SEMISTABLE HOLOMORPHIC BUNDLES OVER COMPACT BI-HERMITIAN MANIFOLDS

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Abstract. In this paper, by using Uhlenbeck-Yau's continuity method, we prove that the existence of approximation α-Hermitian-Einstein strusture and the α-semi-stability on $I_\pm$-holomorphic bundles over compact bi-Hermitian manifolds are equivalent.

1. Introduction

A bi-Hermitian structure on a $2n$-dimensional manifold M consists of a triple (g, I_+, I_-), where g is a Riemannian metric on M and $I_\pm$ are integrable complex structures on M that are both orthogonal with respect to g. Let (M, g, I_+, I_-) be a bi-Hermitian manifold. Let E be a holomorphic vector bundle on M endowed with two holomorphic structures $\bar\partial_+$ and $\bar\partial_-$ with respect to the complex structures I_+ and I_-, respectively. Suppose H is a Hermitian metric on E. Let $F^H_\pm$ be the curvatures of the Chern connections $\nabla^H_\pm$ on E associated to the Hermitian metric H and the holomorphic structures $\bar\partial_\pm$. Motivated by Hitchin [16], Hu *et al.* [18] introduced the following α-Hermitian-Einstein equation, where $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$:

$$\sqrt{-1}(\alpha F^H_+ \wedge \omega_+^{n-1} + (1-\alpha) F^H_- \wedge \omega_-^{n-1}) = (n-1)!\lambda \cdot \mathrm{Id}_E \cdot \mathrm{dvol}_g, \tag{1.1}$$

where $\omega_\pm(\cdot,\cdot) = g(I_\pm \cdot, \cdot)$ are the fundamental 2-forms of g. Once $I_+ = I_-$, (1.1) reduces to the Hermitian-Einstein equation. A Hermitian metric H on E is called α-Hermitian-Einstein if it satisfies (1.1).

Recently, the existence of Hermitian-Einstein metrics on holomorphic vector bundles has attracted a lot of attention. The celebrated Donaldson-Uhlernbeck-Yau theorem states that holomorphic vector bundles over compact Kähler manifolds admit Hermitian-Einstein metrics if they are polystable. It was proved by Narasimhan and Seshadri [32] for compact Riemann surface, by Donaldson [10] for algebraic manifolds and by Uhlenbeck and Yau [40] for general compact Kähler manifolds. The inverse problem is that a holomorphic bundle admitting such a metric must be polystable (that is a direct sum of stable bundles with the same slope). And the problem was solved by Kobayashi [21] and Lübke [28] independently. This is the so-called Hitchin-Kobayashi correspondence for holomorphic vector bundles over compact Kähler manifolds. There are many interesting generalized Hitchin-Kobayashi correspondences (see the References [1, 2, 3, 4, 6, 15, 17, 18, 20, 23, 24, 25, 26, 31, 33, 42], etc.).

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An $I_\pm$-holomorphic bundle $(E, \bar\partial_+, \bar\partial_-)$ over a compact bi-Hermitian manifold (M, g, I_+, I_-) is said to be admitting an approximate α-Hermitian-Einstein structure, if for every $\varepsilon > 0$, there exists a Hermitian metric H_ε on E such that

$$\max_M |\sqrt{-1}(\alpha F_+^{H_\varepsilon} \wedge \omega_+^{n-1} + (1-\alpha) F_-^{H_\varepsilon} \wedge \omega_-^{n-1}) - (n-1)!\lambda \cdot \mathrm{Id}_E \cdot \mathrm{dvol}_g|_{H_\varepsilon} < \varepsilon. \tag{1.2}$$

Kobayashi [22] introduced this notion for holomorphic vector bundles (that is, $I_+ = I_-$). He proved that over a compact Kähler manifold, a holomorphic vector bundle admitting such a structure must be semi-stable. Bruzzo and Graña Otero [5] generalized the above result to Higgs bundles. When X is projective, Kobayashi [22] solved the inverse part that a semi-stable holomorphic vector bundle must admit an approximate Hermitian-Einstein structure and conjectured that this should be true for general Kähler manifolds. This was confirmed in [9, 19, 24]. Later, Nie and Zhang [33] proved that the existence of approximation Hermitian-Einstein strusture and the semi-stability on Higgs bundles over compact Gauduchon manifolds are equivalent. Just very recently, in [42] Zhang *et al.* showed this is also true for a class of non-compact Gauduchon manifolds.

