$(E - (\boldsymbol{\sigma} \cdot \mathbf{p}))\phi_\uparrow = (E - p)\phi_\uparrow = m\chi_\uparrow, \tag{22}$$
$$(E + (\boldsymbol{\sigma} \cdot \mathbf{p}))\chi_\uparrow = (E + p)\chi_\uparrow = m\phi_\uparrow, \tag{23}$$
$$(E - (\boldsymbol{\sigma} \cdot \mathbf{p}))\phi_\downarrow = (E + p)\phi_\downarrow = m\chi_\downarrow, \tag{24}$$
$$(E + (\boldsymbol{\sigma} \cdot \mathbf{p}))\chi_\downarrow = (E - p)\chi_\downarrow = m\phi_\downarrow. \tag{25}$$

If the $\phi_{\uparrow\downarrow}$ spinors are defined by the equation (20) then we can construct the corresponding $u-$ and $v-$ 4-spinors:[3]

$$u_\uparrow(\mathbf{p}) = N_\uparrow^+ \begin{pmatrix} \phi_\uparrow \\ \frac{E_p - p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{E_p + p}{m}}\phi_\uparrow \\ \sqrt{\frac{m}{E_p + p}}\phi_\uparrow \end{pmatrix},$$

$$u_\downarrow(\mathbf{p}) = N_\downarrow^+ \begin{pmatrix} \phi_\downarrow \\ \frac{E_p + p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{m}{E_p + p}}\phi_\downarrow \\ \sqrt{\frac{E_p + p}{m}}\phi_\downarrow \end{pmatrix}, \tag{26}$$

---

[2]This opposes to the choice of the basis of the subsection (2.1), where 4-spinors are the eigenstates of the parity operator, cf. [15].

[3]Alternatively, $\uparrow\downarrow$ may refer to the chiral helicity eigenstates, e.g. $u_\eta = \frac{1}{\sqrt{2}}\begin{pmatrix} N\phi_\eta \\ N^{-1}\phi_{-\eta} \end{pmatrix}$, see next sections and cf. [16, 17].

$$v_\uparrow(\mathbf{p}) = N_\uparrow^- \begin{pmatrix} \phi_\uparrow \\ -\frac{E_p-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{E_p+p}{m}}\phi_\uparrow \\ -\sqrt{\frac{m}{E_p+p}}\phi_\uparrow \end{pmatrix},$$

$$v_\downarrow(\mathbf{p}) = N_\downarrow^- \begin{pmatrix} \phi_\downarrow \\ -\frac{E_p+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{m}{E_p+p}}\phi_\downarrow \\ -\sqrt{\frac{E_p+p}{m}}\phi_\downarrow \end{pmatrix}, \tag{27}$$

where the normalization to the unit (± 1) was used:[4]

$$\bar{u}_h(\mathbf{p})u_{h'}(\mathbf{p}) = \delta_{hh'}, \bar{v}_h(\mathbf{p})v_{h'}(\mathbf{p}) = -\delta_{hh'}, \tag{28}$$

$$\bar{u}_h(\mathbf{p})v_{h'}(\mathbf{p}) = 0 = \bar{v}_h(\mathbf{p})u_{h'}(\mathbf{p}) \tag{29}$$

We define the field operator as follows:

$$\Psi(x^\mu) = \sum_h \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sqrt{m}}{2E_p} [u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip_\mu x^\mu} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})e^{+ip_\mu x^\mu}]. \tag{30}$$

The commutation relations are assumed to be the standard ones [11, 10, 18, 19][5] (compare with [20])

$$\left[a_h(\mathbf{p}), a_{h'}^\dagger(\mathbf{k})\right]_+ = 2E_p\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{hh'}, [a_h(\mathbf{p}), a_{h'}(\mathbf{k})]_+ = 0 = \left[a_h^\dagger(\mathbf{p}), a_{h'}^\dagger(\mathbf{k})\right]_+ \tag{31}$$

$$\left[a_h(\mathbf{p}), b_{h'}^\dagger(\mathbf{k})\right]_+ = 0 = \left[b_h(\mathbf{p}), a_{h'}^\dagger(\mathbf{k})\right]_+, \tag{32}$$

$$\left[b_h(\mathbf{p}), b_{h'}^\dagger(\mathbf{k})\right]_+ = 2E_p\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{hh'}, [b_h(\mathbf{p}), b_{h'}(\mathbf{k})]_+ = 0 = \left[b_h^\dagger(\mathbf{p}), b_{h'}^\dagger(\mathbf{k})\right]_+ \tag{33}$$

Other details of the helicity basis are given in Refs. [21, 14].

However, in this helicity case

$$\Lambda_{\mu\lambda}(p) = \bar{v}_\mu(p)u_\lambda(-p) = i\sigma_{\mu\lambda}^y. \tag{34}$$

So, someone may advise that we should introduce the creation operators by hand in every basis.

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[4]Of course, there are no any mathematical difficulties to change it to the normalization to $\pm m$, which may be more convenient for the study of the massless limit.

[5]The only possible changes may be related to different forms of normalization of 4-spinors, which would have influence on the factor before δ-function.

