Union of two arithmetic sequences. Formulas for rational and real progressions. (3)

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Abstract. We will derive the formulas for the N-th element of the union of two arithmetic progressions with rational and real common differences.

1 Purpose

We will be based on *Definition 1.1* in [1]. In this paper we will deal the union of two arithmetic sequences $S{=}(s_i)$, $T{=}(t_j)$. We will consider only such S, T, that meet the following conditions:
Conditions 1.1
1) S, T are composed of non-negative real numbers: $\forall_{i,j\in\mathbb{N}}\ s_i, t_j{\in}\mathbb{R},\ s_i, t_j{\geq}0$,
2) initial terms are equal to 0: $s_0{=}0,\ t_0{=}0$,
3) common differences $s, t{\in}\mathbb{R}^+$

Using condition 2): $s_i{=}is,\ t_j{=}jt$.

The goal is to derive the formula for the n-th union $U{=}(u_n)$ element in the form: $u_n{=}f(N, s, t)$ in accordance with the definition and conditions.

2 Notation and preliminary findings

We will keep the entire notation from the last version of the previous papers [1] and [2]. The general principles regarding the construction of the union and the derivation of formulas remain in force.
Other symbols:
$\mathbb{N}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$ are positive parts of domains.
Unions arguments (common differences) will be as follows: $a, b{\in}\mathbb{N}^+$, $s, t{\in}\mathbb{R}^+$(or $\mathbb{N}^+$, $\mathbb{Q}^+$, if so marked).

3 Formula with commensurable common differences

Definition 3.1 Commensurability

Let $s, t{\in}\mathbb{R}^+$, $a, b{\in}\mathbb{N}^+$. If exist such a, b that $\dfrac{t}{s}{=}\dfrac{b}{a}$ then s, t are commensurable numbers. Otherwise s, t are incommensurable.

For any formula with commensurable common differences s, t:

$$u_N(s,t)=su_N\left(1,\frac{t}{s}\right)=su_N\left(1,\frac{b}{a}\right)$$

We see that formula does not depend on the type s, t, but on their commensurability only. Continuing:

$$u_N(s,t)=\frac{s}{a}u_N(a,b) \tag{1}$$

where $u_N(a,b)$ is formula [2](5) with natural common differences a, b. This is the simplest version of the formula. Complete notation:

$$u_N(s,t)=\frac{s}{a}\max\left(a\left\lceil\frac{b}{a+b}\left(\left\lfloor\frac{N(a+b)}{a+b-\Theta}\right\rfloor-\frac{a}{b}\right)\right\rceil, b\left\lfloor\frac{a}{a+b}\left\lfloor\frac{N(a+b)}{a+b-\Theta}+1\right\rfloor\right\rfloor\right) \tag{2}$$

Special cases:

1. If $s,t\in\mathbb{N}^+$ then $a=s, b=t$ then $u_N(s,t)=u_N(a,b)$.
2. If $s,t\in\mathbb{Q}^+$, $p_s,q_s,p_t,q_t\in\mathbb{N}^+$ and $s=\frac{p_s}{q_s}$, $t=\frac{p_t}{q_t}$:

$$u_N(s,t)=u_N\left(\frac{p_s}{q_s},\frac{p_t}{q_t}\right)=u_N\left(\frac{p_sq_t}{q_sq_t},\frac{p_tq_s}{q_sq_t}\right)=\frac{1}{q_sq_t}u_N\left(p_sq_t,p_tq_s\right)$$

With $a=p_sq_t$, $b=p_tq_s$, $d=q_sq_t$ we get:

$$u_N(s,t)=\frac{1}{d}u_N\left(a,b\right) \tag{3}$$

Example for $s=\frac{8}{11}$, $t=\frac{12}{14}$:
First we must calculate a, b, d: $a=8\cdot 14=112$, $b=12\cdot 11=132$, $d=11\cdot 14=154$
$u_N\left(\frac{8}{11},\frac{12}{14}\right)=\frac{1}{154}u_N\left(112,132\right)$
This can be simplified to $\frac{1}{77}u_N\left(56,66\right)$, although it is not necessary, the result will be the same.