In this paper, we are interested in the existence of approximate α-Hermitian-Einstein structures on $I_\pm$-holomorphic bundles over compact bi-Hermitian manifolds. In fact, we prove that:

Theorem 1.1. *Let (M, g, I_+, I_-) be a compact bi-Hermitian manifold such that g is Gauduchon with respect to both I_+ and I_-, and $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$. Suppose $(E, \bar\partial_+, \bar\partial_-)$ is an $I_\pm$-holomorphic bundle on M. Then $(E, \bar\partial_+, \bar\partial_-)$ is α-semi-stable if and only if it admits an approximate α-Hermitian-Einstein structure.*

Remark 1.2. Hu *et al.* [18] introduced the $\alpha-$stability on $I_\pm$-holomorphic vector bundles and proved that the $I_\pm$-holomorphic vector bundles admit α-Hermitian-Einstein metrics iff they are α-polystable. We will use Uhlenbeck-Yau's continuity method [40, 29] to prove Theorem 1.1. We can not use the techniques in [18] directly, since the stability condition is not strictly inequality. To fix this, we will adapt Li-Zhang's arguments [24] and Nie-Zhang's arguments [33] to our settings.

Our motivation for studying such bundles also comes from generalized complex geometry. In [13], Gualtieri introduced generalized holomorphic bundles, which are analogues of holomorphic vector bundles on complex manifolds. For instance, on a complex manifold M, a generalized holomorphic bundle corresponds to a co-Higgs bundle, which is a holomorphic vector bundle E on M together with a holomorphic map $\phi : E \to E \otimes T_M$ for which $\phi \wedge \phi = 0$. Some of the general properties of co-Higgs bundles were studied by Hitchin in [16] and moduli spaces of stable co-Higgs bundles were studied in [34, 35, 36, 41], etc. Given the relationship between the generalized complex geometry and the bi-Hermitian geometry, one can study generalized holomorphic bundles in terms of $I_\pm$-holomorphic bundles. Recall that any $\mathbb{J}$-holomorphic bundle over generalized Kähler manifold $(M, \mathbb{J}, \mathbb{J}')$ induces an $I_\pm$-holomorphic bundle on (M, g, I_+, I_-) (see [18, Proposition 2.11]). We will not list the definitions on generalized complex geometry (see [13, 18] for more details). Therefore, combining Theorem 1.1, we have the following result.

Corollary 1.3. Let $(M, \mathbb{J}, \mathbb{J}')$ be a compact generalized Kähler manifold with non-empty boundary ∂M whose associated bi-Hermitian structure (g, I_+, I_-) is such that g is Gauduchon with respect to both I_+ and I_-, and $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$. Moreover, suppose $(E, \bar\partial_+, \bar\partial_-)$ is a $\mathbb{J}$-holomorphic bundle on M. Then $(E, \bar\partial_+, \bar\partial_-)$ is α-semistable if and only if it admits an approximate α-Hermitian-Einstein structure.

Remark 1.4. If M is real $4k$-dimensional and the generalized Kähler structure $(\mathbb{J}, \mathbb{J}')$ is even, then its associated bi-Hermitian structure (g, I_+, I_-) is such that $\mathrm{dvol}_g = \frac{\omega_\pm^n}{n!}$ (see Remark 6.14 in [12]). In this case, one can rewrite (1.2) as

$$\max_M |\alpha\sqrt{-1}\Lambda_+ F_+^{H_\varepsilon} + (1-\alpha)\sqrt{-1}\Lambda_- F_-^{H_\varepsilon} - \lambda \cdot \mathrm{Id}_E|_{H_\varepsilon} < \varepsilon,$$

where $\Lambda_\pm$ are the contraction operators associated to $\omega_\pm$, respectively.

2. Preliminary

Suppose $(E, \bar\partial_+, \bar\partial_-)$ is an $I_\pm$-holomorphic bundle on a bi-Hermitian manifold (M, g, I_+, I_-). Let us fix the $I_\pm$-holomorphic structures $\bar\partial_\pm$ and a Hermitian metric H_0 on $(E, \bar\partial_+, \bar\partial_-)$. For any positive-definite Hermitian endomorphism $h \in \mathrm{Herm}^+(E, H_0)$, let $H := H_0 h$ be the Hermitian metric defined by

$$\langle s, t\rangle_H := \langle hs, t\rangle_{H_0},$$

for $s, t \in C^\infty(E)$. Let $\nabla_\pm^H = \bar\partial_\pm + \partial_\pm^H$ be the corresponding Chern connections. The relation between $\partial_\pm^H$ and $\partial_\pm^{H_0}$ is given by

$$\partial_\pm^H = \partial_\pm^{H_0} + h^{-1}\partial_\pm^{H_0} h. \tag{2.1}$$

Then the curvatures with respect to $\nabla_\pm^H$ and $\nabla_\pm^{H_0}$ satisfy

$$F_\pm^H = F_\pm^{H_0} + \bar\partial_\pm(h^{-1}\partial_\pm^{H_0} h). \tag{2.2}$$

We assumed that the Riemannian metric g to be Gauduchon with respect to both I_+ and I_-, i.e. $dd_\pm^c \omega_\pm^{n-1} = 0$, where $d_\pm^c = I_\pm \circ d \circ I_\pm$ are the twisted differentials with respect to $I_\pm$. Then we can associate to E two degrees $\deg_\pm(E)$ and two slopes $\mu_\pm(E)$ in the standard way [29, Definition 1.4.1]:

$$\deg_\pm(E) = \frac{\sqrt{-1}}{2\pi}\int_M \mathrm{tr}(F_\pm^H) \wedge \frac{\omega_\pm^{n-1}}{(n-1)!}$$

and

$$\mu_\pm(E) = \frac{\deg_\pm(E)}{\mathrm{rank}(E)}.$$

Note that $\deg_\pm(E)$ are independent of the choice of H on E because the curvatures of Chern connections corresponding to different Hermitian metrics on E differ by $\partial_\pm\bar\partial_\pm$-exact forms. Given these degrees and slopes, we now define the α-degree $\deg_\alpha(E)$ and α-slope $\mu_\alpha(E)$ as [18, Definition 3.3]:

$$\deg_\alpha(E) = \alpha \deg_+(E) + (1-\alpha)\deg_-(E)$$

and

$$\mu_\alpha(E) = \alpha\mu_+(E) + (1-\alpha)\mu_-(E),$$

respectively.

Furthermore, we define coherent subsheaves of $(E, \bar\partial_+, \bar\partial_-)$ as follows:

Definition 2.1. [18, Definition 3.4] Let $\mathcal{F}_\pm$ be coherent subsheaves of $(E, \bar\partial_\pm)$, respectively. The pair $\mathcal{F} := (\mathcal{F}_+, \mathcal{F}_-)$ is said to be a coherent subsheaf of $(E, \bar\partial_+, \bar\partial_-)$ if there exist analytic subsets S_+ and S_- of (M, I_+) and (M, I_-), respectively, such that

(1)$S := S_+ \cup S_-$ has codimension at least 2;

(2)$\mathcal{F}_{\pm}|_{M\setminus S_{\pm}}$ are locally free and $\mathcal{F}_{+}|_{M\setminus S} = F_{-}|_{M\setminus S}$.

The α-slope of $\mathcal{F}$ is given by

$$\mu_\alpha(\mathcal{F}) := \alpha\frac{\deg_+(\mathcal{F}_+)}{\mathrm{rank}(\mathcal{F})} + (1-\alpha)\frac{\deg_-(\mathcal{F}_-)}{\mathrm{rank}(\mathcal{F})}.$$

Let us now recall the α-stability for $(E, \bar\partial_+, \bar\partial_-)$.

Definition 2.2. [18, Definition 3.5] An $I_\pm$-holomorphic structure $(\bar\partial_+, \bar\partial_-)$ on E is called α-stable (resp., α-semistable), if, for any proper coherent subsheaf $\mathcal{F}$ of $(E, \bar\partial_+, \bar\partial_+)$, we have

$$\mu_\alpha(\mathcal{F}) < \mu_\alpha(E)(\mathrm{resp.}, \mu_\alpha(\mathcal{F}) \le \mu_\alpha(E)).$$

By using Uhlenbeck-Yau's continuity method [40], we will show that the α-semi-stability implies approximation α-Hermitian-Einstein structure.

Set

$$\mathrm{Herm}(E,H) = \{\eta \in \mathrm{End}(E) | \eta^{*H} = \eta\}$$

and

$$\mathrm{Herm}^+(E,H) = \{\rho \in \mathrm{Herm}(E,H) | \rho \text{ is positive definite}\}.$$

Fixing a proper background Hermitian metric H_0 on E, we consider the following perturbed equation

$$L_\varepsilon(h_\varepsilon) := \Phi(H_\varepsilon) + \varepsilon \log h_\varepsilon = 0, \qquad \varepsilon \in (0,1], \tag{2.3}$$

where

$$\Phi(H_\varepsilon) = \alpha\sqrt{-1}\Lambda_+ F_+^{H_\varepsilon} + (1-\alpha)\sqrt{-1}\Lambda_- F_-^{H_\varepsilon} - \lambda \cdot \mathrm{Id}_E$$

and $h_\varepsilon = H_0^{-1}H_\varepsilon \in \mathrm{Herm}^+(E, H_0)$. It is obvious that h_ε and $\log h_\varepsilon$ are self adjoint with respect to H_0 and H_ε. By the results in [18], (2.3) is solvable for all $\varepsilon \in (0,1]$. Using the assumption of α-semi-stability, we can show that

$$\lim_{\varepsilon\to 0} \varepsilon \max_M |\log h_\varepsilon|_{H_0} = 0. \tag{2.4}$$

This implies that $\max\limits_M |\Phi(H_\varepsilon)|_{H_\varepsilon}$ converges to zero as $\varepsilon \to 0$.

By an appropriate conformal change, we can assume that H_0 satisfies

$$\mathrm{tr}(\Phi(H_0)) = 0.$$

In fact, let $H_0 = e^\varphi H_0'$, where H_0' is an arbitrary metric and φ is a smooth function satisfying

$$\Delta_{\bar\partial,\alpha}\varphi = -\frac{1}{\mathrm{rank}(E)}\mathrm{tr}(\Phi(H_0')), \tag{2.5}$$

where

$$\Delta_{\bar\partial,\alpha} := \alpha\Delta_{\bar\partial_+} + (1-\alpha)\Delta_{\bar\partial_-},$$

and

$$\Delta_{\bar\partial_\pm} := \sqrt{-1}\Lambda_\pm \bar\partial_\pm \partial_\pm.$$

Since $\int_M \mathrm{tr}(\Phi(H_0'))\omega^n = 0$, equation (2.5) is solvable by [29, Corollary 1.2.9].

Fix a background Hermitian metric H_0 satisfying $\mathrm{tr}(\Phi(H_0)) = 0$. From (2.3), we have

$$\begin{aligned}0 =&\mathrm{tr}L_\varepsilon(h_\varepsilon)\\ =&\mathrm{tr}\Phi(H_0) + \mathrm{tr}\Big(\alpha\sqrt{-1}\Lambda_+\overline{\partial}_+(h_\varepsilon^{-1}\partial_+^{H_0}h_\varepsilon)\Big)\\ &+\mathrm{tr}\Big((1-\alpha)\sqrt{-1}\Lambda_-\overline{\partial}_-(h_\varepsilon^{-1}\partial_-^{H_0}h_\varepsilon)\Big) + \varepsilon\mathrm{tr}(\log h_\varepsilon)\\ =&\Delta_{\bar\partial,\alpha}(\mathrm{tr}\log h_\varepsilon) + \varepsilon\mathrm{tr}(\log h_\varepsilon).\end{aligned}$$

Using the maximum principle, we have

$$\det h_\varepsilon = 1.$$

The following lemma was proved in [18].

Lemma 2.3. *If $h_\varepsilon \in \mathrm{Herm}^+(E, H_0)$ satisfies $L_\varepsilon(h_\varepsilon) = 0$ for some $\varepsilon > 0$, then it holds that*

(i) $\frac{1}{2}\Delta_{\bar\partial,\alpha}\left(|\log h_\varepsilon|^2_{H_0}\right) + \varepsilon|\log h_\varepsilon|^2_{H_0} \leq |\Phi(H_0)|_{H_0}|\log h_\varepsilon|_{H_0}$;
(ii) $m = \max_M |\log h_\varepsilon|_{H_0} \leq \frac{1}{\varepsilon}\cdot \max_M |\Phi(H_0)|_{H_0}$;
(iii) $m \leq C\cdot(\|\log h_\varepsilon\|_{L^2} + \max_M |\Phi(H_0)|_{H_0})$, *where C only depends on g and H_0.*

3. Proof of Theorem 1.1

Before giving the detailed proof, we first recall some notations. Fixing $\eta \in \mathrm{Herm}(E, H)$, from [29, p. 237], we can choose an open dense subset $W \subseteq X$ satisfying at each $x \in W$ there exist an open neighbourhood U of x, a local unitary basis $\{e_a\}_{a=1}^r$ with respect to H and functions $\{\lambda_a \in C^\infty(U,\mathbb{R})\}_{a=1}^r$ such that

$$\eta(y) = \sum_{a=1}^r \lambda_a(y)\cdot e_a(y)\otimes e^a(y)$$

for all $y \in U$, where $\{e^a\}_{a=1}^r$ denotes the dual basis of E^*. Let $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$, $\Psi \in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $A = \sum_{a,b=1}^r A_b^a e_a\otimes e^b \in \mathrm{End}(E)$, here we also assume $\mathrm{rank}(E) = r$. We denote $\varphi(\eta)$ and $\Psi(\eta)(A)$ by

$$\varphi(\eta)(y) = \sum_{a=1}^r \varphi(\lambda_a)e_a\otimes e^a$$

and

$$\Psi(\eta)(A)(y) = \Psi(\lambda_a,\lambda_b)A_b^a e_a\otimes e^b. \tag{3.1}$$

Proposition 3.1. If $h_\varepsilon \in \mathrm{Herm}^+(E, H_0)$ solves (2.3) for some $\varepsilon > 0$, then it holds

$$\begin{aligned}&\int_M \mathrm{tr}(\Phi(H_0)s_\varepsilon)\frac{\omega_\pm^n}{n!} + \alpha\int_M\langle\Psi(s_\varepsilon)(\bar\partial_+s_\varepsilon),\bar\partial_+s_\varepsilon\rangle_{H_0}\frac{\omega_\pm^n}{n!}\\ &+(1-\alpha)\int_M\langle\Psi(s)(\bar\partial_-s_\varepsilon),\bar\partial_-s_\varepsilon\rangle_{H_0}\frac{\omega_\pm^n}{n!} = -\varepsilon\|s_\varepsilon\|^2_{L^2},\end{aligned}\tag{3.2}$$

where $s_\varepsilon = \log h_\varepsilon$ and

$$\Psi(x,y) = \begin{cases}\frac{e^{y-x}-1}{y-x}, & x\neq y;\\ 1, & x = y.\end{cases}$$

Proof. By simple calculations,

(3.3)

$$\int_M \Big(\mathrm{tr}(\Phi(H_\varepsilon)s_\varepsilon) - \mathrm{tr}(\Phi(H_0)s_\varepsilon)\Big)$$
$$= \int_M \Big(\alpha\langle\sqrt{-1}\Lambda_+\bar{\partial}_+(h_\varepsilon^{-1}\partial_+^{H_0}h_\varepsilon), s_\varepsilon\rangle_{H_0} + (1-\alpha)\langle\sqrt{-1}\Lambda_-\bar{\partial}_-(h_\varepsilon^{-1}\partial_-^{H_0}h_\varepsilon), s_\varepsilon\rangle_{H_0}\Big).$$

According to [33, Proposition 3.1], we have

$$\int_M \langle\sqrt{-1}\Lambda_\pm\bar{\partial}_\pm(h_\varepsilon^{-1}\partial_\pm^{H_0}h_\varepsilon), s_\varepsilon\rangle_{H_0} = \int_M \langle\Psi(s_\varepsilon)(\bar{\partial}_\pm s_\varepsilon), \bar{\partial}_\pm s_\varepsilon\rangle_{H_0}. \tag{3.4}$$

Combining (3.3) and (3.4), we complete the proof. □

We first prove the following.

Theorem 3.2. *If $(E, \bar{\partial}_+, \bar{\partial}_-)$ is α-semi-stable, then it admits an approximate α-Hermitian-Einstein structure.*

Proof. Let $\{h_\varepsilon\}_{0<\varepsilon\leq 1}$ be the solutions of equation (2.3) with the background metric H_0. Then

$$\|\log h_\varepsilon\|_{L^2}^2 = -\frac{1}{\varepsilon}\int_M \langle\Phi(H_\varepsilon), \log h_\varepsilon\rangle_{H_\varepsilon}\frac{\omega_\pm^n}{n!}.$$

Case 1, There exists a constant $C_1 > 0$ such that $\|\log h_\varepsilon\|_{L^2} < C_1 < +\infty$. From Lemma 2.3, we have

$$\max_M |\Phi(H_\varepsilon)|_{H_\varepsilon} = \varepsilon \cdot \max_M |\log h_\varepsilon|_{H_\varepsilon} < \varepsilon C \cdot (C_1 + \max_M |\Phi(H_0)|_{H_0}).$$

Then it follows that $\max_M |\Phi(H_\varepsilon)|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$.

Case 2, $\overline{\lim_{\varepsilon\to 0}}\|\log h_\varepsilon\|_{L^2} \to \infty$.

Claim If $(E, \bar{\partial}_+, \bar{\partial}_-)$ is α-semi-stable, then it holds

$$\lim_{\varepsilon\to 0}\max_M |\Phi(H_\varepsilon)|_{H_\varepsilon} = \lim_{\varepsilon\to 0}\varepsilon\max_M |\log h_\varepsilon|_{H_\varepsilon} = 0. \tag{3.5}$$

We will follow Simpson's argument ([37, Proposition 5.3]) to show that if the claim does not hold, there exists a subsheaf contradicting the α-semi-stability.

If the claim does not hold, then there exist $\delta > 0$ and a subsequence $\varepsilon_i \to 0,\ i \to +\infty$, such that

$$\|\log h_{\varepsilon_i}\|_{L^2} \to +\infty$$

and

$$\max_M |\Phi(H_{\varepsilon_i})|_{H_{\varepsilon_i}} = \varepsilon_i \max_M |\log h_{\varepsilon_i}|_{H_{\varepsilon_i}} \geq \delta. \tag{3.6}$$

Setting $s_{\varepsilon_i} = \log h_{\varepsilon_i}$, $l_i = \|s_{\varepsilon_i}\|_{L^2}$ and $u_{\varepsilon_i} = s_{\varepsilon_i}/l_i$, it follows that $\mathrm{tr}(u_{\varepsilon_i}) = 0$ and $\|u_{\varepsilon_i}\|_{L^2} = 1$. Then combining (3.6) with Lemma 2.3, we have

$$l_i \geq \frac{\delta}{C\varepsilon_i} - \max_M |\Phi(H_0)|_{H_0} \tag{3.7}$$

and

$$\max_M |u_{\varepsilon_i}| \leq \frac{C}{l_i}(l_i + \max_M |\Phi(H_0)|_{H_0}) < C_2 < +\infty. \tag{3.8}$$

Step 1 We will show that $\|u_{\varepsilon_i}\|_{L_1^2}$ are uniformly bounded. Since $\|u_{\varepsilon_i}\|_{L^2} = 1$, we only need to prove $\|\mathrm{d}u_{\varepsilon_i}\|_{L^2}$ are uniformly bounded.

By Proposition 3.1, for each h_{ε_i}, it holds

$$\begin{aligned}&\int_M \operatorname{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!}+\alpha l_i\int_M\langle\Psi(l_iu_{\varepsilon_i})(\bar\partial_+u_{\varepsilon_i}),\bar\partial_+u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!}\\&+(1-\alpha)l_i\int_M\langle\Psi(l_iu_{\varepsilon_i})(\bar\partial_-u_{\varepsilon_i}),\bar\partial_-u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!}=-\varepsilon_il_i\end{aligned}\tag{3.9}$$

Substituting (3.7) into (3.9), we have

$$\begin{aligned}&\frac{\delta}{C}+\int_M \operatorname{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!}+\alpha l_i\int_M\langle\Psi(l_iu_{\varepsilon_i})(\bar\partial_+u_{\varepsilon_i}),\bar\partial_+u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!}\\&+(1-\alpha)l_i\int_M\langle\Psi(l_iu_{\varepsilon_i})(\bar\partial_-u_{\varepsilon_i}),\bar\partial_-u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!}\le\varepsilon_i\max_M|\Phi(H_0)|_{H_0},\end{aligned}\tag{3.10}$$

Consider the function

$$l\Psi(lx,ly)=\begin{cases}l, & x=y;\\ \frac{e^{l(y-x)}-1}{y-x}, & x\neq y.\end{cases}$$

From (3.8), we may assume that $(x,y)\in[-C_2,C_2]\times[-C_2,C_2]$. It is easy to check that

$$l\Psi(lx,ly)\longrightarrow\begin{cases}(x-y)^{-1}, & x>y;\\ +\infty, & x\le y,\end{cases}\tag{3.11}$$

increases monotonically as $l\to+\infty$. Let $\zeta\in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R}^+)$ satisfying $\zeta(x,y)<(x-y)^{-1}$ whenever $x>y$. From (3.10), (3.11) and the arguments in [37, Lemma 5.4], we have

$$\begin{aligned}&\frac{\delta}{C}+\int_M \operatorname{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!}+\alpha\int_M\langle\zeta(u_{\varepsilon_i})(\bar\partial_+u_{\varepsilon_i}),\bar\partial_+u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!}\\&+(1-\alpha)\int_M\langle\zeta(u_{\varepsilon_i})(\bar\partial_-u_{\varepsilon_i}),\bar\partial_-u_{\varepsilon_i}\rangle_{H_0}\frac{\omega_\pm^n}{n!}\le\varepsilon_i\max_M|\Phi(H_0)|_{H_0}\end{aligned}\tag{3.12}$$

for $i\gg0$. In particular, we take $\zeta(x,y)=\frac{1}{3C_2}$. It is obvious that when $(x,y)\in[-C_2,C_2]\times[-C_2,C_2]$ and $x>y$, $\frac{1}{3C_2}<\frac{1}{x-y}$. This implies that

$$\begin{aligned}&\frac{\delta}{C}+\int_M \operatorname{tr}\{\Phi(H_0)u_{\varepsilon_i}\}\frac{\omega_\pm^n}{n!}+\int_M\frac{1}{3C_2}(\alpha|\bar\partial_+u_{\varepsilon_i}|^2_{H_0}+(1-\alpha)|\bar\partial_-u_{\varepsilon_i}|^2_{H_0})\frac{\omega_\pm^n}{n!}\\&\le\varepsilon_i\max_M|\Phi(H_0)|_{H_0}\end{aligned}\tag{3.13}$$

for $i\gg0$. Then we have

$$\int_M(\alpha|\bar\partial_+u_{\varepsilon_i}|^2_{H_0}+(1-\alpha)|\bar\partial_-u_{\varepsilon_i}|^2_{H_0})\frac{\omega_\pm^n}{n!}\le3C_2^2\max_M|\Phi(H_0)|_{H_0}\mathrm{Vol}(M,g).$$

Thus, u_{ε_i} are bounded in L_1^2. Then we can choose a subsequence $\{u_{\varepsilon_{i_j}}\}$ such that $u_{\varepsilon_{i_j}}\rightharpoonup u_\infty$ weakly in L_1^2, still denoted by $\{u_{\varepsilon_i}\}$ for simplicity. Noting that $L_1^2\hookrightarrow L^2$, we have

$$1=\int_M|u_{\varepsilon_i}|^2_{H_0}\to\int_M|u_\infty|^2_{H_0}.$$

This indicates that $\|u_\infty\|_{L^2}=1$ and u_∞ is non-trivial.

Using (3.12) and following a similar discussion as in [37, Lemma 5.4], it holds

$$\begin{aligned}\frac{\delta}{C}+\int_M \operatorname{tr}\{\Phi(H_0)u_\infty\}\frac{\omega_\pm^n}{n!}+\alpha\int_M\langle\zeta(u_\infty)(\bar\partial_+u_\infty),\bar\partial_+u_\infty\rangle_{H_0}\frac{\omega_\pm^n}{n!}\\ +(1-\alpha)\int_M\langle\zeta(u_\infty)(\bar\partial_-u_\infty),\bar\partial_-u_\infty\rangle_{H_0}\frac{\omega_\pm^n}{n!}\leq 0.\end{aligned}\tag{3.14}$$

Step 2 Using Uhlenbeck and Yau's trick from [40], we construct a subsheaf which contradicts the α-semi-stability of E.

From (3.14) and the technique in [37, Lemma 5.5], we conclude that the eigenvalues of u_∞ are constant almost everywhere. Let $\mu_1<\mu_2<\cdots<\mu_l$ be the distinct eigenvalues of u_∞. The facts that $\operatorname{tr}(u_\infty)=\operatorname{tr}(u_{\varepsilon_i})=0$ and $\|u_\infty\|_{L^2}=1$ force $2\leq l\leq r$. For each μ_j $(1\leq j\leq l-1)$, we construct a function

$$P_j:\mathbb{R}\longrightarrow\mathbb{R}$$

such that

$$P_j=\begin{cases}1, & x\leq\mu_j,\\ 0, & x\geq\mu_{j+1}.\end{cases}$$

Setting $\pi_j=P_j(u_\infty)$, from [18], we have

(i) $\pi_j\in L_1^2$;
(ii) $\pi_j^2=\pi_j=\pi_j^{*H_0}$;
(iii) $(\mathrm{Id}_E-\pi_j)\bar\partial_\pm\pi_j=0$.

By Uhlenbeck and Yau's regularity statement of L_1^2-subbundle [40], $\{\pi_j\}_{j=1}^{l-1}$ determine $l-1$ subsheaves of E. Set $E_j=\pi_j(E)$. Since $\operatorname{tr}(u_\infty)=0$ and $u_\infty=\mu_l\cdot\mathrm{Id}_E-\sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\pi_j$, it holds

$$\mu_l\operatorname{rank}(E)=\sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\operatorname{rank}(E_j).\tag{3.15}$$

Construct

$$\nu=\mu_l\deg_\alpha(E)-\sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\deg_\alpha(E_j).$$

On one hand, substituting (3.15) into ν,

$$\nu=\sum_{\alpha=1}^{l-1}(\mu_{j+1}-\mu_j)\operatorname{rank}(E_j)\left(\frac{\deg_\alpha(E)}{\operatorname{rank}(E)}-\frac{\deg_\alpha(E_j)}{\operatorname{rank}(E_j)}\right).\tag{3.16}$$