2.3 Application of the Majorana anzatzen.

It is well known that "*particle=antiparticle*" in the Majorana theory [22]. So, in the language of the quantum field theory we should have

$$b_\mu(E_p, \mathbf{p}) = e^{i\varphi} a_\mu(E_p, \mathbf{p}) \,. \tag{35}$$

Usually, different authors use $\varphi = 0, \pm\pi/2$ depending on the metrics and on the forms of the 4-spinors and commutation relations.

So, on using (17) and the above-mentioned postulate we come to:

$$a_\mu^\dagger(p) = +ie^{i\varphi}(\boldsymbol{\sigma} \cdot \mathbf{n})_{\mu\lambda} a_\lambda(-p) \,. \tag{36}$$

On the other hand, on using (18) we make the substitutions $E_p \to -E_p$, $\mathbf{p} \to -\mathbf{p}$ to obtain

$$a_\mu(p) = +i(\boldsymbol{\sigma} \cdot \mathbf{n})_{\mu\lambda} b_\lambda^\dagger(-p) \,. \tag{37}$$

The totally reflected (35) is $b_\mu(-E_p, -\mathbf{p}) = e^{i\varphi} a_\mu(-E_p, -\mathbf{p})$. Thus,

$$b_\mu^\dagger(-p) = e^{-i\varphi} a_\mu^\dagger(-p) \,. \tag{38}$$

Combining with (37), we come to

$$a_\mu(p) = +ie^{-i\varphi}(\boldsymbol{\sigma} \cdot \mathbf{n})_{\mu\lambda} a_\lambda^\dagger(-p) \,, \tag{39}$$

and

$$a_\mu^\dagger(p) = -ie^{i\varphi}(\boldsymbol{\sigma}^* \cdot \mathbf{n})_{\mu\lambda} a_\lambda(-p) \,. \tag{40}$$

This contradicts with the equation (36) unless we have the preferred axis in every inertial system.

Next, we can use another Majorana anzatz $\Psi = \pm e^{i\alpha} \Psi^c$ with usual definitions

$$C = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} \mathcal{K}, \quad \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,. \tag{41}$$

Thus, on using $Cu_\uparrow^*(\mathbf{p}) = iv_\downarrow(\mathbf{p})$, $Cu_\downarrow^*(\mathbf{p}) = -iv_\uparrow(\mathbf{p})$ we come to other relations between creation/annihilation operators

$$a_\uparrow^\dagger(\mathbf{p}) = \mp ie^{-i\alpha} b_\downarrow^\dagger(\mathbf{p}) \,, \tag{42}$$

$$a_\downarrow^\dagger(\mathbf{p}) = \pm ie^{-i\alpha} b_\uparrow^\dagger(\mathbf{p}) \,, \tag{43}$$

which may be used instead of (35). Due to the possible signs $\pm$ the number of the corresponding states is the same as in the Dirac case that permits us to have the complete system of the Fock states over the $(1/2, 0) \oplus (0, 1/2)$ representation space in the mathematical sense.[6] However, in this case we deal with the self/anti-self charge conjugate quantum field operator instead of the self/anti-self charge conjugate quantum states. Please remember that it is the latter that answer for the neutral particles; the quantum field operator contains operators for more than one state, which may be either electrically neutral or charged.

We conclude that something is missed in the foundations of both the original Majorana theory and its generalizations in the $(1/2, 0) \oplus (0, 1/2)$ representation.

2.4   Self/anti-self Charge Conjugate States.

We re-write the charge conjugation operator (41) to the form:

$$
C = e^{i\theta} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \mathcal{K} = -e^{i\theta} \gamma^2 \mathcal{K} \,. \tag{44}
$$

It is the anti-linear operator of charge conjugation. $\mathcal{K}$ is the complex conjugation operator. We define the *self/anti-self* charge-conjugate 4-spinors in the momentum space [17]

$$
\begin{align}
C\lambda^{S,A}(\mathbf{p}) &= \pm\lambda^{S,A}(\mathbf{p}) \,, \tag{45} \\
C\rho^{S,A}(\mathbf{p}) &= \pm\rho^{S,A}(\mathbf{p}) \,. \tag{46}
\end{align}
$$

Thus,

$$
\lambda^{S,A}(p^\mu) = \begin{pmatrix} \pm i\Theta \phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix} \,, \tag{47}
$$

and

$$
\rho^{S,A}(\mathbf{p}) = \begin{pmatrix} \phi_R(\mathbf{p}) \\ \mp i\Theta \phi_R^*(\mathbf{p}) \end{pmatrix} \,. \tag{48}
$$

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[6]Please note that the phase factors may have physical significance in quantum field theories as opposed to the textbook nonrelativistic quantum mechanics, as was discussed recently by several authors.

ϕ_L, ϕ_R can be boosted with the Lorentz transformation $\Lambda_{L,R}$ matrices.[7]

The rest λ and ρ spinors are:[8]

$$\lambda_\uparrow^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \; \lambda_\downarrow^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} -i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{50}$$

$$\lambda_\uparrow^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ -i \\ 1 \\ 0 \end{pmatrix}, \; \lambda_\downarrow^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{51}$$

$$\rho_\uparrow^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}, \; \rho_\downarrow^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \tag{52}$$

$$\rho_\uparrow^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}, \; \rho_\downarrow^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{53}$$

Thus, in this basis the explicite forms of the 4-spinors of the second kind $\lambda_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ and $\rho_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ are:

$$\lambda_\uparrow^S(\mathbf{p}) = \frac{1}{2\sqrt{E_p + m}} \begin{pmatrix} ip_l \\ i(p^- + m) \\ p^- + m \\ -p_r \end{pmatrix}, \lambda_\downarrow^S(\mathbf{p}) = \frac{1}{2\sqrt{E_p + m}} \begin{pmatrix} -i(p^+ + m) \\ -ip_r \\ -p_l \\ (p^+ + m) \end{pmatrix} \tag{54}$$

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[7]Such definitions of 4-spinors differ, of course, from the original Majorana definition in x-representation:

$$\nu(x) = \frac{1}{\sqrt{2}}(\Psi_D(x) + \Psi_D^c(x)), \tag{49}$$

$C\nu(x) = \nu(x)$ that represents the positive real $C-$ parity field operator. However, the momentum-space Majorana-like spinors open various possibilities for description of neutral particles (with experimental consequences, see [23]). For instance, "for imaginary C parities, the neutrino mass can drop out from the single β decay trace and reappear in $0\nu\beta\beta$, a curious and in principle experimentally testable signature for a non-trivial impact of Majorana framework in experiments with polarized sources."

[8]The choice of the helicity parametrization (20) for $\mathbf{p} \to \mathbf{0}$ is doubtful in Ref. [24], and it leads to unremovable contradictions, in my opinion.

$$\lambda_\uparrow^A(\mathbf{p}) = \frac{1}{2\sqrt{E_p+m}} \begin{pmatrix} -ip_l \\ -i(p^-+m) \\ (p^-+m) \\ -p_r \end{pmatrix}, \lambda_\downarrow^A(\mathbf{p}) = \frac{1}{2\sqrt{E_p+m}} \begin{pmatrix} i(p^++m) \\ ip_r \\ -p_l \\ (p^++m) \end{pmatrix}$$

(55)

$$\rho_\uparrow^S(\mathbf{p}) = \frac{1}{2\sqrt{E_p+m}} \begin{pmatrix} p^++m \\ p_r \\ ip_l \\ -i(p^++m) \end{pmatrix}, \rho_\downarrow^S(\mathbf{p}) = \frac{1}{2\sqrt{E_p+m}} \begin{pmatrix} p_l \\ (p^-+m) \\ i(p^-+m) \\ -ip_r \end{pmatrix}$$

(56)

$$\rho_\uparrow^A(\mathbf{p}) = \frac{1}{2\sqrt{E_p+m}} \begin{pmatrix} p^++m \\ p_r \\ -ip_l \\ i(p^++m) \end{pmatrix}, \rho_\downarrow^A(\mathbf{p}) = \frac{1}{2\sqrt{E_p+m}} \begin{pmatrix} p_l \\ (p^-+m) \\ -i(p^-+m) \\ ip_r \end{pmatrix}.$$

(57)

As we showed $\lambda-$ and $\rho-$ 4-spinors are not the eigenspinors of the helicity. Moreover, λ and ρ are not the eigenspinors of the parity (if we use the parity matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R$), as opposed to the Dirac case. The indices $\uparrow\downarrow$ should be referred to the chiral helicity quantum number introduced in the 60s, $\eta = -\gamma^5 h$, for λ spinors. While

$$Pu_\sigma(\mathbf{p}) = +u_\sigma(\mathbf{p}), Pv_\sigma(\mathbf{p}) = -v_\sigma(\mathbf{p}),$$

(58)

we have

$$P\lambda^{S,A}(\mathbf{p}) = \rho^{A,S}(\mathbf{p}), P\rho^{S,A}(\mathbf{p}) = \lambda^{A,S}(\mathbf{p})$$

(59)

for the Majorana-like momentum-space 4-spinors on the first quantization level. In this basis one has

$$\rho_\uparrow^S(\mathbf{p}) = -i\lambda_\downarrow^A(\mathbf{p}), \rho_\downarrow^S(\mathbf{p}) = +i\lambda_\uparrow^A(\mathbf{p}),$$

(60)

$$\rho_\uparrow^A(\mathbf{p}) = +i\lambda_\downarrow^S(\mathbf{p}), \rho_\downarrow^A(\mathbf{p}) = -i\lambda_\uparrow^S(\mathbf{p}).$$

(61)

The analogs of the spinor normalizations (for $\lambda_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ and $\rho_{\uparrow\downarrow}^{S,A}(\mathbf{p})$) are the following ones:

$$\overline{\lambda}_\uparrow^S(\mathbf{p})\lambda_\downarrow^S(\mathbf{p}) = -im \quad, \quad \overline{\lambda}_\downarrow^S(\mathbf{p})\lambda_\uparrow^S(\mathbf{p}) = +im \quad,$$

(62)

$$\overline{\lambda}_\uparrow^A(\mathbf{p})\lambda_\downarrow^A(\mathbf{p}) = +im \quad, \quad \overline{\lambda}_\downarrow^A(\mathbf{p})\lambda_\uparrow^A(\mathbf{p}) = -im \quad,$$

(63)

$$\overline{\rho}_\uparrow^S(\mathbf{p})\rho_\downarrow^S(\mathbf{p}) = +im \quad, \quad \overline{\rho}_\downarrow^S(\mathbf{p})\rho_\uparrow^S(\mathbf{p}) = -im \quad,$$

(64)

$$\overline{\rho}_\uparrow^A(\mathbf{p})\rho_\downarrow^A(\mathbf{p}) = -im \quad, \quad \overline{\rho}_\downarrow^A(\mathbf{p})\rho_\uparrow^A(\mathbf{p}) = +im \quad.$$

(65)

All other conditions are equal to zero.

The $\lambda-$ and $\rho-$ spinors are connected with the $u-$ and $v-$ spinors by the following formula

$$
\begin{pmatrix} \lambda_\uparrow^S(\mathbf{p}) \\ \lambda_\downarrow^S(\mathbf{p}) \\ \lambda_\uparrow^A(\mathbf{p}) \\ \lambda_\downarrow^A(\mathbf{p}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i & -1 & i \\ -i & 1 & -i & -1 \\ 1 & -i & -1 & -i \\ i & 1 & i & -1 \end{pmatrix} \begin{pmatrix} u_{+1/2}(\mathbf{p}) \\ u_{-1/2}(\mathbf{p}) \\ v_{+1/2}(\mathbf{p}) \\ v_{-1/2}(\mathbf{p}) \end{pmatrix}, \tag{66}
$$

provided that the 4-spinors have the same physical dimension.[9]

We construct the field operators on using the procedure above with $\lambda_\eta^S(p)$. Thus, the difference with the last equality of the equation (9) is that 1) instead of $u_h(\pm p)$ we have $\lambda_\eta^S(\pm p)$; 2) possible change of the annihilation operators $a_h \rightarrow c_\eta$. Apart, one can make corresponding changes due to normalization factors. Thus, we should have

$$
\sum_{\eta=\pm 1/2} \lambda_\eta^A(p) d_\eta^\dagger(p) = \sum_{\eta=\pm 1/2} \lambda_\eta^S(-p) c_\eta(-p) . \tag{67}
$$

On using the same procedure as in the subsection (2.1), we find surprisingly:

$$
d_\eta^\dagger(p) = -\frac{ip_y}{p} \sigma_{\eta\tau}^y c_\tau(-p) , \quad c_\eta(-p) = -\frac{ip_y}{p} \sigma_{\eta\tau}^y d_\tau^\dagger(p) . \tag{68}
$$

The bi-orthogonal anticommutation relations are given in Ref. [17]. See other details in Ref. [25, 26]. Concerning with the P,C and T properties of the corresponding states see Ref. [26] in this model.

The above-mentioned contradiction may be related to the possibility of the conjugation which is different from that of Dirac. Both in the Dirac-like case and the Majorana-like case ($c_\eta(p) = e^{-i\varphi} d_\eta(p)$) we have difficulties in the construction of field operators.

---

[9]The change of the mass dimension of the field operator has no sufficient foundations because the Lagrangian can be constructed on using the coupled Dirac equations, see Ref. [25]. After that one can play with $\sqrt{m}$ to reproduce all possible mathematical results, which may (or may not) answer to the physical reality.

3 The Spin-1.

3.1 The Standard Basis.

We use the results of Refs. [27, 18, 28] in this Section. The polarization vectors of the standard basis are defined [12]:

$$\epsilon^\mu(\mathbf{0},+1) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon^\mu(\mathbf{0},-1) = +\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \tag{69}$$

$$\epsilon^\mu(\mathbf{0},0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \epsilon^\mu(\mathbf{0},0_t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{70}$$

The Lorentz transformations are ($\hat{p}_i = p_i/|\mathbf{p}|$):

$$\epsilon^\mu(\mathbf{p},\sigma) = L^\mu_\nu(\mathbf{p})\epsilon^\nu(\mathbf{0},\sigma), \tag{71}$$

$$L^0_{\ 0}(\mathbf{p}) = \gamma, \ L^i_{\ 0}(\mathbf{p}) = L^0_{\ i}(\mathbf{p}) = \hat{p}_i\sqrt{\gamma^2 - 1}, \ L^i_{\ k}(\mathbf{p}) = \delta_{ik} + (\gamma - 1)\hat{p}_i\hat{p}_k. \tag{72}$$

Hence, for the particles of the mass m we have:

$$u^\mu(\mathbf{p},+1) = -\frac{N}{\sqrt{2}m}\begin{pmatrix} -p^r \\ m + \frac{p^1 p^r}{E_p+m} \\ im + \frac{p^2 p^r}{E_p+m} \\ \frac{p^3 p^r}{E_p+m} \end{pmatrix}, \quad u^\mu(\mathbf{p},-1) = \frac{N}{\sqrt{2}m}\begin{pmatrix} -p^l \\ m + \frac{p^1 p^l}{E_p+m} \\ -im + \frac{p^2 p^l}{E_p+m} \\ \frac{p^3 p^l}{E_p+m} \end{pmatrix}, \tag{73}$$

$$u^\mu(\mathbf{p},0) = \frac{N}{m}\begin{pmatrix} -p^3 \\ \frac{p^1 p^3}{E_p+m} \\ \frac{p^2 p^3}{E_p+m} \\ m + \frac{(p^3)^2}{E_p+m} \end{pmatrix}, \quad u^\mu(\mathbf{p},0_t) = \frac{N}{m}\begin{pmatrix} E_p \\ -p^1 \\ -p^2 \\ -p^3 \end{pmatrix}. \tag{74}$$

N is the normalization constant. They are the eigenvectors of the parity operator ($\gamma_{00} = \text{diag}(1\ -1\ -1\ -1)$):

$$\hat{P}u_\mu(-\mathbf{p},\sigma) = -u_\mu(\mathbf{p},\sigma), \quad \hat{P}u_\mu(-\mathbf{p},0_t) = +u_\mu(\mathbf{p},0_t). \tag{75}$$

It is assumed that they form the complete orthonormalized system of the $(1/2, 1/2)$ represnetation, $\epsilon^*_\mu(\mathbf{p},0_t)\epsilon^\mu(\mathbf{p},0_t) = 1$, $\epsilon^*_\mu(\mathbf{p},\sigma')\epsilon^\mu(\mathbf{p},\sigma) = -\delta_{\sigma'\sigma}$.

3.2 The Helicity Basis.

The helicity operator is:

$$\frac{(\mathbf{S} \cdot \mathbf{p})}{p} = \frac{1}{p} \begin{pmatrix} 0 & 0 & -ip^3 & ip^2 \\ 0 & ip^3 & 0 & -ip^1 \\ 0 & -ip^2 & ip^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{(\mathbf{S} \cdot \mathbf{p})}{p}\epsilon_{\pm 1}^{\mu} = \pm\epsilon_{\pm 1}^{\mu}, \quad \frac{(\mathbf{S} \cdot \mathbf{p})}{p}\epsilon_{0,0_t}^{\mu} = 0.$$

$$(76)$$

The eigenvectors are:

$$\epsilon_{+1}^{\mu} = \frac{1}{\sqrt{2}}\frac{e^{i\alpha}}{p}\begin{pmatrix} 0 \\ \frac{-p^1 p^3 + ip^2 p}{\sqrt{(p^1)^2 + (p^2)^2}} \\ \frac{-p^2 p^3 - ip^1 p}{\sqrt{(p^1)^2 + (p^2)^2}} \\ \sqrt{(p^1)^2 + (p^2)^2} \end{pmatrix}, \quad \epsilon_{-1}^{\mu} = \frac{1}{\sqrt{2}}\frac{e^{i\beta}}{p}\begin{pmatrix} 0 \\ \frac{p^1 p^3 + ip^2 p}{\sqrt{(p^1)^2 + (p^2)^2}} \\ \frac{p^2 p^3 - ip^1 p}{\sqrt{(p^1)^2 + (p^2)^2}} \\ -\sqrt{(p^1)^2 + (p^2)^2} \end{pmatrix}$$

$$(77)$$

$$\epsilon_0^{\mu} = \frac{1}{m}\begin{pmatrix} p \\ \frac{E}{p}p^1 \\ \frac{E}{p}p^2 \\ \frac{E}{p}p^3 \end{pmatrix}, \quad \epsilon_{0_t}^{\mu} = \frac{1}{m}\begin{pmatrix} E_p \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}. \tag{78}$$

The normalization is the same as in the standard basis. The eigenvectors $\epsilon_{\pm 1}^{\mu}$ are not the eigenvectors of the parity operator $(\gamma_{00}R)$ of this representation. However, the $\epsilon_{1,0}^{\mu}$, $\epsilon_{0,0_t}^{\mu}$ are. Surprisingly, the latter have no well-defined massless limit. In order to get the well-known massless limit one should use the basis of the light-front form reprersentation, cf. [29].

3.3 The Field Operators.

Various-type field operators are possible in this representation. Let us remind the procedure to get them. Again, during the calculations below we have to present $1 = \theta(p_0) + \theta(-p_0)$ in order to get positive- and negative-frequency parts. Meanwhile, the Heaviside $\theta-$ function is not defined in $p_0 = 0$. In general, due to integral theorems this presentation is possible

even for distributions because we use the θ- function in the integrand.[10]

$$A_\mu(x) = \frac{1}{(2\pi)^3} \int d^4p\, \delta(p^2 - m^2) e^{-ip\cdot x} A_\mu(p) =$$

$$= \frac{1}{(2\pi)^3} \sum_\lambda \int d^4p\, \delta(p_0^2 - E_p^2) e^{-ip\cdot x} \epsilon_\mu(p,\lambda) a_\lambda(p) =$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^4p}{2E_p} [\delta(p_0 - E_p) + \delta(p_0 + E_p)][\theta(p_0) + \theta(-p_0)] e^{-ip\cdot x} A_\mu(p) =$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^4p}{2E_p} [\delta(p_0 - E_p) + \delta(p_0 + E_p)] \left[\theta(p_0) A_\mu(p) e^{-ip\cdot x} + \right.$$

$$+ \left. \theta(p_0) A_\mu(-p) e^{+ip\cdot x}\right] = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2E_p} \theta(p_0) [A_\mu(p) e^{-ip\cdot x} + A_\mu(-p) e^{+ip\cdot x}] =$$

$$= \frac{1}{(2\pi)^3} \sum_\lambda \int \frac{d^3\mathbf{p}}{2E_p} [\epsilon_\mu(p,\lambda) a_\lambda(p) e^{-ip\cdot x} + \epsilon_\mu(-p,\lambda) a_\lambda(-p) e^{+ip\cdot x}] . \qquad (79)$$

We should transform the second part to $\epsilon_\mu^*(p,\lambda) b_\lambda^\dagger(p)$ as usual. In such a way we obtain the states which are considered to be the charge-conjugate states. In this Lorentz group representation the charge conjugation operator is just the complex conjugation operator for 4-vectors. Of course, one can try to get P-conjugates or CP-conjugate states too. We posulate

$$\sum_\lambda \epsilon_\mu(-p,\lambda) a_\lambda(-p) = \sum_\lambda \epsilon_\mu^*(p,\lambda) b_\lambda^\dagger(p) . \qquad (80)$$

Then we multiply both parts by $\epsilon^\mu(p,\sigma)$, and use the normalization conditions for polarization vectors.

In the $(\frac{1}{2}, \frac{1}{2})$ representation we can also expand (apart of the equation (80)) in a different way. For example,

$$\sum_\lambda \epsilon_\mu(-p,\lambda) a_\lambda(-p) = \sum_\lambda \epsilon_\mu(p,\lambda) d_\lambda^\dagger(p) . \qquad (81)$$

From the first definition we obtain:

$$\begin{pmatrix} b_{0_t}^\dagger(p) \\ -b_{+1}^\dagger(p) \\ -b_0^\dagger(p) \\ -b_{-1}^\dagger(p) \end{pmatrix} = \sum_{\mu\lambda} \epsilon^\mu(p,\sigma) \epsilon_\mu(-p,\lambda) a_\lambda(-p) = \sum_\lambda \Lambda_{\sigma\lambda}^{(1a)} a_\lambda(-p) =$$

---

[10]However, remember, that we have the $p_0 = 0$ solution of the Maxwell equations. It has the experimental confirmation (for instance, the stationary magnetic field $curl\mathbf{B} = 0$).

$$
= \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{p_r^2}{p^2} & -\frac{\sqrt{2}p_z p_r}{p^2} & \frac{p_z^2}{p^2} \\
0 & -\frac{\sqrt{2}p_z p_r}{p^2} & -1+\frac{2p_z^2}{p^2} & +\frac{\sqrt{2}p_z p_l}{p^2} \\
0 & \frac{p_z^2}{p^2} & +\frac{\sqrt{2}p_z p_l}{p^2} & \frac{p_l^2}{p^2}
\end{pmatrix}
\begin{pmatrix}
a_{00}(-p) \\
a_{11}(-p) \\
a_{10}(-p) \\
a_{1-1}(-p)
\end{pmatrix}. \tag{82}
$$

From the second definition $\Lambda^{1b}_{\sigma\lambda} = \sum_\mu \epsilon^{\mu*}(p,\sigma)\epsilon_\mu(-p,\lambda)$ we have

$$
\begin{pmatrix}
d^\dagger_{0_t}(p) \\
-d^\dagger_{+1}(p) \\
-d^\dagger_0(p) \\
-d^\dagger_{-1}(p)
\end{pmatrix}
= \sum_{\mu\lambda} \epsilon^{\mu*}(p,\sigma)\epsilon_\mu(-p,\lambda)c_\lambda(-p) = \sum_\lambda \Lambda^{(1b)}_{\sigma\lambda}c_\lambda(-p) =
$$

$$
= \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{-p_z^2}{p^2} & -\frac{\sqrt{2}p_z p_l}{p^2} & \frac{-p_l^2}{p^2} \\
0 & -\frac{\sqrt{2}p_z p_r}{p^2} & -1+\frac{2p_z^2}{p^2} & +\frac{\sqrt{2}p_z p_l}{p^2} \\
0 & -\frac{p_r^2}{p^2} & +\frac{\sqrt{2}p_z p_r}{p^2} & -\frac{p_z^2}{p^2}
\end{pmatrix}
\begin{pmatrix}
c_{00}(-p) \\
c_{11}(-p) \\
c_{10}(-p) \\
c_{1-1}(-p)
\end{pmatrix} = -g^{\sigma\lambda} + (\mathbf{S}\cdot\mathbf{n})^2_{\sigma\lambda}.
$$
$$\tag{83}$$

Possibly, we should think about modifications of the Fock space in this case, or introduce several field operators for the $(\frac{1}{2},\frac{1}{2})$ representation. The Majorana-like anzatz is compatible for the 0_t polarization state[11] in this basis of this representation only.

However, the corresponding matrices Λ^2 in the helicity basis are different. Here they are:

$$
\begin{pmatrix}
b^\dagger_{0_t}(p) \\
-b^\dagger_{+1}(p) \\
-b^\dagger_0(p) \\
-b^\dagger_{-1}(p)
\end{pmatrix}
= \sum_{\mu\lambda} \epsilon^\mu(p,\sigma)\epsilon_\mu(-p,\lambda)a_\lambda(-p) = \sum_\lambda \Lambda^{(2a)}_{\sigma\lambda}a_\lambda(-p) =
$$

$$
= -\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{2i\alpha} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{2i\beta}
\end{pmatrix}
\begin{pmatrix}
a_{00}(-p) \\
a_{11}(-p) \\
a_{10}(-p) \\
a_{1-1}(-p)
\end{pmatrix}, \tag{84}
$$

and

$$
\begin{pmatrix}
d^\dagger_{0_t}(p) \\
-d^\dagger_{+1}(p) \\
-d^\dagger_0(p) \\
-d^\dagger_{-1}(p)
\end{pmatrix}
= \sum_{\mu\lambda} \epsilon^{\mu*}(p,\sigma)\epsilon_\mu(-p,\lambda)c_\lambda(-p) = \sum_\lambda \Lambda^{(2b)}_{\sigma\lambda}c_\lambda(-p) =
$$

