An optimization approach to the Riemann Hypothesis

Introduction

A great deal of research has been and still is being devoted to the zeros of the Riemann Zeta function (RZF) that are in the critical strip[1] and known as the nontrivial zeros of RZF. The Riemann Hypothesis (RH) states that these zeros are all located on the critical line[2]. Although a large number of nontrivial zeros have proved to be located on the critical line through numerical computation methods, starting with Riemann's manual computation of the first few zeros [1], no analytical proof or disproof of RH has been found since its conjecture by Riemann in 1859. In this paper, we implement a novel analytical approach to RH based on optimization. This analysis tool proved successful in deriving some important scientific theories and laws [2]. Such a success prompted us to use this tool to analytically derive the location of RZF's nontrivial zeros in order to either prove or disprove the Riemann Hypothesis. This was achieved by formulating and solving the appropriate location optimization problem.

Problem formulation

We denote the Riemann Zeta function as $\zeta(s) = U(\sigma,t) + iV(\sigma,t)$, for complex $s=\sigma+it$. To simplify the notation, we define: $L_\sigma = \partial L/\partial\sigma$, $L_\mu = \partial L/\partial\mu$, $Z_\sigma = \partial Z/\partial\sigma$, $L_{\sigma\sigma} = \partial^2 L/\partial\sigma^2$, $L_{\mu\mu} = \partial^2 L/\partial\mu^2$, Also, the values of some functions at (σ^*, μ^*, t^*) are upper-scripted with a *, *e.g.* U^*, V^*, Z_σ^*.

As a consequence of the properties of RZF and the properties of its nontrivial zeros[3], the search for the location (the real part) of these zeros at imaginary part $t = t^*$ where RZF vanishes, can be limited to the left half of the critical strip $(0\leq \sigma\leq 1/2)$ since zeros on the right half $(1/2\leq \sigma<1)$ can be derived by symmetry about the critical line $(\sigma=1/2)$ as per RZF's property (6)[4].

If RZF has a nontrivial zero at some height $t = t^*$, this search entails finding the value σ^* where $\zeta(\sigma + it^*)$, or equivalently $|\zeta(\sigma, t^*)|^2$, vanishes. Hence, this task can be accomplished by minimizing the univariate function $|\zeta(\sigma; t^*)|^2$ under the constraint $0 \leq \sigma \leq ½$, with t^* being a constant.

The optimization problem (P) of interest is then to:

$$\begin{aligned} &\text{Minimize} \quad f(\sigma) = Z(\sigma; t^*) = |\zeta(\sigma; t^*)|^2 = U^2(\sigma; t^*) + V^2(\sigma; t^*) \\ &\text{Subject to:} \quad g(\sigma) = \sigma - 1/2 \leq 0 \qquad\qquad \text{(P)} \end{aligned}$$

$\sigma \geq 0$, with $f(\sigma)$ and $g(\sigma)$ infinitely differentiable, (RZF property (1), for σ in [0,1/2].

To solve the inequality-constrained problem (P), we use the Karush-Kuhn-Tucker (KKT) method with a nonnegativity condition on the variable σ [3]. The Lagrange function associated with (P) is then:

$L(\sigma, \mu; t^*) = Z(\sigma; t^*) + \mu(\sigma - ½)$, μ being the KKT/Lagrange multiplier for constraint $g(\sigma)$.

For inequality-constrained optimization problems such as problem (P), with continuously differentiable functions, nonnegative variables (σ in our case), and under a regularity qualification of the constraints[5], which is of no concern since we have only one constraint, the necessary optimality conditions are the so called KKT necessary conditions with $\sigma \geq 0$, [3]. For problem (P), these conditions are the following:

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[1] *A strip in the complex plane defined by $0 \leq \sigma \leq 1$*

[2] *The critical line is the line in the complex plane defined by $\sigma = 1/2$*

[3] *See Appendix for the list of properties*

[4] *See Appendix for the list of properties*

[5] *The gradients of the equality and binding nonequality constraints have to be linearly independent at the stationary/critical point(s) of the Lagrangian function. This requirement is of no concern here since we have one constraint only.*

1. Complementary slackness conditions:

$\sigma^* L_\sigma(\sigma^*, \mu^*, t^*) = \sigma^*(Z_\sigma{}^* + \mu^*) = 0$ (1)

$\mu^* L_\mu (\sigma^*, \mu^*, t^*) = \mu^*(\sigma^* - ½) = 0$ (2)

2. Feasibility conditions :

$L_\sigma (\sigma^*, \mu^*, t^*) = Z_\sigma{}^* + \mu^* \geq 0$ (3)

$L_\mu (\sigma^*, \mu^*, t^*) = g(\sigma^* - ½) \leq 0$ (4)

$\mu^* \geq 0$ (5)

$\sigma^* \geq 0$ (6)

Each possible combinations of active/binding (equality) constraints derived from the KKT necessary conditions may enable the identification of candidate solutions to problem (P). To be feasible, candidate solutions have to meet additional feasibility conditions required by the properties[6] of RZF and the properties of its nontrivial zeros as listed below. Feasible candidate solutions, a.k.a critical points, can be minima, maxima or saddle points. Those which meet the sufficient optimality conditions are the sought after minima for problem (P), and therefore are the locations of the nontrivial zeros for $t = t^*$ where RZF vanishes. Candidate and optimal solutions are upper scripted with a *, e.g. σ^*, μ^*.