On the other hand, from [18], we have the following Chern-Weil formula

$$\deg_\alpha(E_j)=\frac{1}{2\pi}\int_M\left(\operatorname{tr}(\pi_j\mathcal{K}(H_0))-\alpha|\bar\partial_+\pi_j|_{H_0}^2-(1-\alpha)|\bar\partial_+\pi_j|_{H_0}^2\right)\frac{\omega^n}{n!},\tag{3.17}$$

where $\mathcal{K}(H_0) = \alpha\sqrt{-1}\Lambda_+ F_+^{H_0} + (1-\alpha)\sqrt{-1}\Lambda_- F_-^{H_0}$. Substituting (3.17) into ν,

$$\begin{aligned}
2\pi\nu =& \mu_l \int_M \operatorname{tr}(\mathcal{K}_{H_0}) \\
&- \sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\left\{\int_M \operatorname{tr}(\pi_j \mathcal{K}_{H_0}) - \int_M \left(\alpha|\bar{\partial}_+\pi_j|^2_{H_0} + (1-\alpha)|\bar{\partial}_+\pi_j|^2_{H_0}\right)\right\} \\
=& \int_M \operatorname{tr}\Big(\mu_l \mathrm{Id}_E - \sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\pi_j\Big)\mathcal{K}_{H_0} \\
&+ \sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\int_M \left(\alpha|\bar{\partial}_+\pi_j|^2_{H_0} + (1-\alpha)|\bar{\partial}_+\pi_j|^2_{H_0}\right) \\
=& \int_M \operatorname{tr}(u_\infty \mathcal{K}_{H_0}) + \int_M \alpha\langle\sum_{\alpha=1}^{l-1}(\mu_{j+1}-\mu_j)(\mathrm{d}P_j)^2(u_\infty)(\bar{\partial}_+ u_\infty), \bar{\partial}_+ u_\infty\rangle_{H_0} \\
&+ \int_M (1-\alpha)\langle\sum_{\alpha=1}^{l-1}(\mu_{j+1}-\mu_j)(\mathrm{d}P_j)^2(u_\infty)(\bar{\partial}_- u_\infty), \bar{\partial}_- u_\infty\rangle_{H_0},
\end{aligned}$$

where the function $\mathrm{d}P_j : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\mathrm{d}P_j(x,y) = \begin{cases} \dfrac{P_j(x)-P_j(y)}{x-y}, & x \neq y; \\ P_j'(x), & x = y. \end{cases}$$

By simple calculation, if $\mu_a \neq \mu_b$,

$$\sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)(\mathrm{d}P_j)^2(\mu_a,\mu_b) = |\mu_a-\mu_b|^{-1}. \tag{3.18}$$

Since $\operatorname{tr}(u_\infty) = 0$, by (3.14) and the same arguments in [24, p. 793-794], it holds that

$$2\pi\nu \leq -\frac{\delta}{C}. \tag{3.19}$$

Combining (3.16) with (3.19), we have

$$\sum_{j=1}^{l-1}(\mu_{j+1}-\mu_j)\operatorname{rank}(E_j)\left(\frac{\deg_\alpha(E)}{\operatorname{rank}(E)} - \frac{\deg_\alpha(E_j)}{\operatorname{rank}(E_j)}\right) < 0,$$

which contradicts the α-semi-stability of E. □

Theorem 3.3. *If $(E, \bar{\partial}_+, \bar{\partial}_-)$ admits an approximate α-Hermitian-Einstein structure, then it is α-semi-stable.*

Proof. Let $\mathcal{F}$ be any saturated subsheaf with rank p. Then by [22, p. 119], $\wedge^p E \otimes \det \mathcal{F}^{-1}$ admits an approximation α-Hermitian-Einstein structure with the constant

$$\lambda = \frac{2p\pi}{\mathrm{Vol}(M)}(\mu_\alpha(E) - \mu_\alpha(\mathcal{F})). \tag{3.20}$$

The injective map $\det(\mathcal{F}) \to \wedge^p E$ induced by the inclusion $\mathcal{F} \hookrightarrow E$, defines a section of $\wedge^p E \otimes \det \mathcal{F}^{-1}$, say s. By construction, s is an $I_\pm$-holomorphic section with respect to the induced $I_\pm$-holomorphic structures. By the vanishing theorem

[18, Theorem 5.4], we have $\lambda \geq 0$. This together with (3.20) gives $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(E)$, i.e. $(E, \bar{\partial}_+, \bar{\partial}_-)$ is α-semi-stable.

□

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