3. If $s,t\in\mathbb{R}^+$ are irrational and commensurable then, after calculating a, b we will use (1).

Example for $s=\frac{8}{11}\pi$, $t=\frac{12}{14}\pi$:

$$\frac{t}{s}=\frac{\frac{12}{14}\pi}{\frac{8}{11}\pi}=\frac{12\cdot 11}{8\cdot 14}=\frac{132}{112}\ \rightarrow\ b=132,\ a=112$$

$$u_N\left(\frac{8}{11}\pi,\frac{12}{14}\pi\right)=\frac{\frac{8}{11}\pi}{112}u_N\left(112,132\right)=\frac{\pi}{154}u_N\left(112,132\right)$$

4 Formula with uncommensurable common differences

Union with uncommensurable common differences with beginnig in the zero has only one common term of union progressions (equal to zero). This means that the whole union is one group G_0. So we'll use relative formulas from [1].

Substituting r in [1](5) with [1](6) and natural a, b with uncommensurable s, t we get:

$$u_n(s,t)=\begin{cases} t\left\lfloor \dfrac{(n+1)s}{s+t} \right\rfloor \\ s\left(n-\left\lfloor \dfrac{(n+1)s}{s+t} \right\rfloor\right) \end{cases} = \begin{cases} t\left\lfloor \dfrac{(n+1)s}{s+t} \right\rfloor & \text{for: } c=0 \\ s\left\lceil \dfrac{nt-s}{s+t} \right\rceil & \text{for: } c>0 \end{cases}$$

Now, we use [2](5):

$$u_n(s,t)=\max\left(s\left\lceil \frac{nt-s}{s+t} \right\rceil, t\left\lfloor \frac{(n+1)s}{s+t} \right\rfloor\right) \tag{4}$$

This is the simplest version of the formula.

5 Unified version of the formula with any real common differences

5.1 Generalized greatest common divisor

From generalized definition of *gcd* for commensurable numbers [3] using Thomae's function $\mathcal{T}$ [4] we get:

$$\gcd(s,t)=s\mathcal{T}\left(\frac{t}{s}\right)=\begin{cases} s\mathcal{T}\left(\dfrac{b}{a}\right) & \text{for commensurable } s,t \\ 0 & \text{for uncommensurable } s,t \end{cases}$$

Upper formula:

$$s\mathcal{T}\left(\frac{b}{a}\right)=s\mathcal{T}\left(\frac{\dfrac{b}{\gcd(a,b)}}{\dfrac{a}{\gcd(a,b)}}\right)=\frac{s}{\dfrac{a}{\gcd(a,b)}}=\frac{s}{a}\gcd(a,b)$$

finally:

$$\gcd(s,t)=\begin{cases} \dfrac{s}{a}\gcd(a,b) & \text{for commensurable } s,t \\ 0 & \text{for uncommensurable } s,t \end{cases} \tag{5}$$

5.2 Generalized symbol Θ

Formula [2](5) contains the greatest common divisor Θ. In [1] we set natural Θ for natural union. According to generalized greatest common divisor we generalize Θ definition:

$$\Theta=\gcd(s,t)=\begin{cases} \dfrac{s}{a}\gcd(a,b) & \text{for commensurable } s,t \\ 0 & \text{for uncommensurable } s,t \end{cases} \tag{6}$$

Conclusions:

1. If $s,t\in\mathbb{N}^+$ then $a=s, b=t$ then $\Theta=\gcd(a,b)$.
2. If $s,t\in\mathbb{Q}^+$, $p_s,q_s,p_t,q_t\in\mathbb{N}^+$ and $s=\dfrac{p_s}{q_s}$, $t=\dfrac{p_t}{q_t}$:

$$\gcd(s,t)=\gcd\left(\frac{p_s}{q_s},\frac{p_t}{q_t}\right)=\gcd\left(\frac{p_sq_t}{q_sq_t},\frac{p_tq_s}{q_sq_t}\right)=\frac{1}{q_sq_t}\gcd\left(p_sq_t,p_tq_s\right)$$

With $a{=}p_s q_t$, $b{=}p_t q_s$, $d{=}q_s q_t$ we get:

$$\Theta{=}\frac{1}{d}\operatorname{gcd}(a,b)$$

3. If $s,t{\in}\mathbb{R}^+$ are irrational and commensurable then $\Theta{=}\frac{s}{a}\operatorname{gcd}(a,b)$
4. If $s,t{\in}\mathbb{R}^+$ and at least one of s,t is irrational and s,t are incommensurable then $\Theta{=}0$.