Problem solution

A. Relevant properties of RZF and its nontrivial zeros

The following properties are relevant for solving problem (P):

1. For any nontrivial zero at $t = t^*$ and $\sigma = \sigma^*$ we have: $U^* = U(\sigma^*, t^*) = 0$ and $V^* = V(\sigma^*, t^*) = 0$,
2. Since RZF is differentiable for σ in $[0,1)$, its derivatives exist, and we have: $U^*U^*_\sigma + V^*V^*_\sigma = 0$, so that $Z^*_\sigma = 2(U^*U^*_\sigma + V^*V^*_\sigma) = 0$, and (7a)
   $U^*U^*_{\sigma\sigma} + V^*V^*\sigma\sigma, = 0$ (7b)
3. RZF's property (7) requires that $\sigma^* > 0$ (8)
4. "Nontrivial zeros occur Either on the critical line Or in pairs[7]" off of it (9)

B. Solution

From condition (8) above, ($\sigma^* > 0$), and KKT condition (1) we get:

$Z^*_\sigma + \mu^* = 0$ (10)

Then, from (7a) and (10) we get $\mu^* = 0$ as the value for μ that meets KKT necessary conditions.

Constraint $g(\sigma)$ can be either active or inactive at a given σ^*. When active, it provides the equality constraint $\sigma^* - ½ = 0$ and the feasible candidate solution $\sigma^* = ½$ which meets the KKT necessary conditions.

It remains to prove that this candidate solution is a minimum for problem (P) and not a maximum or a saddle point. To do so entails proving that the Hessian of the Lagrangian, $L_{\sigma\sigma} (\sigma^*, \mu^*; t^*)$, is positive definite at $\sigma^* = 1/2$ and $\mu^* = 0$ for all directions $u \neq 0$ that are defined by $ug_\sigma(\sigma^*) \geq 0$ [4] , that is for all $u > 0$, since $g_\sigma(\sigma) = 1$ and $u \neq 0$. Hence, the sufficient optimality condition for problem (P) requires that

[6] *Properties of RZF and those of its nontrivial zeros are listed* in *the appendix*

[7] As per property 6 and per Saidak, F, On the Modulus of the Riemann Zeta function*, Mathematica Slovaca, 53 (2003) Vol. 2, pp. 147-148*

for any $u > 0$ we have $u^T L^*_{\sigma\sigma}\, u > 0$. In our case, the u directions are univariate since there is only one variable in problem (P), namely σ. The sufficient condition is then: $u^2 L^*_{\sigma\sigma} > 0$. Since we have:

$L(\sigma, \mu; t^*) = U^2(\sigma; t^*) + V^2(\sigma; t^*) + \mu(\sigma - ½)$; $L_\sigma = 2(UU_\sigma + VV_\sigma) + \mu$, then:

$L_{\sigma\sigma} = 2(U_\sigma^2 + UU_{\sigma\sigma} + V_\sigma^2 + VV_{\sigma\sigma})$.

Based on (7b): $U^*U^*_{\sigma\sigma} + V^*V^*_{\sigma\sigma}, = 0$, the Hessian reduces to $L^*_{\sigma\sigma}(\sigma^*, \mu^*) = 2u^2 (U^{*2}_\sigma + V^{*2}_\sigma)$, thus strictly positive for any direction $u > 0$ as required. This proves that the sufficient condition for optimality is also satisfied for problem (P) at $\sigma^*=1/2$. Hence $\sigma^* = ½$ is a minimum point for (P). This solution being on the critical line rules out the case where $g(\sigma)$ is inactive, i.e. $\sigma^* - ½) < 0$, as per property (9). Hence, it is not possible to have any $\sigma < ½$ as a minimizer of problem (P). Therefore, the only minimizer for (P) is $\sigma^*=1/2$. The symmetric of this zero about the critical strip is also at $\sigma_s^*=1/2$.

Conclusion

The above analysis shows that if RZF were to have a nontrivial zero at any $t=t^*$, then it is necessary and sufficient that $\sigma^* = ½$. Hence, at $t=t^*$, there is a single nontrivial zero located on the critical line. This result proves that all RZF's nontrivial zeros are located on the critical line as stated by the Riemann Hypothesis which is therefore analytically proven true by our optimization approach.

Appendix: Some relevant properties of RZF and its nontrivial zeros

The most important and relevant properties of RZF [5] are listed below:

1. Since RZF is analytic in the complex plane except for a pole at $\sigma=1$, its real and its imaginary parts, $U(s)$ and $V(s)$ respectively, are infinitely differentiable in the critical strip, except for $\sigma=1$, hence:

   $U^*U^*_\sigma + V^*V^*_\sigma = 0$ (1a); and $U^*U^*_{\sigma\sigma} + V^*V^*_{\sigma\sigma} = 0$ (1b)
2. RZF has an infinite number of nontrivial zeros
3. A huge number of nontrivial zeros proved to be located on the critical line
4. Nontrivial zeros are located in the critical strip at different heights $t = t^*$
5. Nontrivial zeros are symmetric about the real line $t = 0$, and about the critical line
6. As per (5), if σ^* is a location of a nontrivial zero at $t = t^*$, then $(1- \sigma^*)$ is also a location of a nontrivial zero at $t = t^*$. Hence noncritical zeros are defined by $\sigma = ½ +/- \alpha$, and occur either in pairs off the critical line for $0 < \alpha < 0$, or in singles on the critical line (for $\alpha=0$)
7. RZF has no zeros on the line $\sigma = 1$. Thus, by symmetry about the critical line, RZF has no zero on the line $\sigma = 0$, hence for nontrivial zeros: $\sigma > 0$
8. $U_\sigma (\sigma = ½) \neq 0$ and $V_\sigma (\sigma = ½) \neq 0$
9. Property (3) and (6) limit the search for nontrivial zeros to the left half of the critical strip. This leads to the constraint: $\sigma \leq ½$

References

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