Schwarschild Metric From Kepler's Law

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Abstract

The simplest non-trivial configuration of spacetime in which gravity plays a role is for the region surrounding a static mass point, for which we can assume that the metric has perfect spherical symmetry and is independent of time. Historically this was first found by Karl Schwarzschild in 1916 as a solution of Einstein's field equations , and all the original empirical tests of general relativity can be inferred from this solution. However, even without knowing the field equations of general relativity, it is possible to give a very plausible (if not entirely rigorous) derivation of the Schwarzschild metric purely from knowledge of the inverse square characteristic of gravity, Kepler's third law for circular orbits, and the null intervals of light paths.

Mathematical Work:

Let r denote the radial spatial coordinate, so that every point on a surface of constant r has the same intrinsic geometry and the same relation to the mass point, which we fix at r = 0. Also, let t denote our temporal coordinate. Any surface of constant r and t must possess the two-dimensional intrinsic geometry of a 2-sphere, and we can scale the radial parameter r such that the area of this surface is 4π r2. (Notice that since the space may not be Euclidean, we don't claim that r is "the radial distance" from the mass point. Rather, at this stage r is simply an arbitrary radial coordinate scaled to give the familiar Euclidean surface area.) With this scaling, we can parameterize the two-dimensional surface at any given r (and t) by means of the ordinary "longitude and latitude" spherical metric

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

where ds is the incremental distance on the surface of an ordinary sphere of radius r corresponding to the incremental coordinate displacements dθ and dϕ. The coordinate θ represents "latitude", with θ = 0 at the north pole and θ = $\pi/2$ at the equator. The coordinate ϕ represents the longitude relative to some arbitrary meridian.

It follows that the complete spacetime metric near a spherically symmetrical mass m can be written in the diagonal form

$$d\tau^2 = g_{tt}\ dt^2\ + g_{rr}\, dr^2 + g_{\theta\theta}\ d\theta^2 + g_{\phi\phi}\, d\phi^2$$

where gθθ = -r2, gϕϕ = -r2 sin(θ)2, and gtt and grr are (as yet) unknown functions of r and the central mass m. Of course, with m = 0 the functions gtt and -grr must both equal 1 in order to give the flat Minkowski metric (in polar form), and we also expect that as r increases to infinity these functions both approach 1, regardless of m, since we expect the metric to approach flatness sufficiently far from the gravitating mass.

This metric is diagonal, so the non-zero components of the contravariant metric tensor are gαα = 1/gαα. In addition, the diagonality of the metric allows us to simplify the definition of the Christoffel symbols to

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2\ g_{\alpha\alpha}}\left[\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right] \qquad \text{(no implied Summation)}$$

Now, the only non-zero partial derivatives of the metric coefficients are

$$\frac{\partial g_{\theta\theta}}{\partial r} = -2r$$

$$\frac{\partial g_{\phi\phi}}{\partial r} = -2r\sin^2\theta$$

$$\frac{\partial g_{\phi\phi}}{\partial \theta} = -2r\sin\theta\sin\theta$$

with ∂gtt/dr and ∂grr/dr, which are yet to be determined. Inserting these values into the preceding equation,

we find that the only non-zero Christoffel symbols are

$$\Gamma^{t}_{tr}=\Gamma^{t}_{rt}=\frac{1}{2g_{tt}}\frac{\partial g_{tt}}{\partial r}$$

$$\Gamma^{r}_{tt}=\frac{-1}{2g_{rr}}\frac{\partial g_{tt}}{\partial r}$$

$$\Gamma^{r}_{rr}=\frac{1}{2g_{rt}}\frac{\partial g_{rr}}{\partial r}$$

$$\Gamma^{r}_{\theta\theta}=\frac{r}{g_{rr}}$$

$$\Gamma^{\theta}_{\phi\phi}=-\sin\theta\cos\theta$$

$$\Gamma^{r}_{\phi\phi}=\frac{r\sin^{2}\theta}{g_{rr}}$$

$$\Gamma^{\phi}_{\phi r}=\Gamma^{\phi}_{r\phi}=\frac{1}{r}$$

$$\Gamma^{\theta}_{r\theta}=\Gamma^{\theta}_{\theta r}=\frac{1}{r}$$

$$\Gamma^{\phi}_{\phi\theta}=\Gamma^{\phi}_{\theta\phi}=\frac{1}{\tan\theta}$$

these are the coefficients of the four geodesic equations near a spherically symmetrical mass, i.e., the equations of paths for which the integrated path length is unchanged by incremental variations of the path. Writing them out in full, we have