5.3 Formula with commensurable common differences

We start with (1) substituting $u(a,b)$ with [2](5):

$$\begin{aligned}u_N(s,t)&{=}\frac{s}{a}\max\left(a\left\lceil\frac{b}{a{+}b}\left(\left\lfloor\frac{N(a{+}b)}{a{+}b{-}\Theta}\right\rfloor-\frac{a}{b}\right)\right\rceil,b\left\lfloor\frac{a}{a{+}b}\left\lfloor\frac{N(a{+}b)}{a{+}b{-}\Theta}{+}1\right\rfloor\right\rfloor\right)\\&=\max\left(\frac{s}{a}a\left\lceil\frac{\frac{s}{a}b}{\frac{s}{a}a{+}\frac{s}{a}b}\left(\left\lfloor\frac{N(\frac{s}{a}a{+}\frac{s}{a}b)}{\frac{s}{a}a{+}\frac{s}{a}b{-}\frac{s}{a}\operatorname{gcd}(a,b)}\right\rfloor-\frac{\frac{s}{a}a}{\frac{s}{a}b}\right)\right\rceil,\frac{s}{a}b\left\lfloor\frac{\frac{s}{a}a}{\frac{s}{a}a{+}\frac{s}{a}b}\left\lfloor\frac{N(\frac{s}{a}a{+}\frac{s}{a}b)}{\frac{s}{a}a{+}\frac{s}{a}b{-}\frac{s}{a}\operatorname{gcd}(a,b)}{+}1\right\rfloor\right\rfloor\right)\end{aligned}$$

Substituting (5) we get:

$$u_N(s,t){=}\max\left(s\left\lceil\frac{t}{s{+}t}\left(\left\lfloor\frac{N(s{+}t)}{s{+}t{-}\operatorname{gcd}(s,t)}\right\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s{+}t}\left\lfloor\frac{N(s{+}t)}{s{+}t{-}\operatorname{gcd}(s,t)}{+}1\right\rfloor\right\rfloor\right)$$

and with generalized Θ:

$$u_N(s,t){=}\max\left(s\left\lceil\frac{t}{s{+}t}\left(\left\lfloor\frac{N(s{+}t)}{s{+}t{-}\Theta}\right\rfloor-\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s{+}t}\left\lfloor\frac{N(s{+}t)}{s{+}t{-}\Theta}{+}1\right\rfloor\right\rfloor\right) \tag{7}$$

5.4 Formula with uncommensurable common differences

We transform (4):

$$\begin{aligned}u_n(s,t)&{=}\max\left(s\left\lceil\frac{nt{-}s}{s{+}t}\right\rceil,t\left\lfloor\frac{(n{+}1)s}{s{+}t}\right\rfloor\right)\\&=\max\left(s\left\lceil\frac{t}{s{+}t}\left(n{-}\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s{+}t}(n{+}1)\right\rfloor\right)\\&=\max\left(s\left\lceil\frac{t}{s{+}t}\left(\lfloor n\rfloor{-}\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s{+}t}(\lfloor n\rfloor{+}1)\right\rfloor\right)\\&=\max\left(s\left\lceil\frac{t}{s{+}t}\left(\left\lfloor\frac{n(s{+}t)}{s{+}t}\right\rfloor{-}\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s{+}t}\left(\left\lfloor\frac{n(s{+}t)}{s{+}t}\right\rfloor{+}1\right)\right\rfloor\right)\end{aligned}$$

hence with $\Theta{=}0$ and $n{=}N$ we get:

$$u_N(s,t){=}\max\left(s\left\lceil\frac{t}{s{+}t}\left(\left\lfloor\frac{N(s{+}t)}{s{+}t{-}\Theta}\right\rfloor{-}\frac{s}{t}\right)\right\rceil,t\left\lfloor\frac{s}{s{+}t}\left(\left\lfloor\frac{N(s{+}t)}{s{+}t{-}\Theta}\right\rfloor{+}1\right)\right\rfloor\right) \tag{8}$$

5.5 Unifed formula

Formulas (7), (8) are identical and both cover the entire $\mathbb{R}^+$ domain, hence they are unified formula. Formula [2](5) is a special case with natural s, t noted as a, b.

It is worth noting that unifed (8) is more difficult to calculate because it usually contains many fractions with real nominators and denominators, while (2) and (4) have natural numbers in these places.

For all formulas (2), (4) and (8) with at least one non-natural s or t, it is necessary to determine their commensurability and for the commensurable s, t - calculation a, b.

References

[1] ZIELIŃSKI, W.: *Union of two arithmetic sequences - Basic calculation formula (1)*, viXra:1712.0636.

[2] ZIELIŃSKI, W.: *Union of two arithmetic sequences - Basic calculation formula (2)*, viXra:1807.0477.

[3] https://en.wikipedia.org/wiki/Greatest_common_divisor#Other_methods

[4] https://en.wikipedia.org/wiki/Thomae's_function