---

[11]00 in other notation.

$$= -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i(\alpha-\beta)} \\ 0 & 0 & 1 & 0 \\ 0 & -e^{+i(\alpha-\beta)} & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{00}(-p) \\ c_{11}(-p) \\ c_{10}(-p) \\ c_{1-1}(-p) \end{pmatrix}. \tag{85}$$

This is compatible with the Majorana-like anzatzen.

Of course, the same procedure can be applied in the construction of the quantum field operator for $F_{\mu\nu}$.

3.4 The $(1,0) \oplus (0,1)$ Representation.

The solutions of the Weinberg-like equation

$$[\gamma^{\mu\nu} \partial_\mu \partial_\nu - \frac{(i\partial/\partial t)}{E} m^2] \Psi(x) = 0. \tag{86}$$

are found in Refs. [30, 27, 31, 32]. Here they are:

$$u_\sigma(\mathbf{p}) = \begin{pmatrix} D^S(\Lambda_R)\xi_\sigma(\mathbf{0}) \\ D^S(\Lambda_L)\xi_\sigma(\mathbf{0}) \end{pmatrix}, \quad v_\sigma(\mathbf{p}) = \begin{pmatrix} D^S(\Lambda_R \Theta_{[1/2]})\xi_\sigma^*(\mathbf{0}) \\ -D^S(\Lambda_L \Theta_{[1/2]})\xi_\sigma^*(\mathbf{0}) \end{pmatrix} = \Gamma^5 u_\sigma(\mathbf{p}), \tag{87}$$

$$\Gamma^5 = \begin{pmatrix} 1_{3\times3} & 0 \\ 0 & -1_{3\times3} \end{pmatrix}, \tag{88}$$

where D^S is the matrix of the $(S,0)$ representation of the spinor group $SL(2,c)$.

In the $(1,0) \oplus (0,1)$ representation the procedure of derivation of the creation operators leads to somewhat different situation:

$$\sum_{\sigma=0,\pm1} v_\sigma(p)b_\sigma^\dagger(p) = \sum_{\sigma=0,\pm1} u_\sigma(-p)a_\sigma(-p), \quad \text{hence } b_\sigma^\dagger(p) = 0. \tag{89}$$

However, if we return to the original Weinberg equations $[\gamma^{\mu\nu} \partial_\mu \partial_\nu \pm m^2]\Psi_{1,2}(x) = 0$ with the field operators:

$$\Psi_1(x) = \frac{1}{(2\pi)^3} \sum_\mu \int \frac{d^3\mathbf{p}}{2E_p} [u_\mu(\mathbf{p})a_\mu(\mathbf{p})e^{-ip_\mu \cdot x^\mu} + u_\mu(\mathbf{p})b_\mu^\dagger(\mathbf{p})e^{+ip_\mu \cdot x^\mu}], \tag{90}$$

$$\Psi_2(x) = \frac{1}{(2\pi)^3} \sum_\mu \int \frac{d^3\mathbf{p}}{2E_p} [v_\mu(\mathbf{p})c_\mu(\mathbf{p})e^{-ip_\mu \cdot x^\mu} + v_\mu(\mathbf{p})d_\mu^\dagger(\mathbf{p})e^{+ip_\mu \cdot x^\mu}], \tag{91}$$

we obtain

$$b_\mu^\dagger(p) = [1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda} a_\lambda(-p) \,, \tag{92}$$

$$d_\mu^\dagger(p) = [1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda} c_\lambda(-p) \,. \tag{93}$$

The application of $\bar{u}_\mu(-p)u_\lambda(-p) = \delta_{\mu\lambda}$ and $\bar{u}_\mu(-p)u_\lambda(p) = [1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda}$ prove that the equations are self-consistent (similarly to the subsection (2.1)). This situation signifies that in order to construct the Sankaranarayanan-Good field operator (which was used by Ahluwalia, Johnson and Goldman [31]) we need additional postulates. One can try to construct the left- and the right-hand side of the field operator separately each other. In this case the commutation relations may also be more complicated.

Is it possible to apply the Majorana-like anzatz to the $(1,0)+(0,1)$ fields? Repeating the procedure of the Section 2.3, on using (92) and the Majorana posulate we come to:

$$a_\mu^\dagger(p) = +e^{+i\varphi}[1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda} a_\lambda(-p) \,. \tag{94}$$

On the other hand, on using the inverse relation, namely, that for $a_\mu(-p)$, we make the substitutions $E_p \to -E_p$, $\mathbf{p} \to -\mathbf{p}$ to obtain

$$a_\mu(p) = +[1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda} b_\lambda^\dagger(-p) \,. \tag{95}$$

The totally reflected Majorana nazatz is $b_\mu(-E_p, -\mathbf{p}) = e^{i\varphi} a_\mu(-E_p, -\mathbf{p})$. Thus,

$$b_\mu^\dagger(-p) = e^{-i\varphi} a_\mu^\dagger(-p) \,. \tag{96}$$

Combining with (95), we come to

$$a_\mu(p) = +e^{-i\varphi}[1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda} a_\lambda^\dagger(-p) \,, \tag{97}$$

and

$$a_\mu^\dagger(p) = +e^{+i\varphi}[1 - 2(\mathbf{S}^* \cdot \mathbf{n})^2]_{\mu\lambda} a_\lambda(-p) \,. \tag{98}$$

In the basis where S_z is diagonal the matrix S_y is imaginary [12]. So, $(\mathbf{S}^* \cdot \mathbf{n}) = S_x n_x - S_y n_y + S_z n_z$, and $(\mathbf{S}^* \cdot \mathbf{n})^2 \neq (\mathbf{S} \cdot \mathbf{n})^2$ in the case of $S = 1$. So, we conclude that there is the same problem in this point, in the aplication of the Majorana-like anzatz, as in the case of spin-1/2. Similarly, one can proceed with (93).

Meanwhile, the attempts of constructing the self/anti-self charge conjugate states failed in Ref. [17]. Instead, the $\Gamma^5 S_{[1]}^c-$ self/anti-self conjugate states have been constructed therein.

4 Conclusions.

We conclude that something is missed in the foundations of both the original Majorana theory and its generalizations. Similar problems exist in the theories of higher spins.

References

[1] V. V. Dvoeglazov, Hadronic J. Suppl. **18**, 239 (2003).

[2] V. V. Dvoeglazov, Int. J. Mod. Phys. B**20**, 1317 (2006).

[3] V. V. Dvoeglazov, in *Einstein and Others: Unification.* Ed. V. V. Dvoeglazov. (Nova Sci. Pubs., Hauppauge, NY, USA, 2015), p. 211.

[4] V. V. Dvoeglazov, Z. Naturforsch. A **71**, 345 (2016).

[5] V. V. Dvoeglazov, SFIN **22**-A1, 157 (2009); J. Phys. Conf. Ser. **284**, 012024 (2011); Bled Workshops **10**-2, 52 (2009); ibid. **11**-2, 9 (2010); in *Beyond the Standard Model. Proceedings of the Vigier Symposium.* (Noetic Press, Orinda, USA, 2005), p. 388.

[6] V. V. Dvoeglazov, Bled Workshops **14**-2, 199 (2013).

[7] P. A. M. Dirac, Proc. Roy. Soc. Lond. A**117**, 610 (1928).

[8] J. J. Sakurai, *Advanced Quantum Mechanics.* (Addison-Wesley, 1967).

[9] L. H. Ryder, *Quantum Field Theory.* (Cambridge University Press, Cambridge, 1985).

[10] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill Book Co., 1980).

[11] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields.* 2nd Edition. (Nauka, Moscow, 1973).

[12] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskiĭ, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988), §6.2.5.

[13] V. V. Dvoeglazov, Fizika B**6**, 111 (1997).

[14] V. V. Dvoeglazov, Int. J. Theor. Phys. **43**, 1287 (2004).

[15] V. B. Berestetskiĭ, E. M. Lifshitz and L. P. Pitaevskiĭ, *Quantum Electrodynamics.* (Pergamon Press, 1982, translated from the Russian), §16.

[16] V. V. Dvoeglazov, Nuovo Cim. B**111**, 483 (1996).

[17] D. V. Ahluwalia, Int. J. Mod. Phys. A**11**, 1855 (1996).

[18] S. Weinberg, *The Quantum Theory of Fields. Vol. I. Foundations.* (Cambridge University Press, Cambridge, 1995).

[19] W. Greiner, *Field Quantization.* (Springer, 1996).

[20] Z. Tokuoka, Prog. Theor. Phys. **37**, 603 (1967).

[21] H. M. Rück and W. Greiner, J. Phys. G: Nucl. Phys. **3**, 657 (1977).

[22] E. Majorana, Nuovo Cim. **14**, 171 (1937).

[23] M. Kirchbach, C. Compean and L. Noriega, Eur. Phys. J. A**22**, 149 (2004).

[24] D. V. Ahluwalia and D. Grumiller, JCAP **0507**, 012 (2005).

[25] V. V. Dvoeglazov, Nuovo Cim. A**108**, 1467 (1995).

[26] V. V. Dvoeglazov, Mod. Phys. Lett. A**12**, 2741 (1997); J. Phys. Conf. Ser. **1010**, 012011 (2018).

[27] Yu. V. Novozhilov, *Introduction to Elementary Particle Physics* (Pergamon Press, 1975).

[28] V. V. Dvoeglazov, in *Photon: Old Problems in Light of New Ideas.* Ed. V. V. Dvoeglazov (Nova Sci. Pubs., Huntington, NY, USA, 2000).

[29] D. V. Ahluwalia and M. Sawicki, Phys. Rev. D**47**, 5161 (1993); Phys. Lett. B**335**, 24 (1994).

[30] A. Sankaranarayanan and R. H. Good, Jr., Nuovo Cim. **36**, 1303 (1965).

[31] D. V. Ahluwalia, M. B. Johnson and T. Goldman, Phys. Lett. B**316**, 102 (1993).

[32] V. V. Dvoeglazov, Int. J. Theor. Phys. **37**, 1915 (1998).