$$\frac{d^2}{d\tau^2}t=\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2$$

$$\frac{d^2}{d\tau^2}r=\frac{1}{2g_{rr}}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2-\frac{1}{2g_{rr}}\frac{\partial g_{rr}}{\partial r}\left(\frac{dr}{d\tau}\right)^2-\frac{r}{2g_{rr}}\left(\frac{d\theta}{d\tau}\right)^2-\frac{r\sin\theta^{2}}{g_{rr}}\left(\frac{d\phi}{d\tau}\right)^2$$

$$\frac{d^2}{d\tau^2}\theta=\frac{-2}{r}\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right)+\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2$$

$$\frac{d^2}{d\tau^2}\phi=\frac{-2}{r}\left(\frac{dr}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right)-\frac{2}{\tan\theta}\left(\frac{d\theta}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right)$$

In the absence of non-gravitational forces, we postulate that any test particle follows a geodesic path, so these equations characterize inertial/gravitational motions of test particles in a spherically symmetrical field. All that remains is to determine the metric coefficients gtt and grr.

We expect that one possible solution should be circular Keplerian orbits, i.e., if we regard r as corresponding (at least approximately) to the Newtonian radial distance from the center of the mass, then there should be a circular geodesic path at constant r that revolves around the central mass m with an angular velocity of ω, and these quantities must be related (at least approximately) in accord with Kepler's third law

$$m = r^3\omega^2$$

(The original deductions of an inverse-square law of gravitation by Hooke, Wren, Newton, and others were all based on this same empirical law). If we consider purely circular motion on the equatorial plane (θ = π/2) at constant r, the metric reduces to

$$d\tau^2 = g_{tt} dt^2 - r^2 d\phi^2$$

and since dr/dτ = 0 the geodesic equations for these circular paths reduce to

$$\frac{d^2}{d\tau^2} t = \frac{d^2}{d\tau^2} \theta = \frac{d^2}{d\tau^2} \phi = 0$$

$$\frac{\partial g_{tt}}{\partial r} \left(\frac{dt}{d\tau} \right)^2 = 2r \left(\frac{d\phi}{d\tau} \right)^2$$

Multiplying through by (dτ/dt)2 and identifying the angular speed ω with the derivative of ϕ with respect to the coordinate time t, the right hand equation becomes

$$\frac{\partial g_{tt}}{\partial r} = 2r\left(\frac{d\phi}{dt}\right)^2 = 2r\omega^2$$

For consistency with Kepler's Third Law we must have ω2 equal (or very nearly equal) to m/r3, so we make this substitution to give

$$\frac{\partial g_{tt}}{\partial r} = \left(\frac{2m}{r^2}\right)$$

Integrating this equation, we find that the metric coefficient gtt must be of the form k - (2m/r) where k is a constant of integration. Since gtt must equal 1 when m = 0 and/or as r approaches infinity, it's clear that k = 1, so we have

$$g_{tt} = \left(1 - \frac{2m}{r}\right)$$

Also, for a photon moving away from the gravitating mass in the purely radial direction, we have $d\tau = 0$, and so our basic metric for a purely radial ray of light gives

$$g_{tt} dt^2 = -g_{rr} dr^2$$

Next we consider a stationary test particle at a radial coordinate r. The metric equation gives the line element for the worldline of this test particle

$$d\tau^2 = g_{tt} dt^2$$

and we also have the radial geodesic equation for this particle

$$\frac{d^2}{d\tau^2} r = \frac{1}{2g_{rr}} \frac{\partial g_{tt}}{\partial r} \left(\frac{dt}{d\tau} \right)^2$$

The left hand side is the acceleration of gravity d2r/dτ2 in geometrical units, which is taken to be the inverse square expression –m/r2. Inserting this expression and substituting from above equations, we get

$$g_{tt}g_{rr}=-1$$

This implies grr = -1/gtt (corresponding to the "perpendicular" factorization gtt = dr/dt and grr = -dt/dr in equation (3)), so we have the complete Schwarzschild metric

$$d\tau^2=\left(1-\frac{2m}{r}\right)dt^2-\left(\frac{1}{1-\frac{2m}{r}}\right)dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2$$

In matrix form the Schwarzschild metric is written as

$$\begin{bmatrix} \frac{r-2m}{r} & 0 & 0 & 0 \\ 0 & \frac{-r}{r-2m} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}$$

Note : In this short derivation I derived Schwarschild Metric By using Kepler's Law without any assumption of General Relativity (Filled with Complexity) . I know these things can be done easily but this is my own derivation so don't bother .

References:

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Kip Thorne Charles Misner (2005) *Gravitation*, 1st edn